OPEN PROBLEM SESSION - OBERWOLFACH, 5-11.09.2010

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1. Fixed Price

1.1. Let (X, μ) be a measure space, and let $A = (\alpha_i, X_i)$, $i = 1, 2, \ldots$, be a countable family of pairs such that X_i is a measurable subset of X and α_i is a partial isomorphism defined on X_i. Such a family gives rise to a graphing $\mathcal{G}(A)$ of X: there is an oriented edge between x and y in X iff there exists (α_i, X_i) such that $x \in X_i$ and $\alpha_i(x) = y$. Let $R(\mathcal{G}(A))$ be the equivalence relation generated by the graphing $\mathcal{G}$.

Given an equivalence relation R on X which can be expressed as $R(\mathcal{G}(A))$ for some family A, define $\operatorname{cost}(R)$, *the cost of* R, to be the infimum of the quantity

$$\sum_{(\alpha_i, X_i) \in B} \mu(X_i)$$

over all families B such that $R = R(\mathcal{G}(B))$.

One can start with an abstract definition of a measurable equivalence relation, but in fact it doesn't increase generality.

1.2. Given a group Γ and a free probability measure preserving action $\rho : \Gamma \curvearrowright X$, define $\operatorname{cost}(\rho)$, *the cost of* ρ, to be the cost of the relation generated by the family $(\rho(\gamma), X)$. Define $\operatorname{cost}(\Gamma)$, *the cost of* Γ, to be the infimum of the costs of all the actions of Γ which are free and probability measure preserving.

1.3. **The Fixed Price Problem for Γ.** Is it the case that for every probability measure preserving action ρ of Γ we have that $\operatorname{cost}(\Gamma) = \operatorname{cost}(\rho)$? (Shortly: *Does Γ has fixed price?*)

If a given group Γ has fixed price, and $\operatorname{cost}\Gamma = r$, then we say that Γ *has fixed price* r.

1.4. **Some known results.** It is not known if there exist groups which don't have fixed price.

(1) F_n, the free group on n generators, has fixed price n.
(2) Amenable group have fixed price 1.
(3) Higher rank non-uniform lattices have fixed price 1.
(4) If Γ has an amenable normal subgroup then Γ has fixed price 1.
(5) $A \times B$, where both A and B contain an element of infinite order, has fixed price 1.
(6) If A, B and C have fixed price and C is amenable then $A *_C B$ has fixed price.
(7) If $\Gamma =< H_1, H_2, \ldots >$ is generated by infinite subgroups $H_1, H_2, \ldots$, such that $[H_i, H_{i+1}] = \{1\}$ for each i then Γ has fixed price 1.

1.5. **Subproblem.** Generalize (5) to products of arbitrary infinite groups.

1.6. **Subproblem.** Do cocompact lattices have fixed price 1? It is known that they have cost equal to 1.

2. Multiplicativity of cost-1

2.1. **The problem.** Let $\rho : \Gamma \curvearrowright X$ be a free, probability measure preserving action of a countable group, and let Λ be a subgroup of Γ of finite index. Let $\rho_{|\Lambda}$ denote the action of Λ on X which is the restricion of ρ. Is it the case that $\operatorname{cost}(\rho_{|\Lambda}) - 1 = [\Gamma : \Lambda] \cdot (\operatorname{cost}(\rho) - 1)$?.

2.2. It is known that $\operatorname{cost}(\Lambda) - 1 = [\Gamma : \Lambda] \cdot (\operatorname{cost}(\Gamma) - 1)$. Also, the inequality $\operatorname{cost}(\rho_{|\Lambda}) - 1 \leq [\Gamma : \Lambda] \cdot (\operatorname{cost}(\rho) - 1)$ always holds.

2.3. **Some known results.**

(1) Multiplicativity of cost -1 is known for treeable relations.
(2) If Γ has fixed price then multiplicativity of cost -1 holds for Γ.

3. Independence of Rank Gradient

3.1. Let Γ be a residually finite group and let (Γ_i) be a chain of normal subgroups of Γ such that $\bigcap_i \Gamma_i = \{1\}$. Define $RG(\Gamma, (\Gamma_i))$, *the rank gradient of* Γ *with respect to* (Γ_i), as the limit

$$\lim_{i \to +\infty} \frac{d(\Gamma_i) - 1}{[\Gamma : \Gamma_i]},$$

where $d(\Gamma_i)$, *the rank of* Γ_i, is the minimal number of generators of Γ_i.

3.2. **The problem.** Is $RG(\Gamma, (\Gamma_i))$ independent of the chain (Γ_i)?

3.3. **Some known results.** If Γ has fixed price than its rank gradient is independent of the chain. The same holds if cost -1 is multiplicative for the actions of Γ (one can even restrict to profinite actions.)

4. Cost vs first l^2-Betti number

The first l^2-Betti number $\beta_1^{(2)}$ can be defined for an arbitrary finitely presented Γ. If Γ is residually finite then by Lueck's approximation theorem it can be defined using ordinary Betti numbers as

$$\lim_{i \to +\infty} \frac{\beta_1(\Gamma_i)}{[\Gamma : \Gamma_i]},$$

where (Γ_i) is any normal chain such that the quotients Γ/Γ_i are finite and $\bigcap_i \Gamma_i = \{1\}$. Note that $\beta_1(\Gamma_i)$ is the $\mathbb{Z}$-rank of the abelianization of Γ_i.

4.1. **The Problem.** Is it true that $\operatorname{cost} \Gamma = \beta_1^{(2)}(\Gamma) + 1$.

4.2. **Subproblem.** It is known that for a group Γ which has Kazhdan'z property (T) we have that $\beta_1^{(2)} = 0$. Is it true that for such Γ we have $\operatorname{cost}(\Gamma) = 1$?

4.3. **Scribe's subproblem (might be trivial).** If Γ is torsion free than it is conjectured, and known in many cases, that $\beta_1^{(2)}$ is an integer (this is known as *Atiyah conjecture for torsion-free groups.*) Is it true for torsion-free Γ that $\operatorname{cost}(\Gamma)$ is an integer?

5. Some infimum

This was something about comparison of $\operatorname{cost}(\langle G \rangle)$, where G is a graphing, with inf $\operatorname{cost}(\langle H \rangle)$, over all subgraphings of G; but I can't figure out the details.

6. An inequality

It is known that

$$\beta_1^{(2)}(\Gamma) \le \mathrm{cost}(\Gamma) - 1 \le RG(\Gamma),$$

where the rank gradient is taken with respect to any normal chain.

6.1. **The problem.** Can it happen that at least one of the inequalities above is strict?

6.2. **Scribe's subproblem (might be trivial).** Analogue of 4.3: For a torsion-free group, is the rank gradient an integer?

7. Treeable actions

If there exists a free probability measure preserving action of Γ which is treeable, is it true that every free probability measure preserving action of Γ is treeable?

7.1. **Known results.** Which groups admit treeable actions? For which groups the answer to the above questions is known to be *yes*?

8. Groupoid cost

Let $\rho\colon \Gamma \curvearrowright X$ be a probability measure preserving, not necessarily free action. Such an action determines a Γ*-labeled relation on* X: we say that to points x and y of X are γ-related, where $\gamma \in \Gamma$ iff $\rho(\gamma)(x) = y$. Call this Γ-labeled relation $R_\Gamma(\rho)$.

Consider a countable family A of pairs (γ_i, X_i), where $\gamma_i \in \Gamma$ and X_i are measurable subsets of X. Such a family generates a Γ*-labeled graphing* $\mathcal{G}_\Gamma(A)$, and this Γ-labeled graphing generates a Γ-labeled relation $R_\Gamma(\mathcal{G}_\Gamma(A))$.

Define $\mathrm{cost}_\Gamma(\rho)$, *the groupoid cost of* ρ, to be the infimum of the quantities

$$\sum_{(\gamma_i, X_i)\in B} \mu(X_i)$$

over all families B such that $R_\Gamma(\rho) = R_\Gamma(\mathcal{G}_\Gamma(B))$.

Note that the groupoid cost of the trivial action on a point is equal to $d(\Gamma)$, the minimal number of generators of Γ.

8.1. **The problem.** Given a probability measure preserving action $\rho\colon F_n \curvearrowright X$ of the free group F_n on n generators, is it true that $\mathrm{cost}_{F_n}(\rho) = n$?

9. l^2-Betti numbers over $\mathbb{F}_p$

Let Γ be residually finite and let Γ_i be a normal chain with trivial intersection and such that the quotients Γ/Γ_i are finite. Let $\beta_1(\Gamma_i, \mathbb{F}_p)$ be the first Betti number of Γ_i over $\mathbb{F}_p$. Note that $\beta_1(\Gamma_i, \mathbb{F}_p)$ is the $\mathbb{F}_p$-rank of the group $\mathbb{F}_p \otimes_{\mathbb{Z}} \Gamma_i^{ab}$ and as such we have

$$\beta_1(\Gamma_i) \le \beta_1(\Gamma_i, \mathbb{F}_p).$$

On the other hand the map $\Gamma_i \to \Gamma_i^{ab}$ is surjective, which easily implies that $d(\Gamma_i)$, the minimal number of generators of Γ_i is at least as big as $\beta_1(\Gamma_i, \mathbb{F}_p)$, i.e. we have

$$\beta_1(\Gamma_i) \le \beta_1(\Gamma_i, \mathbb{F}_p) \le d(\Gamma_i).$$

Consider the limit

$$\lim_{i\to+\infty} \frac{\beta_1(\Gamma_i, \mathbb{F}_p)}{[\Gamma : \Gamma_i]}.$$

9.1. **Problem I.** Does this limit exist? Is it independent of the chosen normal chain (Γ_i)?

If answers to both questions are positive then call this limit *the first l^2-Betti number of Γ over $\mathbb{F}_p$* and denote it by $\beta_1^{(2)}(\Gamma, \mathbb{F}_p)$. From the inequalities above we get that

$$\beta_1^{(2)}(\Gamma) \le \beta_1^{(2)}(\Gamma, \mathbb{F}_p) \le RG(\Gamma).$$

9.2. **Problem II.** Can any of the inequalities above be strict?

9.3. **Known results.** For amenable groups the answer to Problem I is positive.

10. Lueck's approximation theorem for arbitrary coefficients

Let Γ be residually finite and let Γ_i be a normal chain with trivial intersection and such that the quotients Γ/Γ_i are finite. Let $\mathbb{Q}\Gamma$ be the rational group ring of Γ. Note the natural maps $\pi_i : \mathbb{Q}\Gamma \to \mathbb{Q}(\Gamma/\Gamma_i)$. For a given $\theta \in \mathbb{Q}\Gamma$ denote by $\dim_{vN} \ker\theta$ the measure of the atom at 0 of the spectral measure of θ.

Note that $\mathbb{Q}(\Gamma/\Gamma_i)$ acts on the finite dimensional Hilbert space $l^2(\Gamma/\Gamma_i)$ and as such we can consider the standard dimension $\dim\ker\pi_i(\theta)$ of kernel of $\pi_i(\theta)$.

Lueck's approximation theorem says that

$$\dim_{vN}\ker\theta = \lim_{i\to+\infty} \frac{\dim\ker\ \pi_i(\theta)}{[\Gamma:\Gamma_i]}$$

10.1. **Problem.** Does Lueck's approximation theorem hold for $\theta \in \mathbb{C}\Gamma$?

10.2. **Known results.** Lueck's approximation theorem holds for $\theta \in \bar{\mathbb{Q}}\Gamma$, where $\bar{\mathbb{Q}}$ is the algebraic completion of $\mathbb{Q}$.

11. Measurable temperature changes

This problem has some application to a theorem by Chifan and Ioana.

Let Γ be a finitely generated group and let $\rho\colon \Gamma \curvearrowright X$ be the Bernoulli shift action on the product measure space $X := \{0,1\}^\Gamma$. Let $f\colon X \to \mathbb{C}$ be a measurable function such that for each $c \in \mathbb{C}$ we have $\mu(f^{-1}(c)) = 0$. Fix generators of Γ so that we can consider the Cayley graph of Γ with respect to those generators.

Consider the following function $g : X \to \{0,1\}$. First take a subset X' of X on which the action of Γ is free and such that $\mu(X') = 1$.

For $x \in X - X'$ define $g(x)$ to be 0. For $x \in X'$ do the following. Consider the orbit of x in X. Identify x with the neutral element in the Cayley graph of Γ. Since the action is free this gives identification of the Cayley graph of Γ with the orbit of x. Thus consider the set $N(x)$ of those elements in the orbit of x which can be joined with x by a path in the Cayley graph of Γ in such a way, so that for each point p in this path we have $f(p) = f(x)$.

If $N(x)$ is infinite then define $g(x)$ to be 1. Otherwise define it to be 0.

11.1. **The problem.** Is it always the case that g is equal to 0 on a set of measure 1?

11.2. We could define function $g'(x)$ to be 0 iff there are only finitely many points y in the orbit of x such that $f(y) = f(x)$ and ask whether g' is equal to 0 on a set of measure 1. This is not true: now the question is invariant under taking orbit equivalent actions so we can as well take the action of $\mathbb{Z}^2$ on $S^1 \times S^1$ such that the first (resp. second) generator acts on the first (resp. second) copy of S^1 by some irrational rotation. This action is free and ergodic and so it is orbit equivalent to $\mathbb{Z}^2 \curvearrowright \{0,1\}^{\mathbb{Z}^2}$. Consider $f : S^1 \times S^1 \to \mathbb{C}$ given by projection onto the second S^1. In this case $g'(x)$ is equal to the constant function 1.

12. Non-amenable subrelations

This was a question motivated by the fact that the free group can be rand-embedded into any non-amenable group. The question asked whether for any non-amenable ergodic relation there exists a (treeable?) non-amenable ergodic subrelation of some sort.