Seminar on the Atiyah–Singer index theorem

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Abstract

The Atiyah–Singer index theorem was one of the most influencial theorems from the twentieth century. It states that the analytic index of elliptic differential operators (i. e. the difference between kernel and kokernel) can be expressed in purely topological terms. It has numerous applications to geometry and topology, e. g. to the study of Riemannian metrics of positive scalar curvature.

The theorem can be proven in several different ways. In this seminar we follow the excellent textbook [1] and learn about the so called heat equation proof, which works for Dirac operators. Other proofs use cobordism theory (cf. [2]) or K-theory (cf. [3,4]).

Prerequisites

The participants should be familiar with the basics of Riemannian geometry (e. g. from Thomas Schick's lecture course in the winter semester) and differential forms (as it is usually done here in Göttingen in Analysis III). Having further background knowledge in algebraic topology is helpful, but not necessary.

Most of the required knowledge is recapitulated in the first two chapters of the book [1]. Please go through pages 9–29 on your own to see how familiar you are with these topics. Our seminar will start on page 41 with Chapter 3.

Module signatures

B.Mat.3413: Seminar im Zyklus „Differenzialgeometrie“
B.Mat.3414: Seminar im Zyklus „Algebraische Topologie“
B.Mat.3412: Seminar im Zyklus „Analysis Partieller Differenzialgleichungen“
M.Mat.4813: Seminar on differential geometry
M.Mat.4814: Seminar on algebraic topology
M.Mat.4812: Seminar on analysis of partial differential equations

How to participate

The seminar will presumably take place on Tuesdays 2–4 PM. Due to the tight schedule, there will be no preliminary meeting.

If you want to participate, please send an email to christopher.wulff@mathematik.uni-goettingen.de mentioning the three talks in order of preference which you are most interested in.

Bibliography

[1] John Roe: *Elliptic operators, topology and asymptotic methods*
[2] R.S. Palais: *Seminar on the Atiyah-Singer index theorem*
[3] M. Atiyah, I. Singer: *The index of elliptic operators I*
[4] H. B. Lawson, M. L. Michelsohn: *Spin geometry*

List of talks

The schedule is rather tight and in many talks there will not be enough time to present everything in detail. Therefore, you should present the results and focus on the most relevant proofs and ideas. Please consult me in advance about what to leave out! Ideally, the audience will also have gone through the material in advance and will be able to indicate which details they want to see discussed during the talks.

In some cases it might not be possible to squeeze everything into one session. Then we can make use of the few spare days in the schedule to continue the discussion.

All page and chapter numbers in the plan refer to the book [1].

Talk 1: Clifford bundles and Dirac operators: p. 41–46 middle; some examples: p. 49–51 until example 3.24

Talk 2: Spin groups: p. 55–62

Talk 3: Clifford bundles and curvature: p. 46–49; spin structures on manifolds: p. 63–66

Talk 4: Sobolev spaces, Gardings inequality, elliptic estimate: p. 71–77

Talk 5: Further analytic properties of Dirac operators: p. 78–83

Talk 6: The heat and wave equation, part I: p. 95–100

Talk 7: The heat and wave equation, part II: p. 101–107

Talk 8: Trace class operators: p. 110–114 top; the harmonic oscillator: p. 119–124

Talk 9: The index problem: Chapter 11

Talk 10: Filtered algebras and symbols, Getzler symbols: p. 151–157

Talk 11: The Getzler calculus of the heat kernel, the exact solution, the index theorem: p. 157–165