Seminar "Topics in functional analysis"

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Winter Semester 2022/23

Metadata

Interested in participating? Send me an email. I will then register you for the hidden Stud.IP group used to organize the seminar.

Day and time: TBA

Preliminary knowledge: Functional analysis

Module signatures: Presumably the seminar will count for the following modules (TBC):
M.Mat.4812: Seminar on analysis of partial differential equations
M.Mat.4815: Seminar on mathematical methods in physics
M.Mat.4825: Seminar on non-commutative geometry
B.Mat.3412: Seminar im Zyklus "Analysis Partieller Differenzialgleichungen"
B.Mat.3415: Seminar im Zyklus "Mathematische Methoden der Physik"
B.Mat.3425: Seminar im Zyklus "Nichtkommutative Geometrie"

Objective

The objective of the seminar is to learn several topics which normally don't fit into a standard lecture course on functional analysis. In particular they haven't been discussed in the summer semester 2022. As a guiding theme we ask ourselves the question:

How to differentiate non-differentiable functions?

Specifically, we will cover Sobolev spaces, fourier transform, locally convex spaces and, as crowning culmination, distributions.

References

We will use the following two textbooks:

[**W**] *D. Werner*, Funktionalanalysis. 8th revised edition. Berlin: Springer Spektrum (2018)
`https://opac.sub.uni-goettingen.de/DB=1/XMLPRS=N/PPN?PPN=1018141219`

[**G**] *G. Grubb*, Distributions and operators. New York, NY: Springer (2009)
`https://opac.sub.uni-goettingen.de/DB=1/PPN?PPN=1816480398`
The electronic version of the book has just been purchased for us. The library staff hopes that this resource will be used intensely, because the license has been very expensive. Don't disappoint them!

[W] will be our main source for the first 8 talks. It is one of the best textbooks on functional analysis ever written. Unfortunately it is only available in German. Non-german speakers should therefore pick one of the talks on distributions, where we use [G] as our main source. Parts 1 and 2 are also covered by many other standard textbooks on functional analysis. Therefore you will always find good literature to read along.

Plan of talks

Part 1: Sobolev spaces and Fourier transform

Talk 1: Space of test functions, weak derivatives, Sobolev spaces (N.N., October 24–28)

[W] Present the second half of Section V.1, starting with Definition V.1.9. The proof of Lemma V.I.10 makes use of Friedrichs mollifiers, which were introduced in Aufgabe II.5.6. You do not have to solve the exercise completely, because I assume that you won't have enough time to present it in your talk, but you should briefly sketch the key steps.

Talk 2: Fourier transform of L^1-functions, Schwartz space (N.N., October 31–November 4)

[W] Definition V.2.1–Lemma V.2.7.

Talk 3: Fourier–Plancharel transform, Hausdorff–Young inequality, Description of Sobolev spaces via Fourier transform. (N.N., November 7–11)

[W] Satz V.2.8–Lemma V.2.11 and Satz V.2.14.

Talk 4: Sobolev lemma, Relich embedding theorem (N.N., November 14–18)

[W] Satz V.2.12 and Satz V.2.13.

Part 2: Locally convex spaces

Talk 5: Definition and first examples (N.N., November 21–25)

[W] Section VIII.1. There will probably not be enough time to present all examples right away, so we content ourselves with (a),(b),(c) for now. Other examples can be discussed at the end of the talk, if time is left. Otherwise we shift them to the next talk.

Talk 6: Examples and continuous functionals (N.N., November 28–December 2)

[W] Present the remaining examples from Section VIII.1. This helps us recall the notion of locally convex spaces. Afterwards present the first part of Section VIII.2 until the example at the bottom of page 432 (comparing the topologies of pointwise and uniform convergence).

Talk 7: Net convergence and examples (N.N., December 5–9)

[W] Introduce/recall the notion of nets and their convergence (second half of page 558) and present Satz B.2.1 and Satz B.2.1 without proof. Continue in Section VIII.2 with pages 433–435. Try to present at least example (a). The other examples are not so important and hence it is no problem if time runs out.

Talk 8: Hahn–Banach theorems (N.N., December 12–16)

[W] Present Satz VIII.2.8–Corollary VIII.2.13. Note that the Hahn–Banach separation theorems for normed spaces have not been discussed during the functional analysis lecture in the summer semester 2022. Therefore, you should not just cite the results from Section III.2, but give the proofs of Lemma III.2.4 and Theorem III.2.4 in the context of locally convex spaces.

For now, no talk will be planned for the week December 19–23.

We keep this session in reserve in case we need to do some rescheduling.

Part 4: Distributions

Talk 9: Definition and examples (N.N., January 9–13)

[G] Theorem 2.5 and Section 3.1. Here, $C_0^\infty(\Omega)$ is the space which is called $\mathscr{D}(\Omega)$ in [W]. The proof of Theorem 2.5 (except part (c)) can be found in Section VIII.5 in [W].

Alternatively, you can also use Section VIII.5 of [W] up to page 461 as source, if you prefer. Note that there is a slight difference between the sources: [W] considers all compact subsets $K \subset \Omega$ whereas [G] works with an exhaustive sequence of compact subsets $K_i \subset \Omega$ such that $K_i \subset \mathring{K}_{i+1}$.

You can omit the distributions $\Lambda_{f,\alpha}$ on page 30 of [G] and use the dipole distribution (example (d) on page 461 of [W]) as an example of a distribution of order 1. Apart from that, try to present as many examples as possibles (pages 28,29 in [G] and/or pages 460,461 in [W]).

Talk 10: Rules of calculus (N.N., January 16–20)

[G] Theorem 2.6 and Section 3.2 until Lemma 3.7, see also pages 462,463 in [W].

Talk 11: Convolution and methods of approximation (N.N., January 23–27)

[G] Theorems 3.8,3.9 and Section 3.4 until Theorem 3.18.

Talk 12: Realization of differential operators (N.N., January 30–February 3)

[G] Section 4.1.

Talk 13: Temperate distributions and their Fourier transform (N.N., February 6–10)

[G] Sections 5.2 and 5.3 or [W] pages 463–466. The scope of the latter source seems to be more appropriate for a seminar talk.

General comments on the talks

- Please don't hesitate to contact me one or two weeks before you talk (or even earlier) if you have any questions.
- In particular let me know if you feel that the material is too much or not enough for your talk. Then we can discuss what to leave out or what to add.