L^2-INVARIANTS AND RANK METRIC

ANDREAS THOM

Abstract. We introduce a notion of rank completion for bi-modules over a finite tracial von Neumann algebra. We show that the functor of rank completion is exact and that the category of complete modules is abelian with enough projective objects. This leads to interesting computations in the L^2-homology for tracial algebras. As an application, we also give a new proof of a Theorem of Gaboriau on invariance of L^2-Betti numbers under orbit equivalence.

1. Preliminaries

1.1. **Introduction.** The aim of this article is to unify approaches to several results in the theory of L^2-invariants of groups, see [Lüc02, Gab02a], and tracial algebras, see [CS05]. The new approach allows us to sharpen several results that were obtained in [Tho06b]. We also give a new proof of D. Gaboriau's Theorem on invariance of L^2-Betti numbers under orbit equivalence. In order to do so, we introduce the concept of rank metric and rank completion of bi-modules over a von finite tracial von Neumann algebra.

All von Neumann algebras in this article have a separable pre-dual. Recall, a von Neumann algebra is called finite and tracial, if it comes with a fixed positive, faithful and normal trace. Every finite (i.e. Dedekind finite) von Neumann algebra admits such a trace, but we assume that a choice of a trace is fixed.

The rank is a natural measure of the size of the support of an element in a bi-module over a finite tracial von Neumann-algebra. The induced metric endows each bi-module with a topology, such that all bi-module maps are contractions. The main utility of completion with respect to the rank metric is revealed by the observation that the functor of rank completion is exact and that the category of complete modules is abelian with enough projective objects.

Employing the process of rank completion, we aim to proof two main results. First of all, we will show that certain L^2-Betti number invariants of von Neumann algebras coincide with those for arbitrary weakly dense sub-C^*-algebras. The particular case of the first L^2-Betti number was treated in [Tho06b]. The general result required a more conceptual approach and is carried out in this article. The importance of this result was pointed out to the author by D. Shlyakhtenko. Indeed, according to A. Connes and D. Shlyakhtenko, K-theoretic methods might be used to relate the L^2-Euler characteristic of a group C^*-algebra to the ordinary Euler characteristic of the group. This could finally lead to a computation of the L^2-Betti numbers for certain von Neumann algebras and would resolve some longstanding conjectures, as for example the non-isomorphism conjecture for free group factors, see [Voi05]. However, a concrete implementation of this idea is not in reach and a lot preliminary work has still to be carried out.

Secondly, inspired by ideas of R. Sauer from [Sau03], we will give a new and self-contained proof of invariance of L^2-Betti numbers of groups under orbit equivalence. The idea here is very simple. We show that L^2-Betti number invariants cannot see the difference between an $L^\infty(X)$-algebra and its rank completion. Then, we observe the following: If free measure preserving actions of Γ_1 and Γ_2 on a probability space X induce the same equivalence relation, then $L^\infty(X) \rtimes_{\mathrm{alg}} \Gamma_1$ and $L^\infty(X) \rtimes_{\mathrm{alg}} \Gamma_2$ have isomorphic rank completions as bi-modules (with respect to the diagonal left action) over $L^\infty(X)$. It remains to carry out several routine calculations in homological algebra.

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1.2. **Dimension theory.** In his pioneering work, W. Lück was able to describe a lot of the analytic properties of the category of Hilbert-modules over a finite tracial von Neumann algebras (M, τ) in purely algebraic terms. This allowed to employ the maschinery of homological algebra in the study L^2-invariants and lead to substantial results and a conceptional understanding from an algebraic point of view. One important ingredient in his work is a dimension function which is defined for all M-modules, see [Lüc02]. Due to several ring-theoretic properties of M, the natural dimension function for projective modules has an extension to all modules and shares several convenient properies. In particular, it was shown in [Lüc02], that the sub-category of zero-dimensional modules is a Serre sub-category, i.e. is closed under extensions. This implies that there is a 5-Lemma for dimension isomorphisms. The following lemma is immediate from this. (See [Wei94] for the necessary definitions.)

Lemma 1.1. *Let (M, τ) be a finite tracial von Neumann algebra. Let $\mathcal{A}$ be an abelian category with enough projective objects and let $F, G\colon \mathcal{A} \to \mathrm{Mod}^M$ be right exact functors into the category Mod^M of M-modules. If there exists a natural transformation $h\colon F \to G$ which consists of dimension isomorphisms, then the induced natural transformations*

$$h_i\colon L_i(F) \to L_i(G)$$

of left-derived functors consist of dimension isomorphisms too.

L^2-Betti numbers for certain group-actions on spaces were introduced by M. Atiyah in [Ati76]. The domain of definition was extended by J. Cheeger and M. Gromov in [CG86]. For references and most of the main results, see [Lüc02]. An important result of Lück was the following equality, which we take as a basis for our computations is Section 4:

$$\beta_k^{(2)}(\Gamma) = \dim_{L\Gamma} \mathrm{Tor}_k^{\mathbb{C}\Gamma}(L\Gamma, \mathbb{C}). \tag{1}$$

The following observation concerning a characterization of zero-dimensional modules is due to R. Sauer, see [Sau03], and will be of major importance in the sequel.

Theorem 1.2 (Sauer). *Let (M, τ) be a finite tracial von Neumann algebra and let L be a M-module. The following conditions are equivalent:*

(1) *L is zero dimensional.*
(2) $\forall \xi \in L, \forall \varepsilon > 0, \exists p \in \mathrm{Proj}(M)\colon \quad \xi p = \xi \quad$ *and* $\quad \tau(p) \leq \varepsilon$.

The second condition is usually referred to as a local criterion of zero dimensionality. In the next section, we want to exploit this observation further and study completions of bi-modules with respect to a certain metric that measures the size of the support.

Let $a \in M$ be an arbitrary element in a finite tracial von Neumann algebra (M, τ). We denote by $s(a)$ is support projection and by $r(a)$ its range projection. Note that the equality $\tau(s(a)) = \tau(r(a))$ always holds. We denote by $\mathrm{Proj}(M)$ the set of projections of M. Note that $\mathrm{Proj}(M)$ is a complete, complemented modular lattice. We denote the operations of meet, join and complement by $\wedge, \vee$ and $\perp$.

2. Completion of bi-modules

2.1. **Definition.** Let us denote the category of M-bi-modules by Bimod^M. We will loosely identify M-bi-modules with $M \otimes M^o$-modules. In the sequel we regard $M \otimes M^o$ as a bi-module over M, acting by multiplication with $M \otimes M^o$ on the left.

Let (M, τ) be a finite tracial von Neumann algebra and let L be a M-bi-module. Associated to an element $\xi \in L$, there is a real-valued quantity that measures the size of the support. Let us set

$$[\xi] = \inf \left\{ \tau(p) + \tau(q) \colon p, q \in \mathrm{Proj}(M), p^\perp \xi q^\perp = 0 \right\} \in [0, 1].$$

Obviously, for the bi-module M and $x \in M$, we get that $[x]$ equals the trace of the support projection $s(x)$ of x. Indeed, if $p^\perp x q^\perp = 0$, then $\tau(p^\perp) + \tau(s(x)) + \tau(q^\perp) \leq 2$ and thus

$$\tau(p) + \tau(q) \geq \tau(s(x)).$$

We conclude that $[x] \geq \tau(s(x))$. The reverse inequality is obvious.

Lemma 2.1. *Let L be a M-bi-module and let $\xi_1, \xi_2 \in L$. The in equality*

$$[\xi_1 + \xi_2] \leq [\xi_1] + [\xi_2]$$

holds.

Proof. Let $\varepsilon > 0$ be arbitrary. We find projections p_1, q_1, p_2, q_2, such that $\tau(p_i) + \tau(q_i) \leq [\xi_i] + \varepsilon$ and $p_i^{\perp}\xi_i q_i^{\perp} = 0$, for $i = 0, 1$. Since $(p_1^{\perp} \wedge p_2^{\perp})(\xi_1 + \xi_2)(q_1^{\perp} \wedge q_2^{\perp}) = 0$ and, we get that

$$[\xi_1 + \xi_2] \leq \tau(p_1 \vee p_2) + \tau(q_1 \vee q_2) \leq \tau(p_1) + \tau(p_2) + \tau(q_1) + \tau(q_2) \leq [\xi_1] + [\xi_2] + 2\varepsilon.$$

Since ε was arbitrary, the claim follows. □

Note, there is no reason to assume that $[\xi] = 0 \implies \xi = 0$. Indeed, one can easily construct examples where this fails.

Definition 2.2. *Let L be a M-bi-module. The quantity $d(\xi, \zeta) \stackrel{\text{def}}{=} [\xi - \zeta] \in \mathbb{R}$ defines a quasi-metric on L, which we call* rank metric.

Lemma 2.3. *Let $\phi\colon L \to L'$ be a homomorphism of M-bi-modules.*

(1) *The map ϕ is a contraction in the rank-metric.*
(2) *Let $\varepsilon > 0$ be arbitrary. If ϕ is surjective and $\xi' \in L'$, then there exists $\xi \in L$, such that $\phi(\xi) = \xi'$ and $[\xi] \leq [\xi'] + \varepsilon$.*

Proof. (1) Let $\xi \in L$, $\varepsilon > 0$ and $p, q \in \mathrm{Proj}(M)$, such that $\tau(p) + \tau(q) \leq [\xi] + \varepsilon$ and $p^{\perp}\xi q^{\perp} = 0$. Clearly, $p^{\perp}\phi(x)q^{\perp} = 0$, and hence $[\phi(x)] \leq \tau(p) + \tau(q) \leq [\xi] + \varepsilon$. Since ε was arbitrary, the assertion follows.
(2) Let $\xi' \in L'$. There exists $p, q \in \mathrm{Proj}(M)$, such that $\tau(p) + \tau(q) \leq [\xi'] + \varepsilon$ and $p^{\perp}\xi' q^{\perp} = 0$. Let $\xi'' \in L$ be any lift of ξ' and set $\xi = \xi'' - p^{\perp}\xi'' q^{\perp}$. We easily see that $\phi(\xi) = \xi'$ and that $p^{\perp}\xi q^{\perp} = 0$. Hence, ξ is a lift and $[\xi] \leq [\xi'] + \varepsilon$ as required. □

Definition 2.4. *Let L be a M bi-module. The rank metric endows L with a uniform structure.*

(1) *We denote by $CS(L)$ the linear space of Cauchy sequences in L, by $ZS(L) \subset CS(L)$ the sub-space of sequences that converge to $0 \in L$. Finally, we set $c(L) = CS(L)/ZS(L)$ and call it the* completion *of L.*
(2) *There is a natural map $L \to c(L)$ which sends an element to the constant sequence. The bi-module L is called* complete*, if it is an isomorphism. We denote by Bimod_c^M the full sub-category of complete $M \otimes M^o$-modules.*

Lemma 2.5. *Let M be a finite tracial von Neumann algebra.*

(1) *The completion $M \hat{\otimes} M^o$ of $M \otimes M^o$ as a M-bimodule is a unital ring containing $M \otimes M^o$.*
(2) *Let L be a M-bi-module. The completion is naturally a $M \hat{\otimes} M^o$-module and in particular a M-bi-module.*
(3) *The assignment $L \mapsto c(L)$ extends to a functor from the category of $M \otimes M^o$-modules to the category of Bimod_c^M of complete $M \otimes M^o$-modules.*

Proof. Let L be a M-bi-module. Let us first show that the $M \otimes M^o$-module structure extends to $c(L)$. Let $\xi \in M \otimes M^o$. We consider the map $\lambda_\xi : L \to L$ which is defined to be left-multiplication by ξ. λ_ξ is not a module-homomorphism but still to some extend compatible with the rank metric. Let $\eta \in L, \varepsilon > 0$ and $p, q \in \mathrm{Proj}(M)$ with $(p^{\perp} \otimes q^{\perp o})\eta = 0$ and $\tau(p) + \tau(q) \leq [\eta] + \varepsilon$.
Specifying to $\xi = a \otimes b^o$, we get:

$$\lambda_\xi(\eta) = (a \otimes b^o)\eta = (a \otimes b^o)(1 \otimes 1 - p^{\perp} \otimes q^{\perp o})\eta = (ap \otimes b^o)\eta + (ap^{\perp} \otimes (qb)^o)\eta.$$

We compute: $(r(ap)^{\perp} \otimes 1)(ap \otimes b^o)\eta = 0$ and hence $[(ap \otimes b^o)\eta] \leq \tau(r(ap)) = \tau(s(ap)) \leq \tau(p)$. Similarily, $[(ap^{\perp} \otimes (qb)^o)\eta] \leq \tau(q)$ and hence $[(a \otimes b^o)\eta] \leq \tau(p) + \tau(q) \leq [\eta] + \varepsilon$. Again, since $\varepsilon > 0$ was arbitrary, we conclude $[\lambda_{a \otimes b^o}(\eta)] \leq [\eta]$.

If $\xi = a_1 \otimes b_1 + \cdots + a_n \otimes b_n$, we get from Lemma 2.1, that

$$[L_\xi(\eta)] \leq n \cdot [\eta], \quad \forall \eta \in L.$$

We conclude that λ_ξ is Lipschitz for all $\xi \in M \otimes M^o$. Hence, there is an extension $\lambda_\xi : CS(L) \to CS(L)$ which preserves $ZS(L)$. Hence, there exists a bi-linear map $m' : (M \otimes M^o) \times c(L) \to c(L)$ which defines a module structure that is compatible with the module structure on L.
It is clear that $[m(\xi, \eta)] \leq [\xi]$. Indeed, $(p^\perp \otimes q^{\perp o})\xi = 0$ implies $(p^\perp \otimes q^{\perp o})m(\xi, \eta) = 0$. Hence m' has a natural extension

$$m : (M \mathbin{\hat{\otimes}} M^o) \times c(L) \to c(L).$$

Obviously, if $L = M \otimes M^o$, then m defines a multiplication that extends the multiplication on $M \otimes M^o$, i.e. the natural inclusion $M \otimes M^o \hookrightarrow M \mathbin{\hat{\otimes}} M^o$ is a ring-homomorphism. This shows (1) and (2). Assertion (3) is obvious. □

2.2. **Completion is exact.**

Lemma 2.6. *The functor of completion is exact.*

Proof. Let

$$0 \to J \to L \xrightarrow{\pi} K \to 0$$

be an exact sequence of M-bi-modules. We have to show that the induced sequence

$$0 \to c(J) \to c(L) \to c(K) \to 0$$

is exact.
First, we consider the exactness at $c(K)$. Let $(\xi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in K. Without loss of generality, we can assume that $[\xi_n - \xi_{n+1}] \leq 2^{-n}$. Lemma 2.3 implies that we can lift $(\xi_n)_{n\in\mathbb{N}}$ to a sequence $(\xi'_n)_{n\in\mathbb{N}} \subset L$ with $[\xi'_n - \xi'_{n+1}] \leq 2^{1-n}$, hence a Cauchy sequence. This shows surjectivity of $c(\pi)$.
We consider now the exactness at $c(L)$. Obviously, $\mathrm{im}(c(J)) \subset \ker(c(\pi))$. Let $(\xi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in L which maps to zero in $c(K)$. This says that $(\pi(\xi_n))_{n\in\mathbb{N}}$ tends to zero. Again, by Lemma 2.3, we can lift $(\pi(\xi_n))_{n\in\mathbb{N}}$ to a zero-sequence $(\xi'_n)_{n\in\mathbb{N}} \subset L$. Now, $(\xi_n - \xi'_n)_{n\in\mathbb{N}}$ defines a Cauchy sequence in J, that is equivalent to the sequence $(\xi_n)_{n\in\mathbb{N}}$ in the completion of L. Hence $\ker(c(\pi)) \subset \mathrm{im}(c(J))$ and the argument is finished.
The exactness at $c(J)$ is obvious since $J \subset L$ is a contraction in the rank metric by Lemma 2.3. This finishes the proof. □

Theorem 2.7. *Let M be a finite tracial von Neumann algebra. Consider the category* Bimod^M *of M-bi-modules and the full sub-category* Bimod_c^M *of complete modules.*

1. *The completion functor* $c\colon \mathrm{Bimod}^M \to \mathrm{Bimod}_c^M$ *is left-adjoint to the forgetful functor from* Bimod_c^M *to* Bimod^M*, i.e. whenever K is complete:*

$$\hom_{\mathrm{Bimod}^M}(c(L), K) = \hom_{\mathrm{Bimod}^M}(L, K).$$

2. *The category* Bimod_c^M *is abelian and has enough projective objects.*
3. *The completion functor* $c\colon \mathrm{Bimod}^M \to \mathrm{Bimod}_c^M$ *preserves projective objects.*
4. *The kernel of the comparison map $L \to c(L)$ is $\{\xi \in L \colon [\xi] = 0\}$.*

Proof. (1) If K is complete, the natural map $K \to c(K)$ is an isomorphism, so that applying the functor c defines a natural map

$$\hom_{\mathrm{Bimod}^M}(L, K) \to \hom_{\mathrm{Bimod}^M}(c(L), K).$$

A map in the inverse direction is provided by pre-composition with the map $L \to c(L)$. Assertion (1) follows easily by Lemma 2.3 since $\mathrm{im}(L)$ is dense in $c(L)$ and $\{\xi \in K, [\xi] = 0\} = \{0\}$.
(2) If follows from Lemma 2.6, that Bimod_c^M is abelian. Indeed, by exactness, kernels and co-kernels can be formed in Bimod^M and hence all properties of those remain to be true in Bimod_c^M.
Let $L = \oplus_\alpha M \otimes M^o$ be a free $M \otimes M^o$-module. By (1), $c(L)$ is a projective object in Bimod_c^M.

If K is complete and $\oplus_\alpha M \otimes M^o \xrightarrow{\pi} K$ is, using Lemma 2.6, $c(\pi)$ is also a surjection onto K. I.e. there are enough projective objects. (3) follows from (1). (4) is obvious. □

2.3. **Completion is dimension-preserving.**

Lemma 2.8. *Let L be a M bi-module. The natural map*

$$M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} L \to M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} c(L)$$

is a dimension isomorphism.

Proof. By Lemma 2.6, it suffices to show that

$$\ker\left(M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} L \to M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} c(L)\right)$$

is zero-dimensional for all bi-modules L. Indeed $\operatorname{coker}(L \to c(L))$ has vanishing completion, and knowing the assertion for $\operatorname{coker}(L \to c(L))$ in place of L implies that

$$\operatorname{coker}\left(M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} L \to M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} c(L)\right)$$

has dimension zero.

We want to apply the local critertion of Theorem 1.2. Let $\theta = \sum_{i=1}^n \eta_i \otimes \xi_i \in M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} L$ be in the kernel. This is to say that there exists some $l \in \mathbb{N}$ and zero-sequences $(\alpha_{i,k})_{k \in \mathbb{N}}$, for $1 \leq i \leq l$, such that

$$\sum_{i=1}^n \eta_i \otimes \xi_i = \sum_{i=1}^l \zeta_i \otimes \alpha_{i,k}, \quad \forall k \in \mathbb{N}.$$

Indeed, the map factorizes through the split-injection

$$M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} L \to M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} CS(L)$$

and hence $\operatorname{im}(M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} ZS(L)) \subset M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} CS(L)$ contains the image of

$$\ker\left(M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} L \to M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} c(L)\right).$$

Since $\alpha_{i,k} \to 0$, for all $1 \leq i \leq l$, for every $\varepsilon > 0$ there exists k big enough and projections $p_i, q_i \in \operatorname{Proj}(M)$, such that $(p_i^\perp \otimes q_i^{\perp o})\alpha_{i,k} = 0$ and $\tau(p_i) + \tau(q_i) \leq \varepsilon/l$.
Let f_i be projections in $M \mathbin{\overline{\otimes}} M^o$, such that $f_i^\perp \zeta_i = \zeta_i(p_i^\perp \otimes q^{\perp o})$. One can choose f_i to satisfy $\tau(f_i) \leq \tau(p) + \tau(q) \leq \varepsilon/l$, for all $1 \leq i \leq l$. We compute as follows:

$$\bigwedge_{i=1}^l f_i^\perp \theta = \bigwedge_{i=1}^l f_i^\perp \left(\sum_{i=1}^l f_i^\perp \eta_i \otimes \alpha_{i,k} \right) = \bigwedge_{i=1}^l f_i^\perp \left(\sum_{i=1}^l \eta_i \otimes (p_i^\perp \otimes q_i^{\perp o})\alpha_{i,k} \right) = 0.$$

Thus $\tau(\vee_{i=1}^l f_i) \leq \varepsilon$ and and $(\vee_{i=1}^l f_i)\theta = \theta$. Hence

$$\ker\left(M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} L \to M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} c(L)\right)$$

is zero-dimensional by Theorem 1.2. □

Theorem 2.9. *Let $\phi \colon L \to L'$ be a morphism of M-bi-modules. If $c(\phi)$ is an isomorphism, then*

$$\operatorname{Tor}_i^{M \otimes M^o}(M \mathbin{\overline{\otimes}} M^o, L) \xrightarrow{\phi_*} \operatorname{Tor}_i^{M \otimes M^o}(M \mathbin{\overline{\otimes}} M^o, L')$$

is a dimension isomorphism.

Proof. The exactness of $c \colon \mathrm{Bimod}^M \to \mathrm{Bimod}_c^M$ implies the following natural identification among left-derived functors:

$$L_i(M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} ?) \circ c = L_i(M \mathbin{\overline{\otimes}} M^o \otimes_{M \otimes M^o} c(?)).$$

Indeed, this follows from the fact that $c \colon \mathrm{Bimod}^M \to \mathrm{Bimod}_c^M$ maps free modules in Bimod^M to projective objects in Bimod_c^M. This implies the existence of a Grothendieck spectral sequence (see [Wei94, pp. 150]) that yields the desired result.

Lemma 2.8 together with Lemma 1.1 implies the existence of a natural map

$$\operatorname{Tor}_i^{M\otimes M^o}(M\mathbin{\overline{\otimes}} M^o, ?) \to L_i(M\mathbin{\overline{\otimes}} M^o \otimes_{M\otimes M^o} c(?))$$

which is a dimension isomorphism. Combining the preceding two observations, we conclude that

$$\dim_{M\overline{\otimes}M^o}\operatorname{Tor}_i^{M\otimes M^o}(M\mathbin{\overline{\otimes}} M^o, L) = 0, \quad \forall i \geq 0,$$

whenever $c(L) = 0$. This implies the claim, since $c(\ker(\phi)) = 0$ and $c(\operatorname{coker}(\phi)) = 0$ by Lemma 2.6. □

3. L^2-Betti numbers for tracial algebras

3.1. **Preliminaries.** In [CS05], A. Connes and D. Shlyakhtenko introduced a notion of L^2-homology and L^2-Betti numbers for tracial algebras, compare also earlier work of W.L. Paschke in [Pas97]. The definition works well in a situation where the tracial algebra (A, τ) is contained in a finite von Neumann algebra M, to which the trace τ extends. More precisely, using the dimension function of W. Lück, see [Lüc02], they set:

$$\beta_k^{(2)}(A,\tau) = \dim_{M\overline{\otimes}M^o}\operatorname{Tor}_k^{A\otimes A^o}(M\mathbin{\overline{\otimes}} M^o, A)$$

Here, $M\mathbin{\overline{\otimes}} M^o$ denote the spatial tensor product of von Neumann algebras. We equipp $M\mathbin{\overline{\otimes}} M^o$ with the trace $\tau\otimes\tau$. Several results concerning these L^2-Betti numbers where obtained in [CS05] and [Tho06a, Tho06b]. In particular, it was shown in [CS05] that

$$\beta_k^{(2)}(\mathbb{C}\Gamma,\tau) = \beta_k^{(2)}(\Gamma),$$

where the right side denotes the L^2-Betti number of a group in the sense of Atiyah, see [Ati76] and Cheeger-Gromov, see [CG86].

It is conjectured in [CS05] that $\beta_k^{(2)}(M,\tau)$ is an interesting invariant for the von Neumann algebra. Several related quantities where studied in [CS05] as well. In particular,

$$\Delta_k(A,\tau) = \dim_{M\overline{\otimes}M^o}\operatorname{Tor}_k^{M\otimes M^o}(M\mathbin{\overline{\otimes}} M^o, M\otimes_A M)$$

was studied for $k = 1$.

3.2. **Pedersen's Theorem.**

Lemma 3.1. *Let (M,τ) be a finite tracial von Neumann algebra and let $A_1, A_2 \subset M$ be $*$-subalgebras of M. If A_1 and A_2 have the same closure with respect to the rank metric, then*

$$\Delta_k(A_1,\tau) = \Delta_k(A_2,\tau), \quad \forall k \geq 0.$$

Proof. Without loss of generality, $A_1 \subset A_2$. We show that $\pi\colon M\otimes_{A_1} M \to M\otimes_{A_2} M$ induces an isomorphism after completion with respect to the rank metric. The claim follows then from Theorem 2.9.

By Lemma 2.6, it suffices to show that the kernel of π has vanishing completion. An element ξ in the kernel can be written as

$$\xi = \sum_{i=1}^n c_i a_i \otimes d_i - c_i \otimes a_i d_i,$$

for some $c_i, d_i \in M$ and $a_i \in A_2$. Since A_1 is dense in A_2 there exists $a_i' \in A_1$ with $[a_i - a_i'] \leq \varepsilon/n$, for all $1 \leq i \leq n$.

The following equality holds in $M\otimes_{A_1} M$:

$$\xi = \sum_{i=1}^n c_i a_i \otimes d_i - c_i \otimes a_i d_i = \sum_{i=1}^n c_i(a_i - a_i')\otimes d_i - c_i\otimes(a_i - a_i')d_i,$$

Argueing as before, we see that there exists projections p, q of trace less than ε, such that $p^\perp \xi q^\perp = 0$. This implies that $[\xi] \leq \varepsilon$. Since ε was arbitrary, we get that $[\xi] = 0$ for all $\xi \in \ker(\pi\colon M\otimes_{A_1} M \to M\otimes_{A_2} M)$ and thus $c(\ker(\pi)) = 0$. □

The following result by G. Pedersen, see [Ped79, Thm. 2.7.3], is a deep result in the theory of operator algebras, which required a detailed analysis of the precise position of a weakly dense C^*-algebra inside a von Neumann algebra. It is a generalization of a more classical theorem of Lusin in the commutative case.

Theorem 3.2 (Pedersen)**.** *Let (M, τ) be a finite tracial von Neumann algebra and let $A \subset M$ be a weakly dense sub-C^*-algebra. The algebra A is dense in M with respect to the rank metric.*

Corollary 3.3. *Let (M, τ) be a finite tracial von Neumann algebra and let $A \subset M$ be a weakly dense sub-C^*-algebra.*

$$\Delta_k(A, \tau) = \Delta_k(M, \tau), \quad \forall k \geq 0.$$

Remark 3.4. In [Tho06b], it was shown that $\beta_1^{(2)}(A, \tau) = \beta_1^{(2)}(M, \tau)$, whenever A is a weakly dense sub-C^*-algebra. In view of the factorization

$$\mathrm{Tor}\,_1^{A \otimes A^o}(M \,\overline{\otimes}\, M^o, A) \twoheadrightarrow \mathrm{Tor}\,_1^{M \otimes M^o}(M \,\overline{\otimes}\, M^o, M \otimes_A M) \to \mathrm{Tor}\,_1^{M \otimes M^o}(M \,\overline{\otimes}\, M^o, M),$$

the proof also shows that $\Delta_1(A, \tau) = \Delta_1(M, \tau)$ holds. Hence we can view Corollary 3.3 as a generalization of this result from [Tho06b].

4. Equivalence relations and Gaboriau's Theorem

4.1. **Equivalence relations and completion.** Most of the proofs in the section are parallel to proofs in Section 2 and 3 and hence we will give less detail and point to the relevant parts of Section 2 and 3. Let X be a standard Borel space and let μ be a probability measure on X. Given a discrete measurable equivalence relation (see [FM77a, FM77b] for the necessary definitions)

$$R \subset X \times X,$$

we can form a *relation ring* $R(X)$ as follows:

$$R(X) = \left\{ \sum_{i=1}^{n} f_i \phi_i \colon f_i \in L^\infty(X), \phi_i \text{ local isomorphism from } R \right\} \subset L^\infty(R).$$

Here, $L^\infty(R)$ denote the generated von Neumann algebra, *see also* FM1 Note that $R(X)$ is a $L^\infty(X)$-bi-module with respect to the diagonal left action. (All $L^\infty(X)$-modules are bi-modules in this way.) The following observation is the key to our results.

Proposition 4.1. *Let Γ be a discrete group and let $\rho \colon \Gamma \times X \to X$ be a measure preserving free action of G on X. We denote by R_ρ the induced measurable equivalence relation on X. The natural inclusion $\iota \colon L^\infty(X) \rtimes_{\mathrm{alg}} \Gamma \to R_\rho(X)$ induces an isomorphisms after completion.*

Proof. According to the foundational work in [FM77a, FM77b], each local isomorphism ϕ which is implemented by the equivalence relation R_ρ can be decomposed as a infinite sum of local isomorphism ϕ_i, each of which is a cut-down of an isomorphism which is implemented by the action of a group element. The sizes of the supports of the cut-down local isomorphisms ϕ_i in this decomposition sum to the size of the support of ϕ. It clearly implies, that ϕ can be approached by elements of $L^\infty(X) \rtimes \Gamma$ in rank metric. This finishes the proof. □

Definition 4.2. *Two group Γ_1 and Γ_2 are called* orbit equivalent*, if there exists a probability space X and free, measure preserving actions of Γ_1 and Γ_2 on X that induce the same equivalence relation.*

For an excellent survey on the properties of orbit equivalence and related notions, see [Gab02b].

Lemma 4.3. *Let $R \subset X \times X$. We denote the completion of $R(X)$ by $\widehat{R}(X)$. $\widehat{R}(X)$ is a unital $L^\infty(X)$-algebra that contains $R(X)$ as a $L^\infty(X)$-sub-algebra.*

Proof. First of all, $\widehat{R}(X)$ is a $R(X)$-module. Indeed, left multiplication by $\sum_{i=1}^n f_i\phi_i$ is easily seen to be Lipschitz with constant n. In particular, there exists a map $m'\colon R(X)\times\widehat{R}(X)\to\widehat{R}(X)$. As before, we easily see that m' has an extension to an associative and seperately continuous multiplication:

$$m\colon \widehat{R}(X)\times\widehat{R}(X)\to\widehat{R}(X).$$

□

Lemma 4.4. *Let L be a $L^\infty(X)\rtimes_{\rm alg}\Gamma$-module. The completion of L, with respect to the diagonal $L^\infty(X)$-bi-module structure is naturally a $\widehat{R}(X)$-module and in particular a $R(X)$-module.*

Proof. By Proposition 4.1 $L^\infty(X)\rtimes_{\rm alg}\Gamma$ is dense in $R(X)$ and hence in $\widehat{R}(X)$. Again, since $x=\sum_{i=1}^n f_i\phi_i$ acts with Lipschitz constant n on L, the action extends to $c(L)$. Let x_n be a Cauchy sequence in $L^\infty(X)\rtimes\Gamma$ and $\xi\in c(L)$. The rank of $(x_n-x_m)\xi$ is less that the rank of x_n-x_m and if $x_n\to 0$, then $x_n\xi\to 0$. This finishes the proof. □

4.2. **Proof of Gaboriau's Theorem.** The proof of Gaboriau's Theorem which is presented in this section uses the technology of rank completion. It is very much inspired by a proof of Gaboriau's Theorem given by R. Sauer in [Sau03]. In his proof, the local criterion was a crucial ingredient to make the arguments work. We hope that the concept of rank completion will provide a good understanding of why the Theorem is true.

Lemma 4.5. *Let Γ be a discrete group and let $\rho\colon\Gamma\times X\to X$ be a measurable and measure preserving action on a probability space X.*

1. $\mathbb{C}\Gamma\subset L^\infty(X)\rtimes_{\rm alg}\Gamma$ *is flat.*
2. $L\Gamma\subset L^\infty(X)\rtimes\Gamma$ *is flat and dimension preserving.*

Proof. The first assertion is obvious. Indeed, $L^\infty(X)\rtimes_{\rm alg}\Gamma$ is a free $\mathbb{C}\Gamma$-module. The second assertion follows from the fact that $L\Gamma$ is semi-hereditary, see [Sau05]. □

Theorem 4.6. *Let Γ_1 and Γ_2 are orbit equivalent groups, then*

$$\beta_k^{(2)}(\Gamma_1)=\beta_k^{(2)}(\Gamma_2),\quad \forall k\geq 0.$$

Proof. It suffices to write $\beta_k^{(2)}(\Gamma_1)$ entirely in terms of the equivalence relation it generates. Using Lemma 4.5, we rewrite:

$$\begin{aligned}(L^\infty(X)\rtimes\Gamma)\otimes_{L\Gamma}{\rm Tor}_k^{\mathbb{C}\Gamma}(L\Gamma,\mathbb{C}) &= {\rm Tor}_k^{\mathbb{C}\Gamma}(L^\infty(X)\rtimes\Gamma,\mathbb{C})\\ &= {\rm Tor}_k^{L^\infty(X)\rtimes_{\rm alg}\Gamma}(L^\infty(X)\rtimes\Gamma,L^\infty(X)).\end{aligned}$$

Here, the second equality follows since

$$(L^\infty(X)\rtimes_{\rm alg}\Gamma)\otimes_{\mathbb{C}\Gamma}\mathbb{C}=L^\infty(X)$$

as $L^\infty(X)\rtimes_{\rm alg}\Gamma$-module. By Lemma 4.5 and Equation 1, we conclude that

$$\beta_k^{(2)}(\Gamma)=\dim_{L^\infty(X)\rtimes\Gamma}{\rm Tor}_k^{L^\infty(X)\rtimes_{\rm alg}\Gamma}(L^\infty(X)\rtimes\Gamma,L^\infty(X)). \tag{2}$$

There exists an exact functor which completes the category of $L^\infty(X)\rtimes_{\rm alg}\Gamma$-modules with respect to the diagonal left $L^\infty(X)$-bi-module structure. We have shown in Lemma 4.4 that the resulting full subcategory of those $L^\infty(X)\rtimes_{\rm alg}\Gamma$-modules which are complete, is naturally a category of $R(X)$-modules. Let us denote the functor of completion by

$$c\colon {\rm Mod}^{L^\infty(X)\rtimes_{\rm alg}\Gamma}\to{\rm Mod}_c^{R(X)}.$$

Proposition 4.7. *Let L be a $L^\infty(X)\rtimes_{\rm alg}\Gamma$-module. The completion map induces an dimension isomorphism:*

$$(L^\infty(X)\rtimes\Gamma)\otimes_{L^\infty(X)\rtimes_{\rm alg}\Gamma}L\to(L^\infty(X)\rtimes\Gamma)\otimes_{R(X)}c(L).$$

Proof. The map can be factorized as

$$(L^\infty(X)\rtimes\Gamma)\otimes_{L^\infty(X)\rtimes_{\rm alg}\Gamma} L \to (L^\infty(X)\rtimes\Gamma)\otimes_{L^\infty(X)\rtimes_{\rm alg}\Gamma} c(L) \to (L^\infty(X)\rtimes\Gamma)\otimes_{R(X)} c(L).$$

We show that each of the maps is a dimension isomorphism. Let us start with the first one. Again, by exactness of c, it suffices to show that

$$\ker\left(L^\infty(X)\rtimes\Gamma\right)\otimes_{L^\infty(X)\rtimes_{\rm alg}\Gamma} L \to L^\infty(X)\rtimes\Gamma)\otimes_{L^\infty(X)\rtimes_{\rm alg}\Gamma} c(L))$$

is zero-dimensional. As in the proof of Lemma 2.8, an element θ in kernel is of the form:

$$\theta=\sum_{i=1}^{l}\zeta_i\otimes\alpha_{i,k},\quad \forall k\in\mathbb{N},$$

for some zero sequences $(\alpha_{i,k})_{k\in\mathbb{N}}\subset L$. The proof proceeds as the proof of Lemma 2.8.
The second map can be seen to be a dimension isomorphism as follows. Clearly, the map is surjective and it remains to show that the kernel is zero-dimensional. An element of the kernel is of the form:

$$\theta=\sum_{i=1}^{n}\xi_i\eta_i\otimes\zeta_i-\xi_i\otimes\eta_i\zeta_i,$$

for some $\xi_i\in L^\infty(X)\rtimes\Gamma,\eta_i\in R(X)$ and $\zeta_i\in L$. Approximating η_i by elements in $L^\infty(X)\rtimes_{\rm alg}\Gamma$ we can assume (as in the proof of Lemma 3.1) that $[\eta_i]\le\varepsilon/(2n)$. The first summands are smaller than $\varepsilon/(2n)$, since support and range projection have the same trace. The second summand are also smaller than $\varepsilon/(2n)$ by the same argument and since projections in $L^\infty(X)$ can be moved through the tensor product. Hence $[\theta]\le\varepsilon$. Since θ and ε were arbitrary, we conclude by Theorem 1.2 that

$$\ker\big((L^\infty(X)\rtimes\Gamma)\otimes_{L^\infty(X)\rtimes_{\rm alg}\Gamma} c(L)\to(L^\infty(X)\rtimes\Gamma)\otimes_{R(X)} c(L)\big)$$

is zero dimensional. This finishes the proof. □

To conclude the proof of Theorem 4.6, we note that by Lemma 1.1 we get an induced map

$$\operatorname{Tor}_i^{L^\infty(X)\rtimes_{\rm alg}\Gamma}(L^\infty(X)\rtimes\Gamma,?)\to L_i((L^\infty(X)\rtimes\Gamma)\otimes_{R(X)} c(?))$$

which is a dimension isomorphism. The right hand side applied to $L^\infty(X)$ depends only on the generated equivalence relation. Indeed, as in the proof of Theorem 2.9, a Grothendieck spectral sequence shows

$$L_i((L^\infty(X)\rtimes\Gamma)\otimes_{R(X)} c(?))=L_i((L^\infty(X)\rtimes\Gamma)\otimes_{R(X)}?)\circ c.$$

Here, we use implicitly that the category of complete $R(X)$-modules is abelian with enough projective objects. The proof of this fact can be taken verbatim from the proof of Theorem 2.7 and the adjointness relations:

$$\hom_{L^\infty(X)\rtimes\Gamma}(L,K)=\hom_{L^\infty(X)\rtimes\Gamma}(c(L),K)=\hom_{R(X)}(c(L),K).$$

The projective objects are completions of free $R(X)$-modules. □

Remark 4.8. There is a second major result of Gaboriau's on proportionality of L^2-Betti numbers for weakly orbit equivalent groups, see [Gab02b]. Sauer has shown in [Sau03], how homological methods and properties of the dimension function allow to deduce this result. The same arguments apply to our setting.

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Andreas Thom, Mathematisches Institut der Universität Göttingen, Bunsenstr. 3-5, D-37073 Göttingen, Germany
E-mail address: thom@uni-math.gwdg.de
URL: http://www.uni-math.gwdg.de/thom