A-INFINITY FUNCTORS AND HIGHER TORSION

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Abstract. These are notes from a series of lectures given in Göttingen Sept 1-4, 2003 on the topic of higher Franz-Reidemeister torsion. The idea is to give the main ideas without too many technical details. So many of the definitions and statements are loosely worded. Precise statements can be found in [Igu02a], [Igu02b], [Igu03].

1. Lecture I

In the first lecture I gave a rough idea about what kind of thing is higher Franz-Reidemeister torsion. I also showed how calculations can be made using the basic properties of the higher torsion. I began the lecture with the classical degree 0 invariant.

1.1. \(\tau_{0}^{FR}\) : Franz-Reidemeister torsion. Suppose that \(M\) is a compact smooth manifold with boundary \(\partial M = \partial_0 M \cup \partial_1 M\). Suppose also that
\[H_n(M, \partial_0 M; \mathcal{F}) = 0,\]
where \(\mathcal{F}\) is a Hermitian coefficient system\(^1\) on \(M\) given by a unitary representation of the fundamental group
\[\rho : \pi_1 M \to U(m).\]
In that case the Franz-Reidemeister (FR)-torsion is an invariant
\[\tau_{0}^{FR}(M, \partial_0; \mathcal{F}) \in \mathbb{R}.\]

Example 1.1. Take \(M = S^1\) with 1-dimensional unitary representation
\[\rho_u : \pi_1 S^1 = \mathbb{Z} \to U(1)\]
sending the generator to \(\rho_u(1) = u \neq 1 \in U(1)\). Let \(f_1 : S^1 \to \mathbb{R}\) be a Morse function with 2 critical points (a maximum and a minimum). Then the cellular chain complex \(C_*(f_1)\) is given by
\[C_1(f_1) = \mathbb{C} \xrightarrow{1-u} \mathbb{C} = C_0(f_1).\]

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\(^{1}\)When we say that \(\mathcal{F}\) is a Hermitian coefficient system on \(M\) we mean that for each \(x \in M\) we have a complex vector space \(\mathcal{F}_x\) with a Hermitian inner product \(\langle v, w \rangle\) (nondegenerate, linear in \(w\), conjugate linear in \(v\)) so that the vector spaces \(\mathcal{F}_x\) form a flat vector bundle over \(M\). Flat is the same as saying the structure group of the bundle is a discrete group. Hermitian is the same as saying this group is the unitary group \(U(m)\) where \(m = \dim \mathcal{F}_x\).
Figure 1. $L(p, q)$: Take double prism on regular $p$-gon. Identify each face on top with face on bottom $q$ spaces to right.

The FR-torsion is defined to be the real part of the log of the determinant of the boundary map:

$$\tau_{0}^{FR}(S^1; \rho_u) = \Re \log(1 - u) = - \sum_{s=1}^{\infty} \frac{u^s}{s}$$

**Example 1.2.** Take the lens space $M = L(p, q) = S^3/(\mathbb{Z}/p)$ where the generator of $\mathbb{Z}/p$ acts by the unitary transformation

$$
\begin{pmatrix}
  u & 0 \\
  0 & u^q
\end{pmatrix}
$$
on $S^3$ considered as the unit sphere in $\mathbb{C}^2$ where $u$ is a nontrivial $p$-th root of unity: $u^p = 1$. (Assume that $p$ is prime and $q$ is relatively prime to $p$.) Then $L(p, q)$ looks something like the join $S^1 \ast S^1$ of two circles and there is a unitary representation

$$\rho : \pi_1 L(p, q) = \mathbb{Z}/p \rightarrow U(1)$$
which sends the two circles to $u, u^q$ respectively. There is a Morse function $f_2 : L(p, q) \rightarrow \mathbb{R}$ with exactly 4 critical points of index 0,1,2,3 and the cellular chain complex of $f_2$ is

$$C_*(f_2) : \mathbb{C} \xrightarrow{1-u} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{1-u} \mathbb{C}.$$As we explain below, the FR-torsion of this lens space is

$$\tau_{0}^{FR}(L(p, q); \rho) = \Re \log(1 - u) + \Re \log(1 - u^q) \in \mathbb{R}.$$Franz and DeRham used these numbers to classify lens spaces up to homeomorphism. (See [Mil66].)

Franz-Reidemeister torsion has two important properties: Additivity ($\oplus \mapsto +$) and Suspension ($\Sigma \mapsto (-1)$). These are illustrated by the second example which gives a chain complex isomorphic to the chain complex of the first example plus a twice suspended copy with $u$ replaced by $u^q$:

$$C_*(f_2) = \left( \mathbb{C} \xrightarrow{1-u} \mathbb{C} \right) \oplus \Sigma^2 \left( \mathbb{C} \xrightarrow{1-u^q} \mathbb{C} \right)$$

$$\tau_{0}^{FR}(f_2) = \tau_0(1 - u) + (-1)^2 \tau_0(1 - u^q) = \Re \log(1 - u) + \Re \log(1 - u^q).$$
1.2. \( \tau_k^{FR} \): higher FR-torsion. Let \( M_t, t \in B \) be a smooth family of compact smooth manifolds parametrized by \( B \). Let \( \partial_0 M_t \subseteq \partial M_t \) be a smooth family of submanifolds of the boundary of \( M_t \). This is the same as saying that we have a smooth bundle 
\[ (M, \partial_0 M) \to (E, \partial_0 E) \xrightarrow{p} B. \]
with \( M_t = p^{-1}(t) \). Let \( \mathcal{F} \) be a Hermitian coefficient system on the total space \( E \) (given by a unitary representation \( \rho : \pi_1 E \to U(n) \)) and suppose that \( \pi_1 B \) acts trivially on \( H_*(M, \partial_0 M; \mathcal{F}) \).

Under these conditions the higher FR-torsion class 
\[ \tau_k^{FR}(E, \partial_0 E; \mathcal{F}) \in H^{2k}(B; \mathbb{R}) \]
is defined. This is a real cohomology class. It generalizes the FR-torsion which is an invariant of a single manifold \( M \) since 
\[ \tau_0(M; \partial_0 M) \in H^0(*; \mathbb{R}) = \mathbb{R} \]
Without knowing the definition, the higher FR-torsion can be calculated from its basic properties, namely, \( \tau_k^{FR} \) is natural, additive, sends suspension to \((-1)\) and commutes with transfer. We begin with the basic calculation, the only one which was ever done straight from the definition. ([Igu02a], Chapter 7.)

**Lemma 1.3.** Let \( S^1 \to E \to B \) be a circle bundle. Let \( \rho_u : \pi_1 E \to U(1) \) be a unitary representation which sends the generator of \( \pi_1 S^1 = \mathbb{Z} \) to \( u \in U(1) \). Then 
\[ \tau_k^{FR}(E; \rho_u) = -L_{k+1}(u) \frac{c_1(E)^k}{k!} \]
for all \( k > 0 \) (or for \( k = 0 \) and \( u \neq 1 \)) where \( L_{k+1} \) is the polylogarithm 
\[ L_{k+1}(u) := \Re \left( \frac{1}{i^k} \sum_{s=1}^{\infty} \frac{u^s}{s^{k+1}} \right). \]

**Remark 1.4.** This formula is inevitable. By naturality the higher torsion must be a multiple of \( c_1(E)^k \) since \( c_1 \) generates \( H^{2k}(BU(1); \mathbb{R}) \). The factor of \( L_{k+1}(u) \) is also dictated by the transfer formula which comes from the fact that a Morse function on a manifold induces one on any covering space. This will be explained later.

**Theorem 1.5.** If \( S^{2n-1} \to E \to B \) is the sphere bundle of a complex \( n \)-plane bundle \( \xi \) then 
\[ \tau_{2k}^{FR}(E) = (-1)^{k+1} \zeta(2k+1) ch_{2k}(\xi) \]
where \( ch_{2k}(\xi) \) is the Chern character\(^2\) and \( \zeta \) is the Riemann zeta function.

**Remark 1.6.** This was first proved by Ulrich Bunke [Bun] in the case of analytic torsion.

\(^2\)The Chern character \( ch_k(\xi) \in H^{2k}(B; \mathbb{Q}) \) is the characteristic class given by 
\[ ch_k(\xi) = \sum_{i=1}^{n} c_i(\lambda_i)^k / k! \]
if \( \xi \) is the sum of complex line bundles \( \lambda_i \). Since this formula is a symmetric polynomial in the classes \( c_i(\lambda_i) \) it can be expressed as a polynomial (with rational coefficients) in the Chern classes of \( \xi \). The Chern character is a ring homomorphism in the sense that \( ch_k(\xi \oplus \eta) = ch_k(\xi) + ch_k(\eta) \) and \( ch_k(\xi \otimes \eta) = \sum_{i+j=k} ch_i(\xi) \cup ch_j(\eta) \).
Proof. By the splitting principle\footnote{The splitting principle says that there exists a space $B'$ and a mapping $f : B' \to B$ so that the pull-back of $\xi$ to $B'$ splits as a sum of line bundles and so that $f^* : H^*(B) \to H^*(B')$ is a monomorphism. This condition implies that two natural cohomology classes in $B$ will be equal if and only if they are equal in $B'$. Therefore, to prove an equation of characteristic classes we may assume the bundle splits.} we may assume that $\xi$ splits as a sum of complex line bundles $\xi \cong \oplus \lambda_i$. This implies that the sphere bundle $E = S^{n-1}(\xi)$ is the fiberwise join of the circle bundles $S^1(\lambda_i)$. This in turn implies that

$$C_\ast(S^{2n-1}(\xi)) = C_\ast(S^1(\lambda_1)) \oplus \Sigma^2 C_\ast(S^1(\lambda_2)) \oplus \Sigma^4 C_\ast(S^1(\lambda_3)) \oplus \cdots$$

Since torsion sends $\oplus$ to addition and suspension to multiplication by $-1$, we get

$$\tau_{2k}^{FR}(S^{2n-1}(\xi)) = \sum_{i=1}^{n} \tau_{2k}^{FR}(S^1(\lambda_i))$$

$$= - \Re \left( \frac{1}{i^{2k}} \sum_{s=1}^{\infty} \frac{1}{s^{2k+1}} \right) \sum_{i=1}^{n} ch_{2k}(\lambda_i) = -(-1)^k \zeta(2k+1) ch_{2k}(\xi).$$

\[\square\]

1.3. Definition of higher FR-torsion. Outline:

Fiberwise Morse theory: Given a family of manifold pairs

$$p^{-1}(t) = (M_t, \partial_0 M_t), \quad t \in B$$

Assume that $\partial M_t = \partial_0 M_t \bigsqcup \partial_1 M_t$. Then

$$\exists f_t : (M_t, \partial_0 M_t) \to (I, 0)$$

s.t. $\nabla f_t \neq 0$ on $\partial M_t$ points inward along $\partial_0 M_t$ and outward along $\partial_1 M_t$.

The singular set

$$\Sigma(f_t) = \{ x \mid \nabla f_t(x) = 0 \} \subseteq \text{int } M_t$$

is finite.

Assume for a moment that $f_t$ is Morse $\forall t \in B$. (This is the first case in which higher FR-torsion was defined (J. Klein, 1989).)

Cellular chain complex with $\pi = \pi_1 E$ for $M \to E \to B$:

$$C_i(f_t) := \text{free } \mathbb{Z} \pi \text{-module gen by the Morse (nondeg) critical points of } f_t$$

If $\mathcal{F}$ is a coefficient system on $E$ given by the unitary representation $\rho : \pi \to U(m)$ then

$$C_i(f_t; \mathcal{F}) = C_i(f_t) \otimes_\pi \mathbb{C}^m$$

This is a vector space of dimension $m \times \# \text{ critical points of index } i$.

Example $M = S^1 \times S^1$. In this simple example the usual Morse function has four critical points: $a, b, c, d$ with indices $0, 1, 1, 2$ and critical values $f(a) < f(b) < f(c) < f(d)$. There was more discussion of this example in later lectures. Here, I made the first point that the groups $C_i(f)$ are actually given by the stalks of $\mathcal{F}$ but for convenience we write them as vector spaces generated by $a, b, c, d$:

$$C_\ast(f; \mathcal{F}) : \quad \mathbb{C} \to \mathbb{C}^2 \to \mathbb{C}$$

generators : $d \quad b, c \quad a$

Actually: $\mathcal{F}_d \to \mathcal{F}_b \oplus \mathcal{F}_c \to \mathcal{F}_a$
Suppose that $H_*(M_t, \partial_0 M_t; \mathcal{F}) = 0$. Then the chain complex $C_*(f_t; \mathcal{F})$ is acyclic. We can then invoke the two index theorem$^4$ which says that a family of based acyclic chain complexes can be deformed into two indices.

A family of based acyclic chain complexes $/\mathcal{C}$ in two indices is the same as a family of invertible complex matrices. We interpret this as something resembling a mapping to Volodin K-theory

$$\xi : B \rightarrow V(\mathbb{C}) \simeq \Omega BGL(\infty, \mathbb{C})^+$$

End of lecture one.

1.4. Coverings and polylogarithms. I never got to the explanation about why Morse functions on covering spaces leads to polylogarithms. So, I will explain it here.

We haven’t defined the higher FR-torsion yet but we know that one of its basic properties is that the higher torsion of a direct sum of two families of acyclic chain complexes should be equal to the sum of the torsions of the individual families. One case in which this occurs is when the coefficient system $\mathcal{F}$ is a direct sum of two coefficient systems:

**Lemma 1.7.** $\tau^F_k(E, \partial_0; \mathcal{F}_1 \oplus \mathcal{F}_2) = \tau^F_k(E, \partial_0; \mathcal{F}_1) + \tau^F_k(E, \partial_0; \mathcal{F}_2)$.

Now take an $m$-fold covering map

$$q : \widetilde{M} \rightarrow M$$

of compact, connected smooth manifolds. Take a Hermitian coefficient system $\mathcal{F}$ on $\widetilde{M}$ and push it down to $M$. What happens? By definition, the stalks of the coefficient system $q_* \mathcal{F}$ are given by

$$q_* \mathcal{F}_x = \bigoplus \mathcal{F}_y$$

where the direct sum is over all $y \in q^{-1}(x)$.

We can interpret this using group representations. Suppose that $G = \pi_1 M$. Then $H = \pi_1 \widetilde{M}$ is a subgroup of $G$ of index $m$. The coefficient system $\mathcal{F}$ is given by a unitary representation $\rho : H \rightarrow U(d)$. When we push this forward we get the induced representation $\text{Ind}_H^G(\rho) : G \rightarrow U(md)$ which can be determined using Frobenius reciprocity.

Next, look at the Morse theory. Given a Morse function $f : M \rightarrow \mathbb{R}$ we get a Morse function $f \circ q : \widetilde{M} \rightarrow \mathbb{R}$. From (1) we see that the cellular chain complex of $f \circ q$ with coefficients in $q^* \mathcal{F}$ is isomorphic to the cellular chain complex of $f$ with coefficients in $q_* \mathcal{F} = \text{Ind}_H^G(\rho)$.

$$C_*(\widetilde{M}; \rho) \cong C_*(M; \text{Ind}_H^G(\rho)).$$

$^4$There was a discussion during the lecture about the history of the 2-index theorem. Allen Hatcher gave the first proof. However, it had some major flaws, i.e., it was wrong. Hatcher used the 2-index theorem to prove stability of pseudoisotopies in the PL case. This proof had several major flaws and the theorem is still considered to be unknown even for smoothable PL manifolds. I used the general framework of Hatcher’s very short proof and made it into a very long careful proof of a different (and weaker) theorem: stability in the smooth category. One of the main ingredients is the 2-index theorem. Here we are talking about an easier result: the algebraic 2-index theorem. The proof is very similar to my proof of Hatcher’s 2-index theorem in the smooth category.
Isomorphic chain complexes will give the same torsion. So, suppose that \( \tilde{E} \) (with fiber \( \tilde{M} \)) is a covering space of the bundle \( E \) (with fiber \( M \)), \( G = \pi_1 E \) and \( H = \pi_1 \tilde{E} \).

Then
\[
\tau^{FR}_k(\tilde{E}; \rho) = \tau^{FR}_k(E; \text{Ind}^G_H(\rho)).
\]

Now suppose that \( E, \tilde{E} \) are both circle bundles, \( H = \mathbb{Z}/n \) and \( G = \mathbb{Z}/nm \). Let \( \rho = \rho_u : H \to U(1) \) be a 1-dimensional representation which sends the generator of \( \pi_1 \tilde{M} = \pi_1 S^1 = \mathbb{Z} \) to \( u \in U(1) \). By Frobenius reciprocity, the induced representation must be the direct sum of all representations of \( G = \mathbb{Z}/nm \) which restrict to \( \rho_u \):
\[
\text{Ind}^G_H(\rho_u) = \bigoplus \rho_z
\]
where the sum is over all roots of unity \( z \in U(1) \) so that \( z^m = u \). This gives:
\[
(2) \quad \tau^{FR}_k(\tilde{E}; \rho_u) = \sum_{z^m = u} \tau^{FR}_k(E; \rho_z).
\]

However, the higher FR-torsions of \( E, \tilde{E} \) are related by another equation:

**Lemma 1.8.** \( \tau^{FR}_k(E; \rho_z) = m^k \tau^{FR}_k(\tilde{E}; \rho_z) \).

**Proof.** Since \( H^k(BU(1); \mathbb{R}) \) is 1-dimensional, the higher FR-torsion \( \tau^{FR}_k \), whatever it is, must be some fixed multiple of \( c^k_1 \). By looking at circle bundles over \( S^2 \) we see that the first Chern class of the circle bundles \( E, \tilde{E} \) are related by the equation
\[
c_1(E) = mc_1(\tilde{E}).
\]
Therefore, \( \tau^{FR}_k(E) = ac_1(E)^k = atm^k c_1(\tilde{E})^k = m^k \tau^{FR}_k(\tilde{E}) \) if we have the same coefficients.

Combining this lemma with (2) we see that
\[
(3) \quad \tau^{FR}_k(\tilde{E}; \rho_u) = \sum_{z^m = u} m^k \tau^{FR}_k(\tilde{E}; \rho_z).
\]

Now consider \( \tau^{FR}_k(\tilde{E}; \rho_u) \) as a function of \( u \). There is the following conjecture:

**Conjecture 1.9.** Up to multiplication by one scalar, polylogarithms \( L(u) = L_{k+1}(u) \) (defined in Lemma 1.3) are the only functions defined on roots of unity \( u \) satisfying the equation
\[
L(u) = m^k \sum_{z^m = u} L(z).
\]

This conjecture would imply that \( \tau^{FR}_k(E; \rho_u) \) on circle bundles is some fixed multiple of \( L_{k+1}(u)c_1(E)^k \). I don’t know how to prove the conjecture. However, I also proved that \( L(u) \) is a smooth function on \( U(1) \backslash \{1\} \) and the conjecture is easy in this case. This proves the following.

**Theorem 1.10.** For every \( k \geq 1 \) there is a fixed real number \( a_k \) so that
\[
\tau^{FR}_k(E; \rho_u) = a_k L_{k+1}(u)c_1(E)^k
\]
for any oriented circle bundle \( E \).
In my book ([Igu02a], Chapter 7) I calculated the value of $a_k$ by integrating the Kamber-Tondeur form. The result is

$$a_k = -\frac{1}{k!}$$

This also holds for $k = 0$ since $L_1(u) = -\Re \log(1 - u)$.

2. Lecture II

These are notes from my second lecture on higher Franz-Reidemeister torsion. I began with another discussion of $S^1 \times S^1$.

2.1. Upper triangular matrices. The standard Morse function on $S^1 \times S^1$, with critical points $a, b, c, d$ as before, does not have a well defined cellular chain complex since there are two trajectories of $-\nabla f$ going from $c$ down to $b$. We can correct this by tilting the torus. There are four ways to do this. There was a discussion in the lecture about which two we should take. In any two case the cellular chain complexes differ by an upper triangular change of coordinates:

$$\begin{align*}
C & \xrightarrow{g_2} C \\
d_2 & \downarrow d_2^t \\
C^2 & \xrightarrow{g_1} C^2 \\
d_1 & \downarrow d_1^t \\
C & \xrightarrow{g_3} C
\end{align*}$$

where $g_1, g_3$ are the identity maps and

$$g_2 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

is a $\sigma$-upper triangular matrix where $\sigma$ refers to the ordering $b < c$ of the basis of $C_2(f) \cong \mathbb{C}^2$. 

\begin{figure}
\centering
\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\node (c) at (2,0) {$c$};
\node (d) at (3,0) {$d$};
\draw (a) -- (b) -- (c) -- (a);
\draw (b) -- (d) -- (a);
\end{tikzpicture}
\caption{The torus needs to be tilted (or twisted) to get a well-defined cellular chain complex.}
\end{figure}
We see from this example that, in a family of Morse functions, the basis for $C_*(f_t)$ changes by $\sigma$-upper triangular matrices causing the boundary maps to change by right or left multiplication by a $\sigma$-upper triangular matrix.

2.2. Volodin K-theory.

**Definition 2.1** (Volodin). Suppose that $R$ is an associative ring. Then the Volodin space is the direct limit $V_\bullet(R) = \lim_{\to} V_\bullet(n, R)$ where $V_\bullet(n, R)$ is the simplicial set\(^5\) (≃ simplicial complex in this case) with vertex set $V_0(n, R) := GL(n, R)$ (invertible $n \times n$ matrices)

$$V_k(n, R) = \{(g_0, g_1, \cdots, g_k) \in GL(n, R)^{[k]} | \exists \sigma \forall i, j g_i^{-1} g_j \in T_\sigma(n, R)\}$$

where $T_\sigma(n, R) \subseteq GL(n, R)$ is the group of $\sigma$ upper triangular\(^6\) $n \times n$ matrices with 1’s on the diagonal.

Stabilization $V_\bullet(n, R) \to V_\bullet(n + 1, R)$ is given by $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$.

**Theorem 2.2** (Wagoner-Vasserstein (See [Sus81].)). $|V_\bullet(R)| \simeq \Omega BGL(R)^+$

In other words, there is a fibration sequence:

$$|V_\bullet(R)| \xrightarrow{\chi} \mathbb{Z} \to \mathbb{Z} \times BGL(R)^+$$

and $\pi_n|V_\bullet(R)| = K_{n+1}R$.

2.3. Plan. We review our plan for defining the higher FR-torsion. Watch for a choice of two parameters in the definition. We start with a smooth bundle pair $(M, \partial_0 M) \to (E, \partial_0 E) \xrightarrow{\rho} B$.

We interpret this as a family of manifold pairs $(M_t, \partial_0 M_t), t \in B$. We do Morse theory $f_t : (M_t, \partial_0 M_t) \to (I, 0)$.

We get a family of cellular chain complexes $C_*(f_t)$ which are free $R$-complexes ($R = \mathbb{Z}\pi$ where $\pi = \pi_1 E$ or $R = \mathcal{M}_m(\mathbb{C})$ for a representation $\rho : \pi \to U(m)$). We assume the complex is *acyclic* and in 2-indices:

$$\xi(t) = C_*(f_t; \mathcal{F}) : \mathbb{C}^n \xrightarrow{\sim} \mathbb{C}^n \in GL(n, \mathbb{C})$$

This defines a mapping (when $m = 1$)

$$\xi : B \to |Wh_\bullet(C, U(1))|$$

where $Wh_\bullet(C, U(1))$ is a variation of the Volodin space $V_\bullet(C)$. The next step is to use the Borel regulator map

$$b_k : K_{2k+1}C \to \mathbb{R}$$

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\(^5\)Recall from Michael Weiss’ lecture that a simplicial set is a functor $[k] \to V_k(n, R)$.

\(^6\)A matrix $A = (a_{ij})$ is $\sigma$-upper triangular if it has 1’s on the diagonal and for off diagonal entries $a_{ij} \neq 0$ only when $i < j$ in the partial ordering $\sigma$. 

which is given by a continuous cohomology class\(^7\)
\[c_k \in H^{2k+1}_c(GL(n, \mathbb{C}); \mathbb{R})\]

J. DuPont [Dup76] explained how this is given by a differential \(2k + 1\) form called the Kamber-Tondeur form. Since \(K_{2k+1}\mathbb{C} = \pi_{2k}[V_\bullet(\mathbb{C})]\) we can represent \(b_k\) by a \(2k\)-cocycle on the Volodin space which is also given by the Kamber-Tondeur form as we will see. This \(2k\) dimensional cohomology class on Volodin space is also defined on the Whitehead space. The higher FR-torsion is the pull-back of this class by \(\xi = C_\bullet(f_t; F)\):
\[\tau^{FR}_k(E, \partial_0E; F) := \xi^*(b_k)\]

2.4. Whitehead space. There are two basic differences between the Volodin space and the 2-index Whitehead space. In the definition of Volodin space the condition \(g_i^{-1}g_j \in T_\sigma(R)\) means that the matrices \(g_i\) differ by right multiplication by \(\sigma\)-upper triangular matrices. In the Whitehead space we also allow left multiplication by \(\sigma\)-upper triangular matrices. The other difference is that we allow (as morphisms in the category direction) change of basis by monomial matrices with coefficients in a group \(G\) (which we usually take to be the unitary group). For this reason we need to see that the Kamber-Tondeur form is invariant under left and right multiplication by unitary matrices.

2.5. Kamber-Tondeur form. Suppose that \(\partial_t : \mathbb{C}^n \rightarrow \mathbb{C}^n, t \in B\) is a smooth family of invertible matrices. Let
\[h_t := \partial_t \partial_t^*\]
Then, \(h_t = h_t^*\) is p.d. Hermitian\(^8\)
\[\omega := h_t^{-1}dh_t : \text{matrix valued 1-form on } B\]
Note that
\[d\omega = -\omega^2 \quad (\text{since } dh^{-1} = -h^{-1}dhh^{-1})\]
Thus
\[dTr(\omega^{2k+1}) = Tr(\omega^{2k+2}) = 0\]
So, \(c_k = Tr(\omega^{2k+1})\) is a closed \(2k + 1\) form on \(B\) called the Kamber-Tondeur form.

Now we want to get a \(2k\)-cocycle on Volodin space to give a cohomology class
\[b_k \in H^{2k}(V_\bullet(\mathbb{C}); \mathbb{R})\]
then extend to the Whitehead space. The first step is to smoothly interpolate the matrices. In a \(2k\)-simplex \(\sigma \in V_{2k}(n, \mathbb{C})\) we only have matrices \(g_i\) defined on the vertices. If \(t \in \sigma\) then \(t\) is given by barycentric coordinate \(t_i\) (which are \(\geq 0\) and \(\sum t_i = 1\)). We assign to \(t \in \sigma\) the matrix
\[g_t := \sum t_ig_i\]

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\(^7\)Continuous cohomology classes are given by continuous cocycles on the group \(GL(n, \mathbb{C})\). The usual topology on \(GL(n, \mathbb{C}) \subset \mathbb{C}^{n^2}\) is used to say which cocycles are continuous. However, we get a cohomology class for the discrete group \(GL(n, \mathbb{C})\).

\(^8\)When \(h\) is given by \(h = gg^*\) for \(g\) invertible, not only is \(h = h^*\), but all of its eigenvalues must be positive and \(h\) can be written as \(h = UDU^*\) where \(U\) is unitary and \(D\) is diagonal with positive real entries. This implies that \(h^* = U^*DU^*\) is well-defined for all real number \(s\).
This is an element of \( g_\delta T_\sigma(n, \mathbb{C}) \) since \( T_\sigma(n, \mathbb{C}) \) is convex. Let \( h_t = g_t g^*_t \) as before. Then we define \( b_k(\sigma) \) to be:

\[
b_k(\sigma) := \frac{1}{(2k + 1)!2i^k} \int_{(t, s) \in \Delta^{2k+1} \times I} \text{Tr} ((h_t^{-*} d h_t^*)^{2k+1}) + \text{polynomial}
\]

We don’t need to know what the polynomial term is. For the basic example of circle bundle, we know by the covering space argument that the answer is proportional to a polylogarithm which is transcendental. Therefore, all polynomial terms will cancel in the end. The circle bundle is the only case which was computed from the above definition.

Note that there is a random coefficient \( \frac{k!}{(2k + 1)!(2\pi i)^k} \) which is thrown in for some reason. Other people use other coefficients. For example, Sebastian Goette [BG00] uses

\[
\frac{k!}{(2k + 1)!(2\pi i)^k}
\]

which he calls the Chern normalization. Note that \( i = \sqrt{-1} \) in both cases.

To extend this \( 2k \)-cocycle to the Whitehead space one thing we need to do is show that it is invariant under unitary change of coordinates.

Suppose that \( A, B \) are unitary matrices and \( \tilde{g}_t = Ag_t B \) for all \( t \in \sigma \). Then

\[
\tilde{h}_t = \tilde{g}_t \tilde{g}^*_t = Ag_t B^* Bg^*_t A^* = Ag_t g^*_t A^* = Ah_t A^*
\]

\[
\tilde{\omega} = \tilde{h}_t^{-1} d \tilde{h}_t = Ah_t^{-1} d h_t A^{-1}.
\]

So, \( \text{Tr} (\tilde{\omega}^{2k+1}) = \text{Tr} (\omega^{2k+1}) \) does not change.

2.6. Transgression. Here is an abbreviated explanation of why \( b_{2k} \) is the transgression of the degree \( 2k + 1 \) class given by the Kamber-Tondeur form. I promised to explain this in the lectures but I didn’t get to it.

The plus construction gives a homology equivalence

\[
\text{BGL}(R) \to \text{BGL}(R)^+
\]

for any ring \( R \). Let \( X(R) \) be the fiber of this mapping. Then \( X(R) \) is acyclic. The theorem that Volodin K-theory is the loop space of Quillen K-theory is the same as saying that \( V(R) := |V_*(R)| \) is the homotopy fiber of the mapping

\[
X(R) \to \text{BGL}(R).
\]

Equivalently, \( V(R) \) is a covering space of \( X(R) \) and \( X(R) \) can be taken to be the quotient of \( V(R) \) under the free left action of \( GL(R) \).

Now take \( R = \mathbb{C} \). Suppose that \( \alpha : Z^{2k} \to V(\mathbb{C}) \) is a \( 2k \)-cycle and we want to evaluate the Borel regulator map on this cycle. Since \( X(\mathbb{C}) \) is acyclic, the image of \( Z^{2k} \) in \( X(\mathbb{C}) \) is the boundary of a \( 2k + 1 \) chain \( W^{2k+1} \):

\[
\begin{array}{cccc}
Z^{2k} & \longrightarrow & W^{2k+1} & \longrightarrow & W \cup Z \text{C}Z & \longrightarrow & \Sigma Z \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta \\
V(\mathbb{C}) & \longrightarrow & X(\mathbb{C}) & \longrightarrow & \text{BGL}(\mathbb{C}) & \longrightarrow & \text{BGL}(\mathbb{C})^+
\end{array}
\]

In \( \text{BGL}(\mathbb{C}) \), \( Z \) is contracted to a point so we can extend \( W \) by attaching the cone on \( Z \) to form a \( 2k + 1 \) cycle \( W \cup Z \text{C}Z \) in \( \text{BGL}(\mathbb{C}) \). In \( \text{BGL}(\mathbb{C})^+ \), \( W \) goes to a point.
and we get an induced mapping $\delta: \Sigma Z \to BGL(\mathbb{C})^+$ which is adjoint to the original mapping $\alpha: Z \to V(\mathbb{C}) \simeq \Omega(BGL(\mathbb{C})^+)$. The Kamber-Tondeur form gives a $2k + 1$ cocycle $c_k$ on the space $BGL(\mathbb{C})$. Thus we want to evaluate this cocycle on $W \cup Z$. The given part of $b_k$ is the formula for the restriction of this form to the cone $CZ$. The correction term is the restriction of $c_k$ to $W$.

What is the correction term? Since $X(\mathbb{C})$ is acyclic, its cellular chain complex has a chain contraction, say $\eta$. This means that

$$W = d\eta W + \eta dW$$

But $d\eta W$ is a boundary in $X(\mathbb{C})$ which goes to a boundary in $BGL(\mathbb{C})$ so $c_k = 0$ on $d\eta W$. The other term is equal to

$$\eta dW = \eta Z$$

So, we have that $c_k$ transgresses to $b_k$ where

$$b_k(\sigma) = c_k(C(\sigma)) + c_k(\eta(\sigma))$$

for all $2k$-simplices $\sigma$ in $V(\mathbb{C})$. With a lot of work I determined the correction term $c_k(\eta(\sigma))$. But it doesn’t matter. It is just a polynomial.

2.7. Generalized Morse functions. We return to the Morse theory. We assumed that we have a fiberwise Morse function $f_t: (M_t, \partial_0 M_t) \to (I, 0)$. However, such a family of functions does not always exist. We need to introduce degenerate singularities called birth-death or cubic singularities. These are points at which, for a single function $f$, we have the canonical form:

$$f(x) = x_0^3 + \sum \pm x_i^2 + C$$

where $C$ is the critical value and the number of $(-)$ signs is called the index of the birth-death point.

Birth-death points occur generically in 1-parameter families of functions $f_t, t \in \mathbb{R}$ with the formula

$$f_t(x) = x^3 + tx + \sum \pm x_i^2 + C_t$$

where $C_t$ depends only on $t$. When $t = 0$ we have a cubic singularity. When $t > 0$ the function is nonsingular. (The critical points “died.”) When $t < 0$ we have two Morse critical points of index $i, i + 1$. (These points were “born.”)

The cellular chain complex changes at a birth-death point by adding a direct summand of the form

$$0 \to \mathbb{Z}_\pi \xrightarrow{g} \mathbb{Z}_\pi \to 0$$

where $g \in \pi$. In the two-index case the matrix changes by

$$A_t \mapsto \begin{pmatrix} A_t & 0 \\ 0 & g \end{pmatrix}$$

A smooth function having only Morse and birth-death singularities is called a generalized Morse function (GMF).

Theorem 2.3. If $\dim B \leq \dim M$ then a family of functions $f_t: (M_t, \partial_0 M_t) \to (I, 0)$ exists having only Morse and b-d singularities. ($f_t$ is a GMF.)
There are two problems with this theorem. The first is that the dimension condition may not be met. The second is that \( f_t \) may not be unique. We will fix both problems.

First, we make the fiber dimension larger by taking the product of \( E \) with a large dimensional disk \( D^N \). We need to know that this does not change the higher torsion.

**Lemma 2.4** (Stabilization Lemma). \( \tau_k^{FR}(E, \partial_0 E) = \tau_k^{FR}(E \times D^N, \partial_0 E \times D^N) \).

The second problem is more serious. If \( f_t, f'_t \) are two families of GMF’s it often happens that their torsions are different:

\[
\tau_k^{FR}(f_t) \neq \tau_k^{FR}(f'_t).
\]

We need to choose a particular GMF. This gives a choice for the second parameter that I alluded to earlier. Such a choice is given by a **framed function**.

### 3. Lecture III

In this lecture I started to use transparencies. Framed functions were actually defined at the end of Lecture 2 but I reviewed it at the beginning of Lecture 3.

#### 3.1. Framed functions.

**Definition 3.1.** A **framed function** is a GMF

\[
f : M \to \mathbb{R}
\]

together with a tangential framing of the stable manifold\(^9\) at each critical point. At each b-d point the last vector should point in the positive cubic direction.

**Theorem 3.2.** [Igu87] The space of framed functions on \( M^n \) is \((n-1)\)-connected.

**Corollary 3.3.** Given a smooth bundle pair \((M, \partial_0 M) \to (E, \partial_0 E) \to B\) we have

1. If \( \dim B \leq \dim M \) then there exists a smooth family of framed functions

   \[
   f_t : (M_t, \partial_0 M_t) \to (I, 0), \quad t \in B
   \]

2. If \( \dim B < \dim M \) then \( f_t \) is unique up to framed homotopy.

---

\(^9\)The **stable manifold** of a critical point \( x_0 \) is the set of all points \( y \in M \) so that the smooth flow generated by the gradient of \( f \) sends \( y \) to a path converging to \( x \). If \( x \) is a Morse point of index \( i \), the stable manifold will be an \( i \)-cell. If \( x \) is a cubic point of index \( i \) then the stable manifold will look like upper half-space in \( \mathbb{R}^{i+1} \).
3.2. Stabilization Lemma. I didn’t have time to prove the stabilization lemma (2.4) in the lectures so I will do it here.

Proof of Lemma 2.4. Suppose that \( f_t : M_t \to \mathbb{R} \) is a family of framed functions which we use to calculate the higher FR-torsion. Then we take the positive suspension
\[
\sigma_+(f_t) : M_t \times D^N \to \mathbb{R}
\]
given by
\[
\sigma_+(f)(x,y) = f_t(x) + \|y\|^2.
\]
The new function \( \sigma_+(f_t) \) has the same critical points with the same index as \( f_t \). So, \( \sigma_+(f_t) \) is framed and the cellular chain complex of \( \sigma_+(f_t) \) is the same as that of \( f_t \). Therefore, the FR-torsion is unchanged. \( \square \)

Note that the same argument proves the following:

Theorem 3.4. If \( q : D \to E \) is any linear disk bundle over \( E \) and \( \partial_0 D = q^{-1}\partial_0 E \) then
\[
\tau_{FR}^k(D, \partial_0 D; q^*\mathcal{F}) = \tau_{FR}^k(E, \partial_0 E; \mathcal{F}).
\]

3.3. \( A_\infty \)-functors. The mapping to K-theory:
\[
\xi : B \to Wh(\mathbb{Z}\pi, \pi) \to Wh(\mathbb{C}, U(1))
\]
is given by sending \( t \) to \( \xi(t) = C_*(f_t) \) where \( f_t \) is a family of framed functions on \( M_t \times D^N \). In order to make it a family of acyclic chain complexes we need to use \( A_\infty \)-functors. (A good elementary reference is [Kel].)

The idea is simple. Let \( K \) be a field and let \( \mathcal{C}(K) \) be the category of chain complexes/\( K \) and degree 0 chain maps. Let \( \mathcal{X} \) be any category. Then an \( A_\infty \) functor
\[
\Phi : \mathcal{X} \to \mathcal{C}(K)
\]
consists of

(0) For each \( X \in \mathcal{X} \) an object \( \Phi X \in \mathcal{C}(K) \).
(1) For each \( X \xleftarrow{f} Y \) a chain map \( \Phi X \leftarrow \Phi Y : \Phi_1(f) \).
(2) For \( X \xleftarrow{f} Y \xrightarrow{g} Z \) we get \( \Phi_2(f,g) : \Phi Z \to \Phi X \) a chain homotopy \( \Phi_1(f \circ g) \simeq \Phi_1(f) \circ \Phi_1(g) \).
(3) etc.
Definition 3.5. An $A_\infty$-functor $\Phi : \mathcal{X} \to C(K)$ takes each sequence of morphisms

$$X_0 \leftarrow^f X_1 \leftarrow^f \cdots \leftarrow^f X_p$$

to a mapping of degree $p - 1$:

$$\Phi_p(f_1, \cdots, f_p) : \Phi X_p \to \Phi X_0$$

so that $\Phi_0 = d$ is the boundary map for $X_0$ (when $p = 0$) and

$$\sum_{p=1}^{n-1} (-1)^p \Phi_{n-1}(1^{p-1}, m_2, 1^{n-p-1}) = \sum_{p+q=n} (-1)^p m_2(\Phi_p, \Phi_q).$$

where $m_2(f, g) = f \circ g$.

This is an equation of operations which actually means:

$$\sum_{p=1}^{n-1} (-1)^p \Phi_{n-1}(f_1, \cdots, f_{p-1}, f_p f_{p+1}, f_{p+2}, \cdots, f_n) = \sum_{p=0}^{n} (-1)^p \Phi_p(f_1, \cdots, f_p) \Phi_{n-p}(f_{p+1}, \cdots, f_n)$$

For $n = 0, 1, 2$ this says:

$(n = 0)$ $0 = \Phi_0 \Phi_0$ (since $\Phi_0 = d$.)

$(n = 1)$ $0 = \Phi_0 \Phi_1(f) - \Phi_1(f) \Phi_0$ ($df_* = f_* d$ so $f_* = \Phi_1(f)$ is a chain map.)

$(n = 2)$ $-\Phi_1(fg) = \Phi_0 \Phi_2(f, g) - \Phi_1(f) \Phi_1(g) + \Phi_2(f, g) \Phi_0$ or:

$$d\Phi_2(f, g) + \Phi_2(f, g) d = f_* g_* - (fg)_*$$

i.e., $\Phi_2(f, g) : (fg)_* \simeq f_* g_*$.

3.4. Theorem of Kadeishvili. In [Kad80] Kadeishvili proved two theorems which imply the following.

(1) If $A$ is a differential graded algebra (DGA)/$K$ then the homology $H_*(A)$ has an $A_\infty$-structure which is unique up to $A_\infty$-isomorphism so that $A$ is $A_\infty$-quasi-isomorphic to $H_*(A)$ ($A \simeq H_*(A)$)

(2) If $M$ is a DG module over $A$ then $H_*(M)$ has the structure of an $A_\infty$-module over the $A_\infty$-algebra $H_*(A)$ so that $M \simeq H_*(M)$.

If we interpret a $K$-category as an algebra over $K$ with many objects and we interpret a functor as a module with several objects then Kadeishvili’s Theorem 2 for many objects becomes the following.

Homology over $K$ has an $A_\infty$-structure. I.e.,

$$\text{Top} \longrightarrow C(K)$$

$$X \mapsto H_*(X; K)$$

is an $A_\infty$-functor. To distinguish between the ordinary homology functor $H_*(\cdot; K)$ and the $A_\infty$ version we will call the latter the $A_\infty$-homology functor.

---

10A $K$-category is a category in which all Hom sets are vector spaces over $K$ and composition is $K$-bilinear.
3.5. **Eilenberg-MacLane.** The following formula for the $A_{\infty}$ structure on homology is given by Eilenberg and MacLane [EM53] before $A_{\infty}$ structures were defined (by Stasheff [Sta63]).

Given a functor

$$X \to C(K)$$

$$X \mapsto C(X)$$

let $\Phi_X = H_*(C(X))$. Choose chain maps

$$\Phi_X \overset{j}{\to} C(X) \overset{q}{\to} \Phi_X$$

so that $q \circ j = \text{id}$ and $\eta : \text{id} \simeq j \circ q$. Then

$$\Phi_p(f_1, \cdots, f_p) = q_0 C(f_1) \eta_1 C(f_2) \eta_2 \cdots \eta_{p-1} C(f_p) j_p$$

3.6. **Twisting cochains.** For an $A_{\infty}$-functor $\Phi$ such as the $A_{\infty}$-homology functor which have an underlying functor $F$, the difference between $\Phi$ and $F$ is called a **twisting cochain**. For the $A_{\infty}$-homology functor, Kadeishvili’s theorem says that the twisting cochain is zero in degrees 0,1. But in general, a twisting cochain can be nonzero in all nonnegative degrees.

<table>
<thead>
<tr>
<th>F = homology:</th>
<th>$H_*(X)$</th>
<th>$H_*(f)$</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi = \text{twisting cochain}$:</td>
<td>$-$</td>
<td>0</td>
<td>0</td>
<td>$\phi_2$</td>
<td>$\phi_3$</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>

**Definition 3.6.** Suppose that $F : X \to C(K)$ is a functor. Then a **twisting cochain** with coefficients in $F$ is a sum of cochains $\phi = \phi_0 + \phi_1 + \phi_2 + \cdots$ where $\phi_p$ is a $p$-cochain on the nerve of $X$ of the following form. It sends a $p$-simplex in the nerve:

$$X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} \cdots \xleftarrow{f_p} X_p$$

to a map $\phi_p(f_1, \cdots, f_p) : FX_p \to FX_0$ of degree $p - 1$ so that

$$\delta \phi = \phi' \cup \phi$$

where $\phi' = \sum (-1)^p \phi_p$.

The cup product $\phi_p \cup \phi_q$ is given in the usual way by taking $\phi_p$ on the front $p$-face times (composed with) $\phi_q$ on the back $q$-face:

$$(\phi_p \cup \phi_q)(f_1, \cdots, f_{p+q}) = \phi_p(f_1, \cdots, f_p) \phi_q(f_{p+1}, \cdots, f_{p+q}).$$

Twisting cochains were invented by Ed Brown 45 years ago [Bro59]. He defined it as a mapping from a coalgebra to a differential graded algebra. The above definition is the category theoretic version. It is simpler because the differential in the target has been incorporated as part of the twisting cochain (as $d = \phi_0$). Ed Brown used twisting cochains to construct the **twisted tensor product** which gives the homology of the total space of a fibration. Consequently, the twisting cochain contains all of the information in the Serre spectral sequence.
3.7. Flat superconnections. Sebastian Goette pointed out in his papers that twisting cochains are combinatorial versions of the Bismut-Lott flat superconnections. From a topological point of view, flat superconnections are smooth versions of $A_\infty$-functors. In my lecture I gave my definition of a flat superconnection. As Sebastian pointed out, this is a special case where the vector bundle already has an ordinary flat connection. This corresponds to the topological case when the $A_\infty$-functor is equal to an ordinary functor plus a twisting cochain. Thus, in the following definition, $\nabla$ is analogous to $\mathcal{F}$ and $A_p$ is analogous to $\phi_p$.

Suppose that $E^n \to B, n \geq 0$ are smooth complex vector bundles. Suppose that they have a flat connection $\nabla$. (This does not exist in general.) Then a flat superconnection on $E^*$ is a sum

$$\nabla + A_0 + A_1 + A_2 + \cdots$$

where $A_p$ is a $p$-form on $B$ with values in the degree $1 - p$ part of the endomorphism bundle $\text{End}(E^*)$ so that

$$\nabla A_{n-1} = - \sum_{p+q=n} (-1)^{(p-1)q} A_p A_q$$

(If we use supercommutator rules the correct definition is $\nabla A = -A^2$.)

For $n = 0, 1, 2$ the expression (4) becomes the following.

$(n = 0) \quad 0 = A_0$

$(n = 1) \quad \nabla A_0 = A_0 A_1 - A_1 A_0$

$(n = 2) \quad \nabla A_1 = -A_0 A_2 - A_1^2 - A_2 A_0$

Flat superconnections occur in the definition of higher FR-torsion. The reason is that, in order to apply the Kamber-Tondeur form, we need to interpolate the family of matrices to a smooth family over the simplex $\Delta^2_k$. This gives a flat superconnection.

This happens even in the simplest case of our linear interpolation of the matrices in the Volodin construction. Recall that $g_t = \sum t_i g_i$ where $t_i$ are barycentric coordinates on $\Delta^k$. These invertible matrices represent the boundary maps of a family of chain complexes in two indices $g_t : C_1 \to C_0$. So, $A_0 = g_t$. Let $A_1 = (0, g_t^{-1} dg_t)$ ($0$ on $C_0$, $g_t^{-1} dg_t$ on $C_1$) and $A_2 = 0 : C_0 \to C_1$. Then

$$d A_1 = -g_t^{-1} dg_t g_t^{-1} dg_t = -A_1^2$$

$$d A_0 = dg_t = g_t g_t^{-1} dg_t = A_0 A_1$$

where the missing term $A_1 A_0 = 0$ since the target of $A_0$ is $C_0$ and $A_1 = 0$ on $C_0$. Consequently,

$$d + g_t + g_t^{-1} dg_t$$

is a flat superconnection on $\Delta^k$

4. Lecture IV

This is the last lecture of the workshop. I finished the definition of higher FR-torsion and computed the higher torsion of Hatcher’s example.
4.1. **Definition of $Wh_\bullet(R, G)$**. We begin with the definition of the Whitehead category $Wh_\bullet(R, G)$ ([Igu03], Def 2.16) which generalizes the Volodin space $V_\bullet(R)$. The Whitehead category is actually very close to Volodin’s original definition before other people simplified it.

Let $R$ be an associative ring and let $G \subseteq R^\times$ be a subgroup of the group of units of $R$. (E.g., $Wh_\bullet(\mathbb{Z}, \pi), Wh_\bullet(\mathbb{C}, U(1))$) Then the objects of the category $Wh_k(R, G)$ consist of

1. A finite partially ordered basis set which is also graded:

   $$P = P_0 \bigsqcup P_1 \bigsqcup \cdots$$

   (These are the critical points of $f_i$. They are graded by index and partially ordered by critical value.)

2. A strictly upper triangular twisting cochain $\phi$ on $\Delta^k$ with coefficients in the identity (constant) functor $I$. (So, $\Phi = I + \phi_0 + \phi_1 + \phi_2 + \cdots$ is an $A_\infty$-functor on the category of faces of $\Delta^k$ with inclusions as morphisms.)

**Example 4.1.** Take $k = 2$. What does this say?

$P$ is the basis poset.

$R^P$ is the free graded right $R$-module generated by $P = P_0 \bigsqcup P_1 \bigsqcup \cdots$.

For each vertex $i = 0, \cdots, k$ of $\Delta^k$ we have a degree -1 mapping $\phi_0(i) : R^P \rightarrow R^P$ with $\phi_0(i)^2 = 0$. This gives a chain complex:

$$C_*^i : R^P_0 \overset{\phi_0(i)}{\rightarrow} R^P_1 \overset{\phi_0(i)}{\rightarrow} R^P_2 \overset{\phi_0(i)}{\rightarrow} \cdots$$

For all $0 \leq i, j \leq k$ we get a chain map

$$I + \phi_1(i, j) : C_*^j \rightarrow C_*^i$$

Finally, $\phi_2(0, 1, 2)$ is a chain homotopy $(I + \phi_1(0, 1))(I + \phi_1(1, 2)) \simeq (I + \phi_1(0, 2))$.}

Continuing with the definition of $Wh_k(R, G)$, the **morphisms**

$$(\alpha, f) : (P, \phi) \rightarrow (Q, \psi)$$

are given by composing maps of the form:

a) $\alpha : P \rightarrow Q$ an order preserving bijection

b) Elementary collapse

$$x \in P \quad \subseteq \quad C_* \quad \oplus \quad (R \xrightarrow{g} R)$$

$$\alpha(x) \in Q \quad \subseteq \quad C_*$$

where $f(x) = \alpha(x)g_x$, $g_x \in G$.

and $\phi, \psi$ must correspond ($\psi = \alpha \circ \phi \circ \alpha^{-1}$).
4.2. Definition of $\tau_k^{FR}$. Take a fiber bundle $F \to E \to B$ and suppose that $\pi_1 B$ acts trivially on $H_*(F; \mathcal{F})$ where $\mathcal{F}$ is given by a 1-dimensional unitary representation $r : \pi \to U(1)$. Choose a basis $P_n$ for $H_n(F; \mathcal{C}_\rho)$. Take the reverse ordering on $P$:

$$P_0 > P_1 > P_2 > \cdots$$

Then $\phi_0 = \phi_1 = 0$ and $\phi_2, \phi_3,$ etc. will automatically be strictly upper triangular.

The $A_\infty$-homology functor gives a simplicial map on the nerve of the category of simplices of $B$:

$$A_\infty H_* : \mathcal{N}_* \text{simp } B \to \mathcal{W}h_*(\mathbb{C}, U(1)).$$

Suppose next that $E \to B$ is smooth and $f_t$ is a family of framed functions on the fibers. Let $C_*(f_t)$ be the cellular chain complex. This gives another simplicial map

$$C_*(f_t) : \mathcal{N}_* \text{simp } B \to \mathcal{W}h_*(\mathbb{C}, U(1))$$

According to Kadeishvili there must be an $A_\infty$-quasi-isomorphism:

$$\psi_t : C_*(f_t) \xrightarrow{\sim} A_\infty H_*$$

This implies that the mapping cone of $\psi_t$ is acyclic.

$$0 \simeq CC_*(f_t) := A_\infty \text{mapping cone of } \psi_t$$

So, $CC_*(f_t)$ gives a simplicial map

$$\mathcal{N}_* \text{simp } B \xrightarrow{CC_*(f_t)} \mathcal{W}h_*(\mathbb{C}, U(1)).$$

**Definition 4.2.** $\tau_k^{FR}(E, \partial_0 E; \rho) := CC_*(f_t)^*(\tau_k^{FR})$ where $\tau_k^{FR}$ is the universal higher FR-torsion class for $U(1)$ representations.

**Definition 4.3.** $\mathcal{W}h^h_*(R, G)$ is defined to be the simplicial full subcategory of $\mathcal{W}h_*(R, G)$ consisting of acyclic chain complexes. For $i \geq 0$,

$$\mathcal{W}h^h_{[i, i+1]}(R, G) \subseteq \mathcal{W}h^h_*(R, G)$$

is the simplicial full subcategory given by $P = P_i \bigsqcup P_{i+1}$, i.e., these are chain complexes in degree $i, i + 1$.

4.3. Theorems about Whitehead space.

**Theorem 4.4** (2-index theorem [Igu02a]). $|\mathcal{W}h^h_{[i, i+1]}(R, G)| \simeq |\mathcal{W}h^h_*(R, G)|$

We usually take $[i, i + 1] = [0, 1]$.

**Theorem 4.5** (I-Klein). There is a homotopy fiber sequence\footnote{As John Klein pointed out during my lecture, the nomenclature is not correct, or at least not standard. The fibration sequence should look like $Q(BG_+) \to \mathbb{Z} \times BGL(R)^+ \to Wh^h(R, G)$ and the fiber should be called “$Wh^h(R, G)$.”}

$$|\mathcal{W}h^h_*(R, G)| \to Q(BG_+) \to \mathbb{Z} \times BGL(R)^+$$

where $\mathbb{Z}$ is the image of $\mathbb{Z} = K_0 \mathbb{Z}$ in $K_0 R$. The mapping $Q(BG_+) \to \mathbb{Z} \times BGL(R)^+$ is given by the inclusion of monomial matrices into all invertible matrices (same as in Bruce Williams’ talk).

By Borel’s calculation of the rational K-theory of $\mathbb{Z}$ we have:
Corollary 4.6. \(|Wh^h_\bullet(\mathbb{Z}, 1)| \simeq \mathbb{Q} BO\).

The universal FR-torsion class (for \(n\)-dimensional unitary representations) lies in
\[
\tau_{k}^{FR} \in H^{2k}(Wh^h_\bullet(M_n(\mathbb{C}), U(n)); \mathbb{R})
\]
\[
\simeq H^{2k}(Wh^h_\bullet^{(0,1)}(M_n(\mathbb{C}), U(n)); \mathbb{R})
\]
and is given by Kamber-Tondeur.

The following diagram gives my interpretation of how this relates to Bruce Williams’ third lecture:

![Diagram](image)

4.4. Kamber-Tondeur again. To apply Kamber-Tondeur we need to interpolate the discrete family of matrices actually defined on each simplex of \(Wh^h_\bullet^{(0,1)}(M(n), U(n))\) to a smooth family. This was done in my book [Igu02a]. However, later in [Igu03], Remark 2.28, I observed that this interpolation gives a flat superconnection. Using this language I explained more about the polynomial correction term in the formula for the transgression \(b_k\) of the Kamber-Tondeur form.

On a 2\(k\)-simplex \(\sigma\) we take parallel coordinates with respect to the flat connection given by the poset basis. Then the superconnection is
\[
d + \phi_0 - \phi_1 - \phi_2
\]
(\(\phi_3, \text{etc. are zero since we have only two indices.}\)) The maps \(\phi_0, \phi_1, \phi_2\) look like this\(^{12}\)

\[
\begin{array}{c}
C_1 \\
\phi_0 \cong \phi_2 \\
\phi_1 \phi_0 \\
\phi_1^0 \cong \phi_1^1
\end{array}
\]

For \(h = \phi_0^0\) we get:
\[
b_k(\sigma) = \frac{1}{2^k(2k + 1)!} \int_{\Delta^{2k+1}} Tr \left( (h_i^{-s} d h_i^s)^{2k+1} \right) + \text{polynomial}
\]
where the “polynomial” is a linear combination of integrals of traces of products of \(\phi_2 \phi_0, \phi_0^{-1} \phi_1 \phi_0, \phi_1 \) and their adjoints.

\(^{12}\)The right hand vertical arrow is actually: \(\phi_0 + d \phi_0\) since it is at a slightly different place. Thus the commutativity of the square gives \(\phi_0^1 + + d \phi_0 = \phi_1 \phi_0\) or \(d \phi_0 = \phi_1 \phi_0 - \phi_0 \phi_1\), one of the defining equations of a flat superconnection.
4.5. More superconnections. I made some more elementary observations about flat superconnections and their relationship to twisting cochains. This is from [Igu02b], Section 8.

Suppose that $E^* = E^0 \oplus E^1 \oplus E^2 \oplus \cdots$ is a graded smooth vector bundle over $B$ and $D$ is a flat superconnection on $B$ with coefficients in $\text{End}(E)$. Then $D$ operates on $\Omega(B,E)$ and $D^2 = 0$ since $D$ is flat. Therefore $(\Omega(B,E), D)$ is a chain complex which I call the superconnection complex.

Here is a very easy theorem.

**Theorem 4.7.** $(\Omega(B,E), D) \simeq \text{dual of } C_\ast(B) \otimes \phi E^*|_s$, where $E^*|_s$ is the fiber of $E^*$ over one point.

Here $C_\ast(B) \otimes \phi E^*|_s$ is Ed Brown's twisted tensor product given by some twisting cochain $\phi$. (Definition is below.) The dual gives the cohomology of the total space of the bundle of which $(E^*, D)$ is the $A_\infty$-cohomology bundle made smooth. (My interpretation of the above theorem.)

**Definition 4.8.** The twisted tensor product $C_\ast(B) \otimes \phi H_\ast(F)$ is the usual tensor product $C_\ast(B) \otimes H_\ast(F) = C_\ast(B; H_\ast(F))$ with a new boundary operator:

$$\partial_\phi(x \otimes y) = \partial x \otimes y - \sum_{p+q=n} (-1)^p \begin{array}{c} \int_{p\text{-face of } x} \end{array} f_p(x) \otimes \phi_q\left( \begin{array}{c} \int_{q\text{-face of } x} \end{array} b_q(x) \right)(y)$$

where

$$\phi_q(b_q(x)) : H_\ast(F) \to H_{\ast+q-1}(F).$$

**Theorem 4.9.** [Bro59] Assuming $F$ has projective homology, the twisted tensor product gives the homology of the total space:

$$H_\ast(C_\ast(B) \otimes \phi H_\ast(F)) \simeq H_\ast(E).$$

4.6. Hatcher’s Example. My PhD advisor Allen Hatcher suggested this problem to me when I was still a student. It was solved by Marcel Bökstedt but I still continued to work on it until I had my own proof.

**Theorem 4.10.** There exists a smooth disk bundle

$$D^N \to E_H \to S^4$$

which is fiberwise tangentially homeomorphic to $D^N \times S^4$ with

$$\tau^{FR}_2(E_H) = \pm 24\zeta(5)[S^4] \in H^4(S^4; \mathbb{R})$$

F. Waldhausen: showed that there exist such exotic disk bundles over $S^4$.
A. Hatcher: made a specific construction.
M. Bökstedt [Bök84]: proved that $\tau^{FR}_2(E_H) \neq 0$ for Hatcher’s construction.
K.I. [Igu02a]: calculated the value of $\tau^{FR}_2(E_H)$.

The construction goes as follows. Take a generator of the kernel of the $J$-homomorphism

$$\mathbb{Z} \cong \pi_3 O \xrightarrow{\rho} \pi_3 \cong \mathbb{Z}/24$$
Figure 5. Attach $D^m \times D^{n+1}$ to $D^{m+1} \times S^n$ along the maps $\tilde{g}_t : D^m \times S_t^n \to \partial D^{m+1} \times S^n$ to get the fiber of $E_H$ over $t \in S^4$.

This gives a linear sphere bundle which is fiber homotopy equivalent to the trivial bundle:

$$
\begin{align*}
D^{n+1} & \supseteq S^n \\
\downarrow & \\
E & \supseteq \partial E \xrightarrow{g} S^n \times S^4 \\
\downarrow & \\
S^4 & = S^4
\end{align*}
$$

By making the target bigger we can approximate the mapping $g$ by an embedding $\tilde{g}$. We can then thicken the domain to the normal bundle in the target:

$$
\begin{align*}
D^m \times S^n & \quad \partial D^{m+1} \times S^n \times S^4 \\
\downarrow & \\
D^m \times \partial E & \quad \xrightarrow{\tilde{g}} S^n \times S^4 \\
\downarrow & \\
S^4
\end{align*}
$$

Hatcher’s disk bundle $E_H$ is constructed by attaching a handle $D^m \times D^{n+1}$ fiberwise to the trivial bundle with fiber $D^{m+1} \times S^n$ along the map $\tilde{g}$:

$$
\begin{align*}
D^m \times D^{n+1} \cup D^{m+1} \times S^n \\
\downarrow \\
E_H = D^m \times E \cup_{\tilde{g}} D^{m+1} \times S^n \times S^4 \\
\downarrow \\
S^4
\end{align*}
$$

The computation of $\tau^{FR}_2(E_H)$ follows from additivity and the calculation for sphere
bundles (Theorem 1.5).

References


