

L^2 -BETTI NUMBERS OF DISCRETE MEASURED GROUPOIDS

ROMAN SAUER

*FB Mathematik, Universität Münster
Einsteinstr. 62, 48149 Münster, Germany
romansauer@ams.member.org*

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There are notions of L^2 -Betti numbers for discrete groups (Cheeger–Gromov, Lück), for type II_1 -factors (recent work of Connes–Shlyakhtenko) and for countable standard equivalence relations (Gaboriau). Whereas the first two are algebraically defined using Lück’s dimension theory, Gaboriau’s definition of the latter is inspired by the work of Cheeger and Gromov. In this work we give a definition of L^2 -Betti numbers of discrete measured groupoids that is based on Lück’s dimension theory, thereby encompassing the cases of groups, equivalence relations and holonomy groupoids with an invariant measure for a complete transversal. We show that with our definition, like with Gaboriau’s, the L^2 -Betti numbers $b_n^{(2)}(G)$ of a countable group G coincide with the L^2 -Betti numbers $b_n^{(2)}(\mathcal{R})$ of the orbit equivalence relation \mathcal{R} of a free action of G on a probability space. This yields a new proof of the fact the L^2 -Betti numbers of groups with orbit equivalent actions coincide.

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1. Introduction and Statement of Results

L^2 -Betti numbers of (coverings of) manifolds were introduced by Atiyah [2] in 1976 and, subsequently, generalized by several people to L^2 -Betti numbers of arbitrary spaces and groups on the one hand and of foliated manifolds and equivalence relations on the other hand.

In [4] Cheeger and Gromov defined L^2 -Betti numbers for arbitrary countable discrete groups. In a series of papers [14–16] Lück put the theory of L^2 -Betti numbers into a completely algebraic framework by introducing a dimension function $\dim_{\mathcal{N}(G)}$ for arbitrary modules over the group von Neumann algebra $\mathcal{N}(G)$: the

L^2 -Betti numbers $b_n^{(2)}(G)$ of a group G can be read off from the group homology as

$$b_n^{(2)}(G) = \dim_{\mathcal{N}(G)}(\mathrm{Tor}_n^{\mathbb{C}G}(\mathcal{N}(G), \mathbb{C})).$$

Already back in 1979, Connes generalized Atiyah’s definition to L^2 -Betti numbers of compact foliated manifolds with an invariant transverse measure [5]. More recently, in an influential paper of Gaboriau [11] the notion of L^2 -Betti numbers for countable standard equivalence relations was introduced. Gaboriau’s definition is motivated by the one of Cheeger and Gromov. The restriction of the holonomy groupoid of a measured foliation to a complete transversal equipped with an invariant transverse probability measure is an example for a countable standard equivalence relation. If the leaves of the foliation are contractible, Gaboriau’s and Connes’ L^2 -Betti numbers coincide.

This article provides a homological-algebraic definition of L^2 -Betti numbers for countable standard equivalence relations and, more generally, for discrete measured groupoids. For (restricted) holonomy groupoids as above these numbers should coincide with Connes’ L^2 -Betti numbers provided the holonomy coverings of the leaves are contractible.

Since Gaboriau concentrates in his work on applications to ergodic theory, he restricts his definition to equivalence relations for convenience. However, his definition could also be adapted to groupoids, as remarked in [11, p. 103].

In Sec. 3 we will introduce algebraic objects for a discrete measured groupoid \underline{G} like the groupoid ring $\mathbb{C}\underline{G}$ which are analogous to the group case, and we define the L^2 -Betti numbers $b_n^{(2)}(\underline{G})$ of \underline{G} in 5.1 as

$$b_n^{(2)}(\underline{G}) = \dim_{\mathcal{N}(\underline{G})}(\mathrm{Tor}_n^{\mathbb{C}\underline{G}}(\mathcal{N}(\underline{G}), L^\infty(\underline{G}^0))).$$

Here $\underline{G}^0 \subset \underline{G}$ denotes the subset of objects in the groupoid. Denote the source and target maps by s resp. t . We show in 5.3 the following formula for the restriction.

Theorem 1.1. *Let \underline{G} be a discrete measured groupoid, and let $A \subset \underline{G}^0$ be a Borel subset such that $t(s^{-1}(A))$ has full measure in \underline{G}^0 . Then*

$$b_n^{(2)}(\underline{G}) = \mu(A) \cdot b_n^{(2)}(\underline{G}|_A).$$

An essentially free action of a countable group G on a standard Borel probability space X by measure preserving Borel automorphisms is called a *standard action*, and X is called a *probability G -space*. Let \mathcal{R} be the orbit equivalence relation of a standard action of the group G . Then \mathcal{R} is an example of a discrete measured groupoid. The following theorem is proved in 5.5 by homological-algebraic methods, which might be useful in other contexts dealing with homological algebra of finite von Neumann algebras.

Theorem 1.2. $b_n^{(2)}(G) = b_n^{(2)}(\mathcal{R})$.

The standard actions $G \curvearrowright X, H \curvearrowright Y$ are called *orbit equivalent* if there exists a measure-preserving Borel isomorphism $f : X \rightarrow Y$ that maps orbits bijectively onto

orbits. They are called *weakly orbit equivalent* if there are Borel subsets $A \subset X$, $B \subset Y$ (equipped with the normalized measures) meeting almost every orbit and a measure-preserving Borel isomorphism $f : A \rightarrow B$ that maps orbits bijectively onto orbits. The theorems above immediately imply the following:

Theorem 1.3 [11]. *Assume G and H possess weakly orbit equivalent standard actions. Then there is a $C > 0$ such that $b_n^{(2)}(G) = C \cdot b_n^{(2)}(H)$ holds for $n \geq 0$. Further, if the actions are orbit equivalent then $b_n^{(2)}(G) = b_n^{(2)}(H)$ for $n \geq 0$.*

As a remark, two countable groups possess weakly orbit equivalent standard actions if and only if they are measure equivalent [9, 10]. In [11] Gaboriau proves Theorems 1.1–1.3 for his version of L^2 -Betti numbers of countable equivalence relations. We remark that our methods are independent of Gaboriau’s results. In particular, we do *not* show the equality of our definition and Gaboriau’s one to deduce 1.2. Of course, as a consequence of 1.2, Gaboriau’s definition and ours coincide for orbit equivalence relations of free measure-preserving group actions and presumably coincide for every countable standard equivalence relation.

Popa showed that a type II_1 factor \mathcal{B} satisfying some rigidity and compact approximation properties has only one Cartan subalgebra \mathcal{A} [18]. Thus the L^2 -Betti numbers of the countable standard equivalence relation associated to the inclusion $\mathcal{A} \subset \mathcal{B}$ are isomorphism invariants of the factor \mathcal{B} . Recently, Connes and Shlyakhtenko [6] defined L^2 -Betti numbers of an arbitrary type II_1 -factor \mathcal{N} in a very different fashion as

$$b_n^{(2)}(\mathcal{N}) = \dim_{\mathcal{N} \bar{\otimes} \mathcal{N}^\circ} (\text{Tor}_n^{\mathcal{N} \bar{\otimes} \mathcal{N}^\circ} (\mathcal{N} \bar{\otimes} \mathcal{N}^\circ, \mathcal{N})).$$

The motivation for this article is twofold. Using our definition of $b_n^{(2)}(\underline{G})$, Popa’s L^2 -Betti numbers of II_1 -factors (if defined) are brought in line with the L^2 -Betti numbers of groups in the sense of Lück and the Connes–Shlyakhtenko L^2 -Betti numbers of II_1 -factors. Furthermore, our algebraic definition allows to apply tools from homological algebra, which hopefully helps to obtain new computations.

The results in this paper are part of the author’s thesis. I want to thank my adviser Wolfgang Lück for constant support and encouragement.

2. Review of Dimension Theory of Finite von Neumann Algebras

In this section we review the dimension function for arbitrary modules over a finite von Neumann algebra (modules in the algebraic sense) and its basic properties. Further, we prove some additional properties concerning restrictions of von Neumann algebras and give a new criterion for a module to be zero dimensional. In the sequel let \mathcal{A} be a finite von Neumann algebra equipped with a fixed finite faithful normal trace $\text{tr}_{\mathcal{A}}$ which is normalized, i.e. $\text{tr}_{\mathcal{A}}(1) = 1$. The dimension (function) $\dim_{\mathcal{A}}(M) \in [0, \infty]$ of a module M over \mathcal{A} was introduced by Lück in [15, 16]. For a finitely-generated projective \mathcal{A} -module P choose an idempotent

matrix $A = (A_{ij}) \in M_n(\mathcal{A})$ such that $P \cong \mathcal{A}^n \cdot A$. Then the dimension $\dim_{\mathcal{A}}(P)$ is defined as

$$\dim_{\mathcal{A}}(P) = \sum_{i=1}^n \operatorname{tr}_{\mathcal{A}}(A_{ii}) \in [0, \infty).$$

For an arbitrary \mathcal{A} -module N the dimension is then defined as

$$\dim_{\mathcal{A}}(N) = \sup\{\dim_{\mathcal{A}}(P); P \subset N \text{ finitely-generated projective submodule}\} \in [0, \infty].$$

For the following fundamental theorem see [15, Theorem 0.6], also [17, Theorem 6.7, p. 239].

Theorem 2.1. *The dimension function $\dim_{\mathcal{A}}$ satisfies the following properties.*

- (i) *A projective \mathcal{A} -module P is trivial if and only if $\dim_{\mathcal{A}}(P) = 0$.*
- (ii) *Additivity. If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence of \mathcal{A} -modules then*

$$\dim_{\mathcal{A}}(B) = \dim_{\mathcal{A}}(A) + \dim_{\mathcal{A}}(C)$$

holds, where we put $\infty + r = r + \infty = \infty$ for $r \in [0, \infty]$.

- (iii) *Cofinality. Let $M = \bigcup_{i \in I} M_i$ be a directed union of submodules $M_i \subset M$. Then*

$$\dim_{\mathcal{A}}(M) = \sup_{i \in I} \{\dim_{\mathcal{A}}(M_i)\}.$$

Definition 2.2. An \mathcal{A} -homomorphism $f : M \rightarrow N$ between \mathcal{A} -modules M, N is called a $\dim_{\mathcal{A}}$ -isomorphism if $\dim_{\mathcal{A}}(\ker f) = \dim_{\mathcal{A}}(\operatorname{coker} f) = 0$.

There is a suitable localization of the category of \mathcal{A} -modules in which $\dim_{\mathcal{A}}$ -isomorphisms become isomorphisms. Let us recall the relevant notions.

Let \mathcal{C} be an abelian category. A non-empty full subcategory \mathcal{D} of \mathcal{C} is called a *Serre subcategory* if for all short exact sequences in \mathcal{C}

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

the central term M belongs to \mathcal{D} if and only if both M' and M'' do. Then there is a quotient category \mathcal{C}/\mathcal{D} with the same objects as \mathcal{C} and a functor $\pi : \mathcal{C} \rightarrow \mathcal{C}/\mathcal{D}$. Moreover, \mathcal{C}/\mathcal{D} is abelian, π is exact and $\pi(f)$ is an isomorphism if and only if $\ker(f)$ and $\operatorname{coker}(f)$ lie in \mathcal{D} for a morphism f in \mathcal{C} . The properties in 2.1 imply that the subcategory of zero-dimensional \mathcal{A} -modules is a Serre subcategory. This has useful consequences. For instance, there is a 5-lemma for $\dim_{\mathcal{A}}$ -isomorphisms because there is one for general abelian categories.

In [14, Lemma 3.4] it is proved that any finitely-generated \mathcal{A} -module N splits as $N = \mathbf{P}N \oplus \mathbf{T}N$ where $\mathbf{P}N$ is finitely-generated projective and $\mathbf{T}N$ is the kernel of the canonical homomorphism $N \rightarrow N^{**}$ into the double \mathcal{A} -dual, mapping $x \in N$ to $N^* \rightarrow \mathcal{A}, f \mapsto f(x)$. Furthermore, $\dim_{\mathcal{A}}(\mathbf{T}N) = 0$ holds. The modules $\mathbf{P}N$ and $\mathbf{T}N$

are called the *projective* resp. *torsion part* of N . Further, if N is a finitely-presented \mathcal{A} -module then the torsion part \mathbf{TN} possesses an exact resolution of the form

$$0 \rightarrow \mathcal{A}^n \xrightarrow{r_A} \mathcal{A}^n \rightarrow \mathbf{TN} \rightarrow 0,$$

where r_A is right multiplication with a positive $A \in M_n(\mathcal{A})$.

The next lemma is Exercise 6.3 (with solution on p. 530) in [17, p. 289]. It is formulated for group von Neumann algebras there, but the proof is exactly the same for arbitrary finite von Neumann algebras.

Lemma 2.3. *Let M be a submodule of a finitely-generated projective \mathcal{A} -module P . For every $\epsilon > 0$, there exists a submodule $P' \subset M$ that is a direct summand in P and satisfies $\dim_{\mathcal{A}}(M) \leq \dim_{\mathcal{A}}(P') + \epsilon$.*

The following theorem is a local criterion for a module to be zero dimensional.

Theorem 2.4. *Let M be an \mathcal{A} -module. Its dimension $\dim_{\mathcal{A}}(M)$ vanishes if and only if for every element $m \in M$ there is a sequence $p_i \in \mathcal{A}$ of projections such that*

$$\lim_{i \rightarrow \infty} \operatorname{tr}_{\mathcal{A}}(p_i) = 1 \quad \text{and} \quad p_i \cdot m = 0 \quad \text{for all } i \in \mathbb{N}.$$

Furthermore, if $q \in \mathcal{A}$ is a given projection with $qm = 0$ for an element m in M with $\dim_{\mathcal{A}}(M) = 0$, then the sequence p_i can be chosen such that $q \leq p_i$.

Proof. First assume $\dim_{\mathcal{A}}(M) = 0$. Consider an element $m \in M$, and let $q \in \mathcal{A}$ be a projection such that $qm = 0$. For a given $\epsilon > 0$, we want to construct a projection $p \in \mathcal{A}$ such that $\operatorname{tr}_{\mathcal{A}}(p) \geq 1 - \epsilon$, $p \cdot m = 0$ and $p \geq q$. Let $\langle m \rangle \subset M$ be the submodule generated by m . We have the epimorphism

$$\phi : \mathcal{A}(1 - q) \rightarrow \langle m \rangle, \quad n(1 - q) \mapsto nm$$

and $\dim_{\mathcal{A}}(\ker \phi) = \dim_{\mathcal{A}}(\mathcal{A}(1 - q)) - \dim_{\mathcal{A}}(\langle m \rangle) = 1 - \operatorname{tr}_{\mathcal{A}}(q)$. By 2.3 there is a submodule $P \subset \ker \phi$ such that P is a direct summand in $\mathcal{A}(1 - q)$ and $\dim_{\mathcal{A}}(\ker \phi) \leq \dim_{\mathcal{A}}(P) + \epsilon$. Hence $\mathcal{A}q \oplus P \subset \mathcal{A}q \oplus \mathcal{A}(1 - q) = \mathcal{A}$ is a direct summand in \mathcal{A} , i.e. it has the form $\mathcal{A}p$ for a projection p . Its trace is $\operatorname{tr}_{\mathcal{A}}(p) = \operatorname{tr}_{\mathcal{A}}(q) + \dim_{\mathcal{A}}(P) \geq 1 - \epsilon$. Moreover, $\mathcal{A}q \subset \mathcal{A}p$ implies $qp = q$, i.e. $q \leq p$, and $pm = 0$ is obvious.

Now we prove the converse. It suffices to prove that M has no non-trivial finitely-generated projective submodules. Suppose $P \subset M$ is a non-trivial finitely-generated projective submodule. Then there is a non-trivial \mathcal{A} -homomorphism $\phi : P \rightarrow \mathcal{A}$. Choose non-zero element $y = \phi(x) \neq 0$ in the image of ϕ . There is a sequence of projections $p_i \in \mathcal{A}$ such that $\operatorname{tr}_{\mathcal{A}}(p_i) \rightarrow 1$ and $p_i \cdot x = 0$. In particular, $p_i \cdot y = \phi(p_i \cdot x) = 0$ yielding $y = 0$. Hence no such non-trivial P can exist. \square

Theorem 2.5. *Let \mathcal{A} and \mathcal{B} be finite von Neumann algebras, and let F be an exact functor from the category of \mathcal{A} -modules to the category of \mathcal{B} -modules which preserves colimits. Assume there is a constant $C > 0$ such that*

$$\dim_{\mathcal{B}}(F(P)) = C \cdot \dim_{\mathcal{A}}(P) \tag{1}$$

holds for every finitely-generated projective \mathcal{A} -module P . Then $\dim_{\mathcal{B}}(F(M)) = C \cdot \dim_{\mathcal{A}}(M)$ holds for every \mathcal{A} -module M .

Proof. We prove this for finitely-presented, finitely-generated and arbitrary modules, subsequently.

Step 1: Let M be a finitely-presented \mathcal{A} -module. Then M splits as $M = \mathbf{P}M \oplus \mathbf{T}M$, where $\mathbf{P}M$ is projective and $\mathbf{T}M$ admits an exact resolution

$$0 \rightarrow \mathcal{A}^n \rightarrow \mathcal{A}^n \rightarrow \mathbf{T}M \rightarrow 0.$$

Additivity yields $\dim_{\mathcal{A}}(\mathbf{T}M) = 0$. Applying the exact functor F to this resolution, additivity also implies $\dim_{\mathcal{B}}(F(\mathbf{T}M)) = 0$. We have $C \cdot \dim_{\mathcal{A}}(\mathbf{P}M) = \dim_{\mathcal{B}}(F(\mathbf{P}M))$ by assumption. Hence, $C \cdot \dim_{\mathcal{A}}(M) = \dim_{\mathcal{B}}(F(M))$.

Step 2: Let M be finitely generated. Then there is a finitely-generated free \mathcal{A} -module P with an epimorphism $P \rightarrow M$. Let K be its kernel. K can be written as the directed union of its finitely-generated submodules $K = \bigcup_{i \in I} K_i$. By cofinality and additivity (2.1) we conclude

$$\begin{aligned} \dim_{\mathcal{A}}(M) &= \dim_{\mathcal{A}}(P) - \dim_{\mathcal{A}}(K) = \dim_{\mathcal{A}}(P) - \sup_{i \in I} \{ \dim_{\mathcal{A}}(K_i) \} \\ &= \inf_{i \in I} \{ \dim_{\mathcal{A}}(P) - \dim_{\mathcal{A}}(K_i) \} \\ &= \inf_{i \in I} \{ \dim_{\mathcal{A}}(P/K_i) \}. \end{aligned}$$

We have $F(K) = \text{colim}_{i \in I} F(K_i)$ with injective structure maps because F is colimit-preserving and exact. Thus we can conclude similarly to obtain

$$\dim_{\mathcal{B}}(F(M)) = \inf_{i \in I} \{ \dim_{\mathcal{B}}(F(P/K_i)) \}.$$

Then the claim follows from the first step.

Step 3: Let M be an arbitrary module. Every module is the directed union of its finitely-generated submodules, which reduces the claim to the preceding step due to cofinality. □

The following theorem is proved in [15] for inclusions of group von Neumann algebras induced by group inclusions. Using the previous theorem it suffices to prove it for finitely-generated projective modules, which is easy.

Theorem 2.6. *Let $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a trace-preserving $*$ -homomorphism between finite von Neumann algebras \mathcal{A} and \mathcal{B} . Then for every \mathcal{A} -module N we have*

$$\dim_{\mathcal{A}}(N) = \dim_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} N).$$

We recall some definitions and easy facts about Morita equivalence of rings. For details and proofs we refer to [13, Sec. 18]. Two rings are called *Morita equivalent* if there exists a category equivalence, called *Morita equivalence*, between their module categories. Every Morita equivalence is exact and preserves projective modules. An idempotent p in a ring R is called *full* if the additive subgroup in R generated by the elements rpr' with $r, r' \in R$, denoted by RpR , coincides with R . In this case R and pRp are Morita equivalent, and the mutual inverse category equivalences are given by tensoring with the bimodules Rp resp. pR .

Definition 2.7. Let \mathcal{A} be a finite von Neumann algebra, and let R be a ring containing \mathcal{A} as a subring. An idempotent $p \in R$ is called *dim $_{\mathcal{A}}$ -full* if the inclusion of RpR into R is a $\text{dim}_{\mathcal{A}}$ -isomorphism of \mathcal{A} -modules.

Note for the following that if $p \neq 0$ is a projection in the finite von Neumann algebra \mathcal{A} then $p\mathcal{A}p = \{pap; a \in \mathcal{A}\}$ is again a finite von Neumann algebra equipped with the normalized trace $\text{tr}_{p\mathcal{A}p}(x) = \frac{1}{\text{tr}_{\mathcal{A}}(p)}\text{tr}_{\mathcal{A}}(x)$.

Theorem 2.8. *Let p be a $\text{dim}_{\mathcal{A}}$ -full projection in \mathcal{A} . Then $\mathcal{A}p$ is a right flat $p\mathcal{A}p$ -module.*

Proof. Consider the image $\bar{1}$ of 1 in the cokernel of the inclusion $\mathcal{A}p\mathcal{A} \subset \mathcal{A}$. By assumption, the cokernel has dimension zero. By the local criterion 2.4 there is a sequence $(p_i)_{i \in \mathbb{N}}$ of projections in \mathcal{A} such that $p_i\bar{1} = 0$, i.e. $p_i \in \mathcal{A}p\mathcal{A}$, $\text{tr}_{\mathcal{A}}(p_i) \rightarrow 1$ and $p_i \geq p$. From $p \leq p_i$ we get $p = p_i p p_i \in p_i \mathcal{A} p_i$. Furthermore, $p = p_i p p_i$ and $p_i \in \mathcal{A} p \mathcal{A}$ imply

$$p_i \in (p_i \mathcal{A} p_i) p (p_i \mathcal{A} p_i),$$

hence p is a full idempotent in $p_i \mathcal{A} p_i$, and the rings $p_i \mathcal{A} p_i$ and $p\mathcal{A}p$ are Morita equivalent. Thus the right $p\mathcal{A}p$ -module $p_i \mathcal{A} p$ is projective. The $p\mathcal{A}p$ -homomorphism

$$\mathcal{A}p \rightarrow \prod_{i \in \mathbb{N}} p_i \mathcal{A} p, \quad n \mapsto (p_i n)_{i \in \mathbb{N}}$$

is injective. Now we use the fact that a von Neumann algebra is a semihereditary ring (see 4.1). Over a semihereditary ring the property of being flat is inherited to products and submodules by 4.4. Therefore the product on the right is flat, and its submodule $\mathcal{A}p$ is flat as a $p\mathcal{A}p$ -module. □

Theorem 2.9. *Let p be a $\text{dim}_{\mathcal{A}}$ -full projection in \mathcal{A} . For every $p\mathcal{A}p$ -module M we have*

$$\text{dim}_{\mathcal{A}}(\mathcal{A}p \otimes_{p\mathcal{A}p} M) = \text{tr}_{\mathcal{A}}(p) \text{dim}_{p\mathcal{A}p}(M).$$

Proof. Due to the preceding theorem, the functor $\mathcal{A}p \otimes_{p\mathcal{A}p} \text{---}$ is exact. By 2.5 it suffices to check the claim for a finitely-generated projective $p\mathcal{A}p$ -module M . Let $A \in M_n(p\mathcal{A}p)$ be an idempotent matrix such that

$$M \cong (p\mathcal{A}p)^n \cdot A.$$

Then $\dim_{p\mathcal{A}p}(M) = \sum_{i=1}^n \text{tr}_{p\mathcal{A}p}(A_{ii})$ by definition. The \mathcal{A} -module $\mathcal{A}p \otimes_{p\mathcal{A}p} M$ is finitely-generated projective, and we have

$$\mathcal{A}p \otimes_{p\mathcal{A}p} M \cong (\mathcal{A}p)^n \cdot A = \mathcal{A}^n \cdot A.$$

For the right equal sign note that $(1 - p)A = 0$. Hence we can conclude

$$\begin{aligned} \dim_{\mathcal{A}}(\mathcal{A}p \otimes_{p\mathcal{A}p} M) &= \sum_{i=1}^n \text{tr}_{\mathcal{A}}(A_{ii}) = \text{tr}_{\mathcal{A}}(p) \cdot \sum_{i=1}^n \text{tr}_{p\mathcal{A}p}(A_{ii}) \\ &= \text{tr}_{\mathcal{A}}(p) \cdot \dim_{p\mathcal{A}p}(M). \end{aligned} \quad \square$$

Theorem 2.10. *Let R be a ring containing \mathcal{A} as a subring, and let p be a $\dim_{\mathcal{A}}$ -full idempotent in R . Then*

$$\phi : Rp \otimes_{pRp} pM \rightarrow M, \quad n \otimes m \mapsto nm \tag{2}$$

is a $\dim_{\mathcal{A}}$ -isomorphism for every R -module M .

Proof. First we show that ϕ is $\dim_{\mathcal{A}}$ -surjective. The local criterion 2.4 applied to the cokernel of the inclusion $RpR \subset R$ provides a sequence p_i of projections in \mathcal{A} , which lie in RpR and satisfy $\text{tr}_{\mathcal{A}}(p_i) \rightarrow 1$. But every element in the cokernel of ϕ is annihilated by the p_i , so this yields $\dim_{\mathcal{A}}(\text{coker } \phi) = 0$, again due to the local criterion.

Now we can prove that ϕ is a $\dim_{\mathcal{A}}$ -isomorphism. Consider the exact sequence

$$0 \rightarrow \ker \phi \rightarrow Rp \otimes_{pRp} pM \rightarrow M \rightarrow \text{coker } \phi \rightarrow 0.$$

Applying the exact functor $pR \otimes_R -$ produces an isomorphism in the middle because $pR \otimes_R (Rp \otimes_{pRp} pM) = pRp \otimes_{pRp} pM = pR \otimes_R M$. Hence $p \ker \phi = 0$. Because ϕ is already shown to be $\dim_{\mathcal{A}}$ -surjective, we obtain $\dim_{\mathcal{A}}(\ker \phi) = 0$. Hence ϕ is a $\dim_{\mathcal{A}}$ -isomorphism. □

Theorem 2.11. *Let p be a $\dim_{\mathcal{A}}$ -full projection in \mathcal{A} . Then for every \mathcal{A} -module M we have*

$$\dim_{\mathcal{A}}(M) = \text{tr}_{\mathcal{A}}(p) \cdot \dim_{p\mathcal{A}p}(pM).$$

Proof. By the preceding theorem we have $\dim_{\mathcal{A}}(\mathcal{A}p \otimes_{p\mathcal{A}p} pM) = \dim_{\mathcal{A}}(M)$. Now the claim is obtained by 2.9. □

3. Discrete Measured Groupoids

Discrete measured groupoids are generalizations of countable standard equivalence relations. The reference for the definitions and basic properties of the groupoid ring and von Neumann algebra associated to a countable standard equivalence relation is [8]. We review the definitions in the more general setting of discrete measured groupoids.

In [12, pp. 82, 88, 89] the following standard measure-theoretical facts are proved: any measurable subset of a standard Borel space is a standard Borel space. A bijective Borel map between standard Borel spaces is a Borel isomorphism. The image of an injective Borel map between standard Borel spaces is Borel. We omit the proof of the following technical lemma. The case for countable standard equivalence relations is implicit in [8, Proposition 2.3]. A detailed proof can be found in [20, Theorem 1.3].

Lemma 3.1. *Let $f : X \rightarrow Y$ be a Borel map between standard Borel spaces such that the preimages $f^{-1}(\{y\})$, $y \in Y$, are countable. Then the image $f(X)$ is measurable in Y , and there is a countable partition of X into measurable subsets X_i , $i \in \mathbb{N}$, such that all $f|_{X_i}$ are injective, and $f|_{X_1} : X_1 \rightarrow f(X)$ is a Borel isomorphism. If, in addition, there is some $N \in \mathbb{N}$ such that $\#f^{-1}(\{y\}) \leq N$ for all $y \in Y$, then the partition can be chosen to have at most N sets.*

Recall that a *groupoid* is a small category where all morphisms are invertible. We usually identify a groupoid \underline{G} with the set of its morphisms. The set of objects \underline{G}^0 can be considered as a subset (via the identity morphisms). There are four canonical maps, namely

- the *source map* $s : \underline{G} \rightarrow \underline{G}^0$, $(f : x \rightarrow y) \mapsto x$,
- the *target map* $t : \underline{G} \rightarrow \underline{G}^0$, $(f : x \rightarrow y) \mapsto y$,
- the *inverse map* $i : \underline{G} \rightarrow \underline{G}$, $f \mapsto f^{-1}$ and
- the *composition* $\circ : \underline{G}^{(2)} := \{(f, g) \in \underline{G} \times \underline{G}; s(f) = t(g)\} \rightarrow \underline{G}$, $(f, g) \mapsto f \circ g$.

The composition will also be denoted by g_1g_2 instead of $g_2 \circ g_1$. A *discrete measurable groupoid* \underline{G} is a groupoid \underline{G} equipped with the structure of a standard Borel space such that the composition and the inverse map are Borel and $s^{-1}(\{x\})$ is countable for all $x \in \underline{G}^0$. We remark that the source and target maps of a discrete measurable groupoid \underline{G} are measurable, $\underline{G}^0 \subset \underline{G}$ is a Borel subset, and $t^{-1}(\{x\})$ is countable. Now let μ be a probability measure on the set of objects \underline{G}^0 of a discrete measurable groupoid \underline{G} . Then, for any measurable subset $A \subset \underline{G}$, the function $\underline{G}^0 \rightarrow \mathbb{C}$, $x \mapsto \#(s^{-1}(x) \cap A)$ is measurable, and the measure μ_s on \underline{G} defined by

$$\mu_s(A) = \int_{\underline{G}^0} \#(s^{-1}(x) \cap A) d\mu(x)$$

is σ -finite. The analogous statement holds if we replace s by t . The following conditions on μ are equivalent.

- (i) $\mu_s = \mu_t$,
- (ii) $i_*\mu_s = \mu_s$,
- (iii) for every Borel subset $E \subset \underline{G}$ such that $s|_E$ and $t|_E$ are injective we have $\mu(s(E)) = \mu(t(E))$.

Definition 3.2. A discrete measurable groupoid \underline{G} together with an invariant probability measure on \underline{G}^0 , i.e. satisfying one of (i)–(iii), is called a *discrete measured groupoid*.

In the sequel \underline{G} will always be a discrete measured groupoid with invariant measure μ . The measure on \underline{G} induced by μ is denoted by $\mu_{\underline{G}}$. For a Borel subset $A \subset \underline{G}^0$, $\underline{G}|_A = s^{-1}(A) \cap t^{-1}(A)$ equipped with the normalized measure $\frac{1}{\mu(A)}\mu|_A$ is a discrete measured groupoid, called the *restriction of \underline{G} to A* . The *orbit equivalence relation* on a probability G -space (X, μ) defined by

$$\mathcal{R}(G \curvearrowright X) = \{(x, gx); x \in X, g \in G\} \subset X \times X$$

is a discrete measured groupoid. The composition is given by $(x, y)(y, z) = (x, z)$. Another example is given by the restriction of the *holonomy groupoid* of a foliation to a complete transversal with an invariant measure. For a function $\phi : \underline{G} \rightarrow \mathbb{C}$ and $x \in \underline{G}^0$ we put

$$\begin{aligned} S(\phi)(x) &= \#\{g \in \underline{G}; \phi(g) \neq 0, s(g) = x\} \in \mathbb{N} \cup \{\infty\}, \\ T(\phi)(x) &= \#\{g \in \underline{G}; \phi(g) \neq 0, t(g) = x\} \in \mathbb{N} \cup \{\infty\}. \end{aligned}$$

As usual, the set of complex-valued, measurable, essentially bounded functions (modulo almost null functions) on \underline{G} with respect to $\mu_{\underline{G}}$ is denoted by $L^\infty(\underline{G}, \mu_{\underline{G}})$. The *groupoid ring* $\mathbb{C}\underline{G}$ of \underline{G} is defined as

$$\mathbb{C}\underline{G} = \{\phi \in L^\infty(\underline{G}, \mu_{\underline{G}}); S(\phi) \text{ and } T(\phi) \text{ are essentially bounded on } \underline{G}^0\}.$$

The set $\mathbb{C}\underline{G}$ is a ring with involution containing $L^\infty(\underline{G}^0) = L^\infty(\underline{G}^0, \mu)$ as a subring. The addition is the pointwise addition in $L^\infty(\underline{G}, \mu_{\underline{G}})$, the multiplication is given by the convolution product

$$(\phi\eta)(g) = \sum_{\substack{g_1, g_2 \in \underline{G} \\ g_1 g_2 = g}} \phi(g_1)\eta(g_2), \quad \phi, \psi \in \mathbb{C}\underline{G}, \quad g \in \underline{G},$$

and the involution is defined by $(\phi^*)(g) = \overline{\phi(i(g))}$. The groupoid ring of the restriction $\mathbb{C}\underline{G}|_A$ of \underline{G} to A , called the *restricted groupoid ring*, is canonically isomorphic to $\chi_A \mathbb{C}\underline{G} \chi_A$. Next we explain how $L^\infty(\underline{G}^0)$ becomes a left $\mathbb{C}\underline{G}$ -module equipped with a $\mathbb{C}\underline{G}$ -epimorphism from $\mathbb{C}\underline{G}$ to $L^\infty(\underline{G}^0)$. The *augmentation homomorphism* is defined by $\epsilon : \mathbb{C}\underline{G} \rightarrow L^\infty(\underline{G}^0)$ by

$$\epsilon : \mathbb{C}\underline{G} \rightarrow L^\infty(\underline{G}^0), \quad \epsilon(\phi)(x) = \sum_{g \in s^{-1}(x)} \phi(g) \text{ for } x \in \underline{G}^0.$$

It becomes a homomorphism of $\mathbb{C}\underline{G}$ -modules when we equip $L^\infty(\underline{G}^0)$ with the $\mathbb{C}\underline{G}$ -module structure defined below, but it is not a homomorphism of rings unless \underline{G} is a group. In the language of [3], this means that $\mathbb{C}\underline{G}$ is an augmented ring with the augmentation module $L^\infty(\underline{G}^0)$. One checks easily that the augmentation homomorphism ϵ induces a $\mathbb{C}\underline{G}$ -module structure on $L^\infty(\underline{G}^0)$ by

$$\eta \cdot f = \epsilon(\eta f) \quad \text{where } \eta \in \mathbb{C}\underline{G}, \quad f \in L^\infty(\underline{G}^0).$$

Here ηf is the product in $\mathbb{C}\underline{G}$.

Lemma 3.3. *As a $L^\infty(\underline{G}^0)$ -module the groupoid ring $\mathbb{C}\underline{G}$ is generated by the characteristic functions χ_E of Borel subsets $E \subset \underline{G}$ with the property that $s|_E$ and $t|_E$ are injective.*

Proof. Consider $\phi \in \mathbb{C}\underline{G}$. Note that there is some $N > 0$ such that the preimages of almost all points in $\phi^{-1}(\mathbb{C} - \{0\})$ under s and t contain at most N elements. By 3.1 there is a finite Borel partition $X_i, i \in I$, of $\phi^{-1}(\mathbb{C} - \{0\})$ such that all $s|_{X_i}, t|_{X_i}$ are injective. Hence ϕ can be written as a finite sum $\phi = \sum_{i \in I} \phi_i$, where the support of ϕ_i lies in X_i . Since $s|_{X_i}, t|_{X_i}$ are injective, every ϕ_i is of the form $f\chi_{X_i}$ (convolution product) with $f \in L^\infty(\underline{G}^0)$. □

The groupoid ring $\mathbb{C}\underline{G}$ of discrete measured groupoid \underline{G} lies as a weakly dense involutive \mathbb{C} -subalgebra in the von Neumann algebra $\mathcal{N}(\underline{G})$ of the groupoid \underline{G} . For the construction of $\mathcal{N}(\underline{G})$ see [7, 8], also [1].

The von Neumann algebra $\mathcal{N}(\underline{G})$ has a finite trace $\text{tr}_{\mathcal{N}(\underline{G})}$ induced by the invariant measure μ . For $\phi \in \mathbb{C}\underline{G} \subset \mathcal{N}(\underline{G})$ we have

$$\text{tr}_{\mathcal{N}(\underline{G})}(\phi) = \int_{\underline{G}^0} \phi(g) d\mu(g).$$

Let R be a ring and G be a group. Given a homomorphism $c : G \rightarrow \text{Aut}(R), g \mapsto c_g$, we define the crossed product $R *_c G$ of R with G as the free R -module with basis G . It carries a ring structure that is uniquely defined by the rule $gr = c_g(r)g$ for $g \in G, r \in R$. For a probability G -space X we denote by $L^\infty(X) *_c G$ the crossed product $L^\infty(X) *_c G$ obtained by the homomorphism $c : G \rightarrow \text{Aut}(L^\infty(X))$ that maps g to $f \mapsto f \circ l_{g^{-1}}$. Here $l_g(x) = gx$ is left translation by $g \in G$. The ring homomorphism

$$L^\infty(X) *_c G \rightarrow \mathbb{C}\mathcal{R}(G \curvearrowright X), \quad \sum_{g \in G} f_g \cdot g \mapsto ((gx, x) \mapsto f_g(gx))$$

is injective and $\phi \in \mathbb{C}\mathcal{R}(G \curvearrowright X)$ is in the image if and only if there is a finite subset $F \subset G$ such that $g \notin F$ implies $\phi(gx, x) = 0$ for almost all $x \in X$.

Note that the map is well defined because the action is essentially free. In the sequel we regard $L^\infty(X) *_c G$ as a subring of $\mathbb{C}\mathcal{R}(G \curvearrowright X)$.

Remark 3.4. The restriction of the $\mathbb{C}\mathcal{R}(G \curvearrowright X)$ -module structure on $L^\infty(X)$ to $L^\infty(X) *_c G$ is isomorphic to the $L^\infty(X) *_c G$ -module structure obtained by the isomorphism $L^\infty(X) \cong (L^\infty(X) *_c G) \otimes_{\mathbb{C}G} \mathbb{C}$.

4. Homological Algebra and Dimension Theory

The objects of study in this section are Tor-groups of the type $\text{Tor}_n^R(B, M)$ where $\mathcal{B} \subset R \subset \mathcal{A}$ are ring inclusions, \mathcal{A}, \mathcal{B} finite von Neumann algebras, B is a \mathcal{A} - R -bimodule and M is a R -module. Among the questions we deal with are: what

happens to the \mathcal{A} -dimension of $\text{Tor}_n^R(B, M)$ if we replace M by a $\dim_{\mathcal{B}}$ -isomorphic module, R by a $\dim_{\mathcal{B}}$ -isomorphic subring $R' \subset R$ or B by a $\dim_{\mathcal{A}}$ -isomorphic bimodule?

Recall that a ring R is called *semihereditary* if every finitely-generated submodule of a projective R -module is projective. A large class of examples for semihereditary rings is given by the following theorem. See [17, Theorem 6.7 on pp. 239, and 288].

Theorem 4.1. *Every von Neumann algebra \mathcal{N} is a semihereditary ring.*

Theorem 4.2 [14, Theorem 1.2]. *The category of finitely-presented modules over a semihereditary ring is abelian. In particular, the category of finitely-presented modules over a von Neumann algebra is abelian.*

The following theorem is a generalization of [17, Theorem 6.29, p. 253] from group von Neumann algebras to arbitrary finite von Neumann algebras. A short proof based on the fact that a von Neumann algebra is semihereditary can be found in [20, Theorem 1.48; 19, Theorem 6.8(i)].

Theorem 4.3. *Any trace-preserving $*$ -homomorphism between finite von Neumann algebras is a faithfully flat ring extension.*

Theorem 4.4 [13, pp. 139–146]. *Let R be a semihereditary ring. Then the following holds.*

- (i) *All torsionless R -modules are flat.*
- (ii) *Any direct product of flat R -modules is flat.*
- (iii) *Submodules of flat R -modules are flat.*

Lemma 4.5. *The groupoid ring $\mathbb{C}\underline{G}$ of a discrete measured groupoid is flat over $L^\infty(\underline{G}^0)$.*

Proof. There is an inclusion of rings $L^\infty(\underline{G}^0) \subset \mathbb{C}\underline{G} \subset \mathcal{N}(\underline{G})$ where $\mathcal{N}(\underline{G})$ is a finite von Neumann algebra whose trace extends that of $L^\infty(\underline{G}^0)$. By 4.3 $\mathcal{N}(\underline{G})$ is a flat module over $L^\infty(\underline{G}^0)$. By the previous theorem $\mathbb{C}\underline{G}$ is a flat $L^\infty(\underline{G}^0)$ -module. □

Definition 4.6. An \mathcal{A} - \mathcal{B} -bimodule M is called *dimension-compatible* if for every \mathcal{B} -module N the following implication holds:

$$\dim_{\mathcal{B}}(N) = 0 \Rightarrow \dim_{\mathcal{A}}(M \otimes_{\mathcal{B}} N) = 0.$$

We record some easy facts about dimension-compatible bimodules.

Lemma 4.7.

- (i) *If M is a dimension-compatible \mathcal{A} - \mathcal{B} -bimodule and N is a dimension-compatible \mathcal{B} - \mathcal{C} -bimodule, then $M \otimes_{\mathcal{B}} N$ is a dimension-compatible \mathcal{A} - \mathcal{C} -bimodule.*

- (ii) Quotients and direct summands of dimension-compatible bimodules are dimension-compatible.
- (iii) Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of finite von Neumann algebras. Then \mathcal{A} is a dimension-compatible \mathcal{A} - \mathcal{B} -bimodule.

Here the third assertion follows from 2.6. Next we show that groupoid rings provide examples of dimension-compatible bimodules.

Lemma 4.8. $\mathbb{C}\underline{G}$ is a dimension-compatible $L^\infty(\underline{G}^0)$ - $L^\infty(\underline{G}^0)$ -bimodule.

Proof. Let M be an $L^\infty(\underline{G}^0)$ -module with $\dim_{L^\infty(\underline{G}^0)}(M) = 0$. We have to show

$$\dim_{L^\infty(\underline{G}^0)}(\mathbb{C}\underline{G} \otimes_{L^\infty(\underline{G}^0)} M) = 0. \tag{3}$$

By the local criterion 2.4, Eq. (3) follows, if for a given $x \in \mathbb{C}\underline{G} \otimes_{L^\infty(\underline{G}^0)} M$ a sequence of annihilating projections $\chi_{A_i} \in L^\infty(\underline{G}^0)$ exists such that

$$\chi_{A_i}x = 0 \quad \text{and} \quad \text{tr}_{L^\infty(\underline{G}^0)}(\chi_{A_i}) = \mu(A_i) \rightarrow 1. \tag{4}$$

Suppose this is true for a set S of $L^\infty(\underline{G}^0)$ -generators of $\mathbb{C}\underline{G} \otimes_{L^\infty(\underline{G}^0)} M$. Then (4) holds for any element in $\mathbb{C}\underline{G} \otimes_{L^\infty(\underline{G}^0)} M$ by the following observation.

If χ_{A_i} and χ_{B_i} are annihilating projections for the elements r resp. s , whose traces converge to 1, then the $\chi_{A_i \cap B_i}$ are annihilating projections for $f \cdot r + g \cdot s$, $f, g \in L^\infty(\underline{G}^0)$, whose traces also converge to 1.

A set S of $L^\infty(\underline{G}^0)$ -generators of $\mathbb{C}\underline{G} \otimes_{L^\infty(\underline{G}^0)} M$ is given by elements of the form $\chi_E \otimes m$, where χ_E is the characteristic function of a Borel subset $E \subset \underline{G}$, such that $s|_E$ and $t|_E$ are injective, and m is an element in M . This is Lemma 3.3. Before we prove (4) for the elements in S , we show that for any Borel subset $A \subset \underline{G}^0$ there is a Borel subset $A' \subset \underline{G}^0$ such that

$$\mu(A') \geq \mu(A) \quad \text{and} \quad \chi_{A'} \cdot \chi_E = \chi_E \cdot \chi_A. \tag{5}$$

We have the identities $\chi_{A'}\chi_E = \chi_{s^{-1}(A') \cap E}$ and $\chi_E\chi_A = \chi_{t^{-1}(A) \cap E}$. Put

$$A' = s(E \cap t^{-1}(A)) \cup (\underline{G}^0 - s(E)).$$

Because $s|_E$ is injective we get

$$s^{-1}(A') \cap E = s^{-1}(s(E \cap t^{-1}(A))) \cap E = E \cap t^{-1}(A).$$

This yields $\chi_{A'} \cdot \chi_E = \chi_E \cdot \chi_A$. The invariance of μ yields

$$\begin{aligned} \mu(A') &= \mu(s(E \cap t^{-1}(A))) + \mu(\underline{G}^0 - s(E)) \\ &= \mu(t(E \cap t^{-1}(A))) + \mu(\underline{G}^0 - t(E)) \\ &= \mu(t(E) \cap A) + \mu(\underline{G}^0 - t(E)) \\ &\geq \mu(A). \end{aligned}$$

Now we can prove (4) for $x = \chi_E \otimes m \in S$ as follows. Because of $\dim_{L^\infty(\underline{G}^0)}(M) = 0$ there are $A_i \subset \underline{G}^0$ with $\chi_{A_i}m = 0$ and $\mu(A_i) \rightarrow 1$, due to the local criterion 2.4.

By (5) there are $A'_i \subset \underline{G}^0$ with $\chi_{A'_i} \chi_E = \chi_E \chi_{A_i}$ and $\text{tr}_{L^\infty(\underline{G}^0)}(\chi_{A'_i}) = \mu(A'_i) \rightarrow 1$. Now (4) is obtained from $\chi_{A'_i}(\chi_E \otimes m) = \chi_E \chi_{A_i} \otimes m = \chi_E \otimes \chi_{A_i} m = 0$. \square

Lemma 4.9. *Let \mathcal{N} be a finite von Neumann algebra, R a ring and B_1, B_2 \mathcal{N} - R -bimodules. A bimodule map $B_1 \rightarrow B_2$ that is a $\dim_{\mathcal{N}}$ -isomorphism induces $\dim_{\mathcal{N}}$ -isomorphisms*

$$\text{Tor}_{\bullet}^R(B_1, M) \rightarrow \text{Tor}_{\bullet}^R(B_2, M)$$

for every R -module M .

Proof. Let B an \mathcal{N} - R -bimodule with $\dim_{\mathcal{N}}(B) = 0$. Let M be an arbitrary R -module and P_{\bullet} a free R -resolution of M . Then $\dim_{\mathcal{N}}(B \otimes_R P_{\bullet}) = 0$ follows from the additivity and cofinality of $\dim_{\mathcal{N}}$ (see 2.1). Hence

$$\dim_{\mathcal{N}}(H_{\bullet}(B \otimes_R P_{\bullet})) = \dim_{\mathcal{N}}(\text{Tor}_{\bullet}^R(B, M)) = 0.$$

In the general case of a $\dim_{\mathcal{N}}$ -isomorphism $\phi : B_1 \rightarrow B_2$, we consider the short exact sequences

$$\begin{aligned} 0 &\rightarrow \ker \phi \rightarrow B_1 \rightarrow \text{im } \phi \rightarrow 0, \\ 0 &\rightarrow \text{im } \phi \rightarrow B_2 \rightarrow \text{coker } \phi \rightarrow 0. \end{aligned}$$

$\ker \phi$ and $\text{coker } \phi$ have vanishing dimension. We obtain long exact sequences for the Tor-terms:

$$\begin{aligned} \dots &\rightarrow \text{Tor}_1^R(B_1, M) \rightarrow \text{Tor}_1^R(\text{im } \phi, M) \\ &\rightarrow \underbrace{\ker \phi \otimes M}_{=\text{Tor}_0^R(\ker \phi, M)} \rightarrow B_1 \otimes M \rightarrow \text{im } \phi \otimes M \rightarrow 0 \\ \dots &\rightarrow \text{Tor}_1^R(B_2, M) \rightarrow \text{Tor}_1^R(\text{coker } \phi, M) \rightarrow \text{im } \phi \otimes M \rightarrow B_2 \otimes M \\ &\rightarrow \underbrace{\text{coker } \phi \otimes M}_{=\text{Tor}_0^R(\text{coker } \phi, M)} \rightarrow 0. \end{aligned}$$

We already know $\dim_{\mathcal{N}}(\text{Tor}_{\bullet}^R(\ker \phi, M)) = 0$ and $\dim_{\mathcal{N}}(\text{Tor}_{\bullet}^R(\text{coker } \phi, M)) = 0$, hence

$$\begin{aligned} \text{Tor}_{\bullet}^R(B_1, N) &\rightarrow \text{Tor}_{\bullet}^R(\text{im } \phi, N), \\ \text{Tor}_{\bullet}^R(\text{im } \phi, N) &\rightarrow \text{Tor}_{\bullet}^R(B_2, N) \end{aligned}$$

are $\dim_{\mathcal{N}}$ -isomorphisms, and so is their composition. \square

Lemma 4.10. *Let $\mathcal{B} \subset R \subset \mathcal{A}$ be an inclusion of rings where \mathcal{A}, \mathcal{B} are finite von Neumann algebras. Let B be an \mathcal{A} - R -bimodule. We assume the following.*

- (i) R is dimension-compatible as a \mathcal{B} - \mathcal{B} -bimodule.
- (ii) B is dimension-compatible as an \mathcal{A} - \mathcal{B} -bimodule.
- (iii) B is flat as a right \mathcal{B} -module.

Then every R -homomorphism $M \rightarrow N$, which is a $\dim_{\mathcal{B}}$ -isomorphism, induces a $\dim_{\mathcal{A}}$ -isomorphism

$$\mathrm{Tor}_{\bullet}^R(B, M) \rightarrow \mathrm{Tor}_{\bullet}^R(B, N).$$

Proof. First we get the following flatness properties.

- The \mathcal{A} - R -bimodule $B \otimes_{\mathcal{B}} R$ is flat as a right R -module because B is flat as a right \mathcal{B} -module.
- R is a \mathcal{B} -submodule of the flat \mathcal{B} -module \mathcal{A} (4.3). Hence R is flat as a right \mathcal{B} -module by 4.4 and 4.1.
- Therefore $B \otimes_{\mathcal{B}} R$ is also flat as a right \mathcal{B} -module.

Multiplication yields a surjective \mathcal{A} - R -bimodule homomorphism

$$m : B \otimes_{\mathcal{B}} R \rightarrow B.$$

$B \otimes_{\mathcal{B}} R$ is dimension-compatible (as an \mathcal{A} - \mathcal{B} -bimodule) because B and R are dimension-compatible. The map m splits as an \mathcal{A} - \mathcal{B} -homomorphism by the map $B \rightarrow B \otimes_{\mathcal{B}} R, b \mapsto b \otimes 1$. Hence, as an \mathcal{A} - \mathcal{B} -bimodule, $\ker m$ is a direct summand in $B \otimes_{\mathcal{B}} R$ and therefore also dimension-compatible. We record the properties of $\ker m$:

- $\ker m$ is an \mathcal{A} - R -bimodule;
- $\ker m$ is dimension-compatible as an \mathcal{A} - \mathcal{B} -bimodule;
- $\ker m$ is flat as a right \mathcal{B} -module because it is the submodule of a flat \mathcal{B} -module (4.4, 4.1).

Notice that $\ker m$ satisfies all properties imposed on B . Let M be an R -module with $\dim_{\mathcal{B}}(M) = 0$. So we have $\dim_{\mathcal{A}}(\ker m \otimes_{\mathcal{B}} M) = 0$ and hence for its quotient

$$\dim_{\mathcal{A}}(\ker m \otimes_R M) = 0.$$

The short exact sequence $0 \rightarrow \ker m \rightarrow B \otimes_{\mathcal{B}} R \rightarrow B \rightarrow 0$ induces a long exact sequence of Tor-terms

$$\begin{aligned} \cdots \rightarrow 0 \rightarrow \mathrm{Tor}_2^R(B, M) \rightarrow \mathrm{Tor}_1^R(\ker m, M) \rightarrow \underbrace{\mathrm{Tor}_1^R(B \otimes_{\mathcal{B}} R, M)}_{=0} \\ \rightarrow \mathrm{Tor}_1^R(B, M) \rightarrow \ker m \otimes_R M \rightarrow (B \otimes_{\mathcal{B}} R \otimes_R M) \rightarrow B \otimes_R M \rightarrow 0, \end{aligned}$$

where the zero terms are due to $B \otimes_{\mathcal{B}} R$ being R -flat. We obtain

$$\begin{aligned} \dim_{\mathcal{A}}(B \otimes_R M) &= \dim_{\mathcal{A}}(\mathrm{Tor}_1^R(B, M)) = 0, \\ \mathrm{Tor}_{i+1}^R(B, M) &\cong \mathrm{Tor}_i^R(\ker m, M) \quad i \geq 1. \end{aligned} \tag{6}$$

Now we apply (6) to $\ker m$ instead of B and get

$$\dim_{\mathcal{A}}(\mathrm{Tor}_1^R(\ker m, M)) = 0, \quad \dim_{\mathcal{A}}(\mathrm{Tor}_2^R(B, M)) = 0.$$

Repeating this (“dimension shifting”) yields

$$\dim_{\mathcal{A}}(\mathrm{Tor}_i^R(B, M)) = 0 \quad \text{for } i \geq 0.$$

Deducing the general case of a $\dim_{\mathcal{B}}$ -isomorphism $\phi : M \rightarrow N$ from the case $\dim_{\mathcal{B}}(M) = 0$ uses exactly the same method as in the proof of 4.9. \square

Theorem 4.11. *Let $\mathcal{B} \subset R_1 \subset R_2 \subset \mathcal{A}$ be an inclusion of rings where \mathcal{A}, \mathcal{B} are finite von Neumann algebras. We assume the following.*

- (i) R_2 is dimension-compatible as a \mathcal{B} - \mathcal{B} -bimodule.
- (ii) The inclusion $R_1 \subset R_2$ is a $\dim_{\mathcal{B}}$ -isomorphism.

Then

$$\dim_{\mathcal{A}}(\mathrm{Tor}_{\bullet}^{R_1}(\mathcal{A}, M)) = \dim_{\mathcal{A}}(\mathrm{Tor}_{\bullet}^{R_2}(\mathcal{A}, M))$$

holds for every R_2 -module M .

Proof. By 4.9 the induced map

$$\mathrm{Tor}_i^{R_2}(R_2, M) \leftarrow \mathrm{Tor}_i^{R_1}(R_1, M) = \begin{cases} M & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} \tag{7}$$

is a $\dim_{\mathcal{B}}$ -isomorphism. Let $P_{\bullet} \rightarrow M$ be a projective R_1 -resolution of M . The Künneth spectral sequence, applied to \mathcal{A} and the complex $R_2 \otimes_{R_1} P_{\bullet}$ (see [21, Theorem 5.6.4, p. 143]), has the E^2 -term

$$E_{pq}^2 = \mathrm{Tor}_p^{R_2}(\mathcal{A}, H_q(R_2 \otimes_{R_1} P_{\bullet})) = \mathrm{Tor}_p^{R_2}(\mathcal{A}, \mathrm{Tor}_q^{R_1}(R_2, M))$$

and converges to

$$H_{p+q}(\mathcal{A} \otimes_{R_2} (R_2 \otimes_{R_1} P_{\bullet})) = \mathrm{Tor}_{p+q}^{R_1}(\mathcal{A}, M).$$

Now we can apply the preceding Lemma 4.10 to the inclusion $\mathcal{B} \subset R_2 \subset \mathcal{A}$. Here recall that \mathcal{A} is flat as a right \mathcal{B} -module (4.3) and dimension-compatible as an \mathcal{A} - \mathcal{B} -bimodule. So we obtain

$$\dim_{\mathcal{A}}(E_{pq}^2) = \begin{cases} \dim_{\mathcal{A}}(\mathrm{Tor}_p^{R_2}(\mathcal{A}, M)) & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases}.$$

Hence additivity (2.1) yields

$$\dim_{\mathcal{A}}(\mathrm{Tor}_p^{R_1}(\mathcal{A}, M)) = \dim_{\mathcal{A}}(E_{p0}^{\infty}) = \dim_{\mathcal{A}}(E_{p0}^2) = \dim_{\mathcal{A}}(\mathrm{Tor}_p^{R_2}(\mathcal{A}, M)).$$

Recall that there is a localization of the category of \mathcal{A} -modules with respect to $\dim_{\mathcal{A}}$ -isomorphisms, which is itself an abelian category. One could also say that in this localization the spectral sequence collapses at its E^2 -term. \square

Actually, the theorem above provides more than just an equality of the dimensions. There is a natural zig-zag of $\dim_{\mathcal{A}}$ -isomorphisms between $\mathrm{Tor}_{\bullet}^{R_1}(\mathcal{A}, M)$ and $\mathrm{Tor}_{\bullet}^{R_2}(\mathcal{A}, M)$.

Theorem 4.12. *Let $\mathcal{B} \subset R \subset \mathcal{A}$ be an inclusion of rings, where \mathcal{A}, \mathcal{B} are finite von Neumann algebras. Let $p \in \mathcal{B}$ be a projection. We assume the following.*

- (i) R is dimension-compatible as a \mathcal{B} - \mathcal{B} -bimodule.
- (ii) p is $\dim_{\mathcal{B}}$ -full in R .

Then the equality

$$\dim_{p\mathcal{A}p} (p\mathrm{Tor}_{\bullet}^R(\mathcal{A}, M)) = \dim_{p\mathcal{A}p} (\mathrm{Tor}_{\bullet}^{pRp}(p\mathcal{A}p, pM))$$

holds for every R -module M .

Proof. We have $p\mathrm{Tor}_i^{pRp}(Rp, pM) \cong \mathrm{Tor}_i^{pRp}(pRp, pM) = 0$ for $i > 0$. Theorem 2.10 implies $\dim_{\mathcal{B}}(\mathrm{Tor}_i^{pRp}(Rp, pM)) = 0, i > 0$. By the same theorem the multiplication map m

$$\mathrm{Tor}_0^{pRp}(Rp, pM) = Rp \otimes_{pRp} pM \rightarrow M$$

is a $\dim_{\mathcal{B}}$ -isomorphism. Now let $P_{\bullet} \rightarrow pM$ be a pRp -projective resolution of pM . The Künneth spectral sequence applied to \mathcal{A} and the complex $Rp \otimes_{pRp} P_{\bullet}$ (see [21, Theorem 5.6.4, p. 143]) has the E^2 -term

$$E_{ij}^2 = \mathrm{Tor}_i^R(\mathcal{A}, H_j(Rp \otimes_{pRp} P_{\bullet})) = \mathrm{Tor}_i^R(\mathcal{A}, \mathrm{Tor}_j^{pRp}(Rp, pM))$$

and converges to

$$H_{i+j}(\underbrace{\mathcal{A} \otimes_R (Rp \otimes_{pRp} P_{\bullet})}_{= \mathcal{A}p \otimes_{pRp} P_{\bullet}}) = \mathrm{Tor}_{i+j}^{pRp}(\mathcal{A}p, pM).$$

We know $\mathrm{Tor}_{\bullet}^{pRp}(Rp, pM)$ up to $\dim_{\mathcal{B}}$ -isomorphism. An application of Lemma 4.10 to that module yields

$$\dim_{\mathcal{A}}(E_{ij}^2) = \begin{cases} \dim_{\mathcal{A}}(\mathrm{Tor}_i^R(\mathcal{A}, M)) & \text{if } j = 0 \\ 0 & \text{if } j > 0. \end{cases}$$

The spectral sequence collapses up to dimension. This implies

$$\dim_{\mathcal{A}}(\mathrm{Tor}_i^{pRp}(\mathcal{A}p, pM)) = \dim_{\mathcal{A}}(E_{i0}^{\infty}) = \dim_{\mathcal{A}}(E_{i0}^2) = \dim_{\mathcal{A}}(\mathrm{Tor}_i^R(\mathcal{A}, M)).$$

It follows from 2.4 that $\mathcal{A}p\mathcal{A} \subset \mathcal{A}$ is $\dim_{\mathcal{A}}$ -surjective since $RpR \subset R$ is $\dim_{\mathcal{B}}$ -surjective. Hence p is $\dim_{\mathcal{A}}$ -full in \mathcal{A} and the claim is obtained from 2.11. □

5. L^2 -Betti Numbers of Discrete Measured Groupoids

Definition 5.1. Let \underline{G} be a discrete measured groupoid. Its n th L^2 -Betti number $b_n^{(2)}(\underline{G})$ is defined as

$$b_n^{(2)}(\underline{G}) = \dim_{\mathcal{N}(\underline{G})}(\mathrm{Tor}_n^{\mathbb{C}\underline{G}}(\mathcal{N}(\underline{G}), L^{\infty}(\underline{G}^0))).$$

Lemma 5.2. *Let \underline{G} be a discrete measured groupoid, and let $A \subset \underline{G}^0$ be a Borel subset such that $t(s^{-1}(A))$ has full measure in \underline{G}^0 . Then the characteristic function $\chi_A \in L^\infty(\underline{G}^0) \subset \mathbb{C}\underline{G}$ of A is a $\dim_{L^\infty(\underline{G}^0)}$ -full idempotent in $\mathbb{C}\underline{G}$.*

Proof. We have to show that the inclusion $\mathbb{C}\underline{G}\chi_A\mathbb{C}\underline{G} \subset \mathbb{C}\underline{G}$ is a $\dim_{L^\infty(\underline{G}^0)}$ -isomorphism. Let $s^{-1}(A) = \bigcup_{n \in \mathbb{N}} E_n$ be a partition into Borel sets E_n with the property that $s|_{E_n}$ is injective. The partition exists by Lemma 3.1. We have

$$\chi_{t(E_n)} \cdot \chi_A \cdot \chi_{E_n} = \chi_{t(E_n)}\chi_{s^{-1}(A) \cap E_n} = \chi_{t(s^{-1}(A) \cap E_n)}.$$

The last equality is due to the injectivity of $s|_{E_n}$. Hence $\sum_{n=1}^N \chi_{t(s^{-1}(A) \cap E_n)} \in \mathbb{C}\underline{G}\chi_A\mathbb{C}\underline{G}$. So we get

$$\chi_{t(s^{-1}(A) \cap \bigcup_{n=1}^N E_n)} = f \cdot \left(\sum_{n=1}^N \chi_{t(s^{-1}(A) \cap E_n)} \right) \in \mathbb{C}\underline{G} \cdot \chi_A \cdot \mathbb{C}\underline{G}$$

for a suitable $f \in L^\infty(\underline{G}^0)$. This implies $\chi_{t(s^{-1}(A) \cap \bigcup_{n=1}^N E_n)} \cdot \phi = 0$ for every element ϕ in the quotient $\mathbb{C}\underline{G}/\mathbb{C}\underline{G}\chi_A\mathbb{C}\underline{G}$. Because $t(s^{-1}(A))$ has full measure, we get

$$\lim_{N \rightarrow \infty} \mu \left(t \left(s^{-1}(A) \cap \bigcup_{n=1}^N E_n \right) \right) = \mu(\underline{G}^0) = 1.$$

So $\mathbb{C}\underline{G}\chi_A\mathbb{C}\underline{G} \subset \mathbb{C}\underline{G}$ is a $\dim_{L^\infty(\underline{G}^0)}$ -isomorphism by 2.4. □

Theorem 5.3. *Let \underline{G} be a discrete measured groupoid, and let $A \subset \underline{G}^0$ be a Borel subset such that $t(s^{-1}(A))$ has full measure in \underline{G}^0 . Then*

$$b_n^{(2)}(\underline{G}) = \mu(A) \cdot b_n^{(2)}(\underline{G}|_A).$$

Proof. By 5.2 and 2.11 we get

$$b_n^{(2)}(\underline{G}) = \mu(A) \cdot \dim_{\chi_A \mathcal{N}(\underline{G}) \chi_A} (\chi_A \text{Tor}_n^{\mathbb{C}\underline{G}}(\mathcal{N}(\underline{G}), L^\infty(\underline{G}^0))).$$

By 4.8 the groupoid ring $\mathbb{C}\underline{G}$ is dimension-compatible as a $L^\infty(\underline{G}^0)$ - $L^\infty(\underline{G}^0)$ -bimodule whence by 4.12 we obtain that $b_n^{(2)}(\underline{G})$ equals

$$\mu(A) \cdot \dim_{\chi_A \mathcal{N}(\underline{G}) \chi_A} (\text{Tor}_n^{\chi_A \mathbb{C}\underline{G} \chi_A} (\chi_A \mathcal{N}(\underline{G}) \chi_A, \chi_A L^\infty(\underline{G}^0))) = \mu(A) \cdot b_n^{(2)}(\underline{G}|_A).$$

For the right equal sign note that $\mathcal{N}(\underline{G}|_A) = \chi_A \mathcal{N}(\underline{G}) \chi_A$, $\mathbb{C}\underline{G}|_A = \chi_A \mathbb{C}\underline{G} \chi_A$ and $\chi_A L^\infty(\underline{G}^0) = L^\infty(A)$. □

Lemma 5.4. *Let X be a probability G -space and be \mathcal{R} the orbit equivalence relation on X . Then the inclusion $L^\infty(X) * G \subset \mathbb{C}\mathcal{R}$ is a $\dim_{L^\infty(\underline{G}^0)}$ -isomorphism.*

Proof. We apply the local criterion (2.4) to show that the quotient $\mathbb{C}\mathcal{R}/L^\infty(X) * G$ has dimension zero. Let $\phi \in \mathbb{C}\mathcal{R}$ and $[\phi]$ be its image in the quotient. Choose an enumeration $G = \{g_1, g_2, \dots\}$. Define Borel subsets $X_n \subset X$ by $X_n = \{x \in X; \phi(g_i x, x) = 0 \text{ for all } i > n\}$, and let χ_n be the characteristic function of X_n . Then

$\chi_n \phi \in L^\infty(X) * G$ holds, hence $\chi_n[\phi] = 0 \in \mathbb{C}\mathcal{R}/L^\infty(X) * G$. Because of $\mu(X_n) \rightarrow \mu(X) = 1$ the local criterion yields $\dim_{L^\infty(X)}(\mathbb{C}\mathcal{R}/L^\infty(X) * G) = 0$. □

Theorem 5.5. *Let X be a probability G -space. Then the L^2 -Betti numbers of G and the orbit equivalence relation \mathcal{R} on X coincide.*

$$b_n^{(2)}(G) = b_n^{(2)}(\mathcal{R}).$$

Proof. The crossed product ring $L^\infty(X) * G$ is flat as a right $\mathbb{C}G$ -module because of the equality $(L^\infty(X) * G) \otimes_{\mathbb{C}G} M = L^\infty(X) \otimes_{\mathbb{C}} M$. Hence we obtain

$$\begin{aligned} b_n^{(2)}(G) &= \dim_{\mathcal{N}(\mathcal{R})}(\mathcal{N}(\mathcal{R}) \otimes_{\mathcal{N}(G)} \text{Tor}_n^{\mathbb{C}G}(\mathcal{N}(G), \mathbb{C})) \quad \text{by (2.6)} \\ &= \dim_{\mathcal{N}(\mathcal{R})}(\text{Tor}_n^{\mathbb{C}G}(\mathcal{N}(\mathcal{R}), \mathbb{C})) \quad \text{by (4.3)} \\ &= \dim_{\mathcal{N}(\mathcal{R})}(\text{Tor}_n^{L^\infty(X)*G}(\mathcal{N}(\mathcal{R}), (L^\infty(X) * G) \otimes_{\mathbb{C}G} \mathbb{C})) \\ &= \dim_{\mathcal{N}(\mathcal{R})}(\text{Tor}_n^{L^\infty(X)*G}(\mathcal{N}(\mathcal{R}), L^\infty(X))). \end{aligned}$$

Now we will apply Theorem 4.11 to the ring inclusions

$$L^\infty(X) \subset L^\infty(X) * G \subset \mathbb{C}\mathcal{R} \subset \mathcal{N}(\mathcal{R}).$$

As an $L^\infty(X)$ - $L^\infty(X)$ -bimodule, $\mathbb{C}\mathcal{R}$ is dimension-compatible by 4.8. The inclusion $L^\infty(X) * G \subset \mathbb{C}\mathcal{R}$ is a $\dim_{L^\infty(\underline{G}^0)}$ -isomorphism by 5.4. Hence by 4.11 we obtain

$$\begin{aligned} \dim_{\mathcal{N}(\mathcal{R})}(\text{Tor}_n^{L^\infty(X)*G}(\mathcal{N}(\mathcal{R}), L^\infty(X))) &= \dim_{\mathcal{N}(\mathcal{R})}(\text{Tor}_n^{\mathbb{C}\mathcal{R}}(\mathcal{N}(\mathcal{R}), L^\infty(X))) \\ &= b_n^{(2)}(\mathcal{R}). \end{aligned} \quad \square$$

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