Seminar on Counterexamples in Topology

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Metadata

Interested in participating? Register in Stud.IP and send me an email.

Day and time: Thursdays 10-12 AM in lecture hall 4.

Preliminary knowledge: The course is intended to supplement the cycle on algebraic topology, but it should also be possible to participate if you have some basic knowledge about point-set topology from other courses.

Module signatures: The module signatures are:

M.Mat.4814.Mp: Seminar on algebraic topology
B.Mat.3414.Mp: Seminar im Zyklus „Algebraische Topologie“

Objectives

In topology, many theories work only for spaces whose topology is nice enough in certain ways. Among the properties that one might need are separation axioms like the Hausdorff property, regularity or normality; different type of compactness assumptions such as (local) compactness or paracompactness; countability properties such as separability or first/second countabilty; and (local/path/simple) connectedness.

In order to understand why they are needed, it is convenient to have some spaces at hand which are counterexamples to some of the properties but not to others. The objective of the seminar is to become acquainted with a couple of standard counterexamples.

References

We will use mainly the following textbooks:

[W] *S. Willard*, General Topology, Addison-Wesley (1970).

[SS] *L. A. Steen, J. A. Seebach Jr.*, Counterexamples in Topology, 2nd edition, Springer (1978).

Plan of talks

Unless stated otherwise, the description of the talks refer to the book [W].

Talk 1: Bases and subbases, weak and strong topologies

Introduce bases and subbases (Section 5). Theorem 5.4 can be skipped. Theorem 5.6 is important, since topologies will often be given by specifying a subbase $\mathscr{C}$. Also mention that the topology generated by $\mathscr{C}$ in this way is the coarsest in which all sets of $\mathscr{C}$ are open and that continuity of functions can be checked on a subbase.

Introduce the weak topology (Definition 8.9 and Theorem 8.10) and present the product topology (Definition 8.3–Theorem 8.8) as a special case.

Dually, introduce the strong topology (Problem 9H). Note that the disjoint union of spaces X_α can be introduced as the set $\coprod X_\alpha$ equipped with the strong topology with respect to the collection of inclusions $X_\alpha \hookrightarrow \coprod X_\alpha$. Then present the quotient topology (Definition 9.1–Theorem 9.4) as a special case.

Since many of these notions have already been introduced in Algebraic Topology I, you do not have to work out everything in detail.

Talk 2: Convergence

Discuss inadequacy of sequences (Section 10). The main counterexample of your talk should be Example 10.6(a). Part (b) is equally interesting, but we probably need more time to understand it in detail, so it is better to postpone it to a later day.

Afterwards present nets and how they solve the problem (Section 11). Example 11.4(a) is central, since it is the prototypical type of net appearing in proofs.

In your talk you will also have to introduce some definitions from Section 4 along the way (neighbourhoods, neighbourhood systems, neighbourhood bases).

If you have time left, you can fill it up with problems from Sections 11 and 12 or briefly adress filters (Section 12) as a competing concept to nets.

Talk 3: Separation axioms

Seciton 13 about T_0-, T_1- and Hausdorff spaces. Note that Theorems 13.10–13.13 are actually trivial corollaries of Theorem 13.7(c), i. e. the proofs are even simpler than the ones given in the book. Problems 13C, 13D and 13H seem to be interesting.

Since Section 13 is rather short and there is not a lot of interesting stuff to say about T_0- and T_1-spaces, you could use spare time to already start with regularity: Definition 14.1–Theorem 14.3.

Talk 4: Regularity and complete regularity

Section 14. Note that the previous speaker might already have presented Definition 14.1–Theorem 14.3.

Talk 5: Normal spaces

Section 15. Urysohn's Lemma (15.6) and Tietze's extension theorem (15.8) are the heart of this topic, because they are the reason why one considers normal spaces after all.

Talk 6: Countability properties

Section 16.

Disscussion session 1

Let us take a break from the normal talks and spend a whole session discussing further examples and counterexamples for the topological properties we covered so far.

In particular, it is now time to discuss the ordinal space (1.19-1.22 with the topology of 6D) and catch up on the omitted Example 10.6(b).

We can carry out this session in two different ways: Either we have one new speaker presenting all the examples or several of the previous speakers present individual examples from their sections.

Talk 7: Compactness

Section 17. This section is somewhat lengthy and hence there won't be enough time to discuss everything. Luckily that is not necessary, because all of us already know quite a bit about compactness from the basic lectures or Algebraic Topology I. In particular, you can kick out all the trivial examples (17.2(a)+(b),17.9(c)).

The Tychonoff plank (Example 17.12 together with Example 17.2(c)), however, are a very important source for counterexamples.

Talk 8: Local compactness

Section 18. Make sure to also present the beautiful conterexample from problem 18G. In this context you can also mention complete Hausdorffness (problem 14G).

Talk 9: Paracompactness

Section 20. In addition to Example 20.11, you can also give the definition of the famous long line.

Another nice example is the Prüfer surface: It is a two-dimensional manifold except that it is not second countable but only separable. In the definition of manifolds, second countability is equivalent to separability plus paracompactness and here the latter fails.

The definitions of the long line and the Prüfer surface can be found on Wikipedia, but I can also try to find other sources.

Talk 10: Connectedness

Sections 26 and 27. Since the notions of (local) (path-)connectedness have been discussed in Algebraic topology I, we do not want to see all the details. Focus on giving the definitions and the most illustrative examples and counterexamples. We want to see pictures, pictures, pictures!

We should also see the related concept of (local) simple connectedness and see the Hawaiian earrings. It is not covered in [W], but in many other standard textbooks.

Disscussion session 2

We want to conclude the semester with another discussion session in which we can see even more beautiful counterexamples. In particular, we should now also take a look at [SS] to find more interesting spaces which were not covered by [W].

General comments on the talks

- Please contact me one or two weeks in advance to discuss the exact material of your talk. Not everything in the theory parts of the sections in [W] has to be covered, whereas many of the problems are interesting and could very well be discussed.

- Please try to avoid the T_n-notation for the separation axioms as much as possible. I really dislike it, because it is hard to remember what they actually mean. In fact, almost no topologist I know uses it. I prefer calling them by their properties:

  - T_0- and T_1-spaces unfortunately do not have a different name.
  - T_2-spaces are *Hausdorff spaces.*
  - T_3-spaces are defind as regular T_1-spaces, but this is equivalent to being *regular Hausdorff spaces.*
  - $T_{3\frac{1}{2}}$-spaces are called *Tychonoff spaces* and are defined as completely regular T_1-spaces. This is equivalent to being *completely regular Hausdorff spaces.*
  - T_4-spaces are defined as normal T_1-spaces, but I like to call them *normal Hausdorff spaces.*