

Atiyah-Singer Revisited

Göttingen Mathematical Institute

Paul Baum
Penn State

7 February, 2017

ATIYAH-SINGER REVISITED

This is an expository talk about the Atiyah-Singer index theorem.

- 1 Two low-dimensional examples.
- 2 Proof (based on Bott periodicity) of the Dirac case.
- 3 Outline of proof that the Dirac case implies the full theorem.

Atiyah-Singer Index theorem

M compact C^∞ manifold without boundary

D an elliptic differential (or elliptic pseudo-differential) operator on M

$E^0, E^1, \dots, C^\infty$ \mathbb{C} vector bundles on M

$C^\infty(M, E^j)$ denotes the \mathbb{C} vector space of all C^∞ sections of E^j .

$D: C^\infty(M, E^0) \longrightarrow C^\infty(M, E^1)$

D is a linear transformation of \mathbb{C} vector spaces.

Atiyah-Singer Index theorem

M compact C^∞ manifold without boundary

D an elliptic differential (or elliptic pseudo-differential) operator on M

$$\text{Index}(D) := \dim_{\mathbb{C}} (\text{Kernel } D) - \dim_{\mathbb{C}} (\text{Cokernel } D)$$

Theorem (M. Atiyah and I. Singer)

Index (D) = (a topological formula)

Example

$$M = S^1 = \{(t_1, t_2) \in \mathbb{R}^2 \mid t_1^2 + t_2^2 = 1\}$$

$D_f: L^2(S^1) \longrightarrow L^2(S^1)$ is

T_f	0
0	I

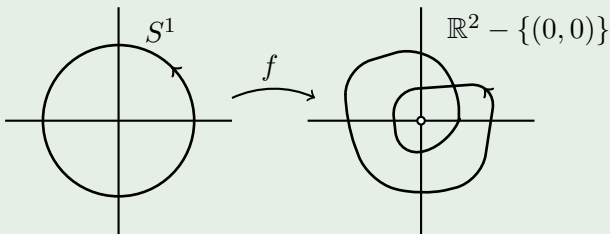
where $L^2(S^1) = L_+^2(S^1) \oplus L_-^2(S^1)$.

$L_+^2(S^1)$ has as orthonormal basis $e^{in\theta}$ with $n = 0, 1, 2, \dots$

$L_-^2(S^1)$ has as orthonormal basis $e^{in\theta}$ with $n = -1, -2, -3, \dots$

Example

$f: S^1 \rightarrow \mathbb{R}^2 - \{0\}$ is a C^∞ map.



$T_f: L_+^2(S^1) \rightarrow L_+^2(S^1)$ is the composition

$$L_+^2(S^1) \xrightarrow{\mathcal{M}_f} L^2(S^1) \rightarrow L_+^2(S^1)$$

$T_f: L_+^2(S^1) \rightarrow L_+^2(S^1)$ is the Toeplitz operator associated to f

Example

Thus T_f is the composition

$$T_f: L_+^2(S^1) \xrightarrow{\mathcal{M}_f} L^2(S^1) \xrightarrow{P} L_+^2(S^1)$$

where $L_+^2(S^1) \xrightarrow{\mathcal{M}_f} L^2(S^1)$ is $v \mapsto fv$

$$fv(t_1, t_2) := f(t_1, t_2)v(t_1, t_2) \quad \forall (t_1, t_2) \in S^1 \quad \mathbb{R}^2 = \mathbb{C}$$

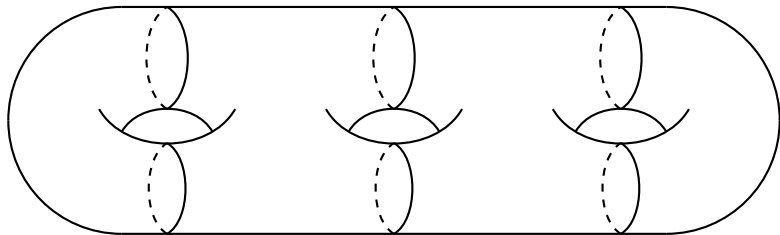
and $L^2(S^1) \xrightarrow{P} L_+^2(S^1)$ is the Hilbert space projection.

$$D_f(v + w) := T_f(v) + w \quad v \in L_+^2(S^1), \quad w \in L_-^2(S^1)$$

$\text{Index}(D_f) = \text{-winding number}(f)$.

RIEMANN - ROCH

M compact connected Riemann surface



$$\begin{aligned}\text{genus of } M &= \# \text{ of holes} \\ &= \frac{1}{2} [\text{rank } H_1(M; \mathbb{Z})]\end{aligned}$$

D a divisor of M

D consists of a finite set of points of M p_1, p_2, \dots, p_l and an integer assigned to each point n_1, n_2, \dots, n_l

Equivalently

D is a function $D: M \rightarrow \mathbb{Z}$ with finite support

$$\text{Support}(D) = \{p \in M \mid D(p) \neq 0\}$$

$\text{Support}(D)$ is a finite subset of M

D a divisor on M

$$\deg(D) := \sum_{p \in M} D(p)$$

Remark

D_1, D_2 two divisors

$$D_1 \geq D_2 \text{ iff } \forall p \in M, D_1(p) \geq D_2(p)$$

Remark

D a divisor, $-D$ is

$$(-D)(p) = -D(p)$$

Example

Let $f: M \rightarrow \mathbb{C} \cup \{\infty\}$ be a meromorphic function.

Define a divisor $\delta(f)$ by:

$$\delta(f)(p) = \begin{cases} 0 & \text{if } p \text{ is neither a zero nor a pole of } f \\ \text{order of the zero} & \text{if } f(p) = 0 \\ -(\text{order of the pole}) & \text{if } p \text{ is a pole of } f \end{cases}$$

Example

Let w be a meromorphic 1-form on M . Locally w is $f(z)dz$ where f is a (locally defined) meromorphic function. Define a divisor $\delta(w)$ by:

$$\delta(w)(p) = \begin{cases} 0 & \text{if } p \text{ is neither a zero nor a pole of } w \\ \text{order of the zero} & \text{if } w(p) = 0 \\ -(\text{order of the pole}) & \text{if } p \text{ is a pole of } w \end{cases}$$

D a divisor on M

$$H^0(M, D) := \left\{ \begin{array}{l} \text{meromorphic functions} \\ f: M \rightarrow \mathbb{C} \cup \{\infty\} \end{array} \middle| \delta(f) \geq -D \right\}$$
$$H^1(M, D) := \left\{ \begin{array}{l} \text{meromorphic 1-forms} \\ w \text{ on } M \end{array} \middle| \delta(w) \geq D \right\}$$

Lemma

$H^0(M, D)$ and $H^1(M, D)$ are finite dimensional \mathbb{C} vector spaces

$$\dim_{\mathbb{C}} H^0(M, D) < \infty$$

$$\dim_{\mathbb{C}} H^1(M, D) < \infty$$

Theorem (R. R.)

Let M be a compact connected Riemann surface and let D be a divisor on M . Then:

$$\dim_{\mathbb{C}} H^0(M, D) - \dim_{\mathbb{C}} H^1(M, D) = d - g + 1$$

$$d = \text{degree } (D)$$

$$g = \text{genus } (M)$$

EXAMPLE. Let M be a compact even-dimensional Spin^c manifold without boundary.

$$D_{\mathcal{S}^*}^+ : C^\infty(M, \mathcal{S}^+ \otimes \mathcal{S}^*) \longrightarrow C^\infty(M, \mathcal{S}^- \otimes \mathcal{S}^*)$$

is the Hirzebruch signature operator of M .

If the dimension of M is divisible by 4, the index of this operator is the signature of the quadratic form

$$H^r(M; \mathbb{R}) \otimes_{\mathbb{R}} H^r(M, \mathbb{R}) \longrightarrow \mathbb{R} \quad n = 2r \quad r \text{ even}$$

$$a \otimes b \mapsto (a \cup b)[M]$$

BOTT PERIODICITY

$$\pi_j GL(n, \mathbb{C}) = \begin{cases} \mathbb{Z} & j \text{ odd} \\ 0 & j \text{ even} \end{cases}$$

$$j = 0, 1, 2, \dots, 2n - 1$$

Why ???? does Bott periodicity imply

DIRAC CASE OF ATIYAH-SINGER

Let M be a compact even-dimensional Spin^c manifold without boundary. Let E be a \mathbb{C} vector bundle on M . D_E denotes the Dirac operator of M tensored with E .

THEOREM $\text{Index}(D_E) = (ch(E) \cup Td(M))[M]$.

$$K_0(\cdot)$$

Definition

Define an abelian group denoted $K_0(\cdot)$ by considering pairs (M, E) such that:

- 1 M is a compact even-dimensional Spin^c manifold without boundary.
- 2 E is a \mathbb{C} vector bundle on M .

Set $K_0(\cdot) = \{(M, E)\} / \sim$ where the equivalence relation \sim is generated by the three elementary steps

- Bordism
- Direct sum - disjoint union
- Vector bundle modification

Addition in $K_0(\cdot)$ is disjoint union.

$$(M, E) + (M', E') = (M \sqcup M', E \sqcup E')$$

In $K_0(\cdot)$ the additive inverse of (M, E) is $(-M, E)$ where $-M$ denotes M with the Spin^c structure reversed.

$$-(M, E) = (-M, E)$$

Isomorphism (M, E) is isomorphic to (M', E') iff \exists a diffeomorphism

$$\psi: M \rightarrow M'$$

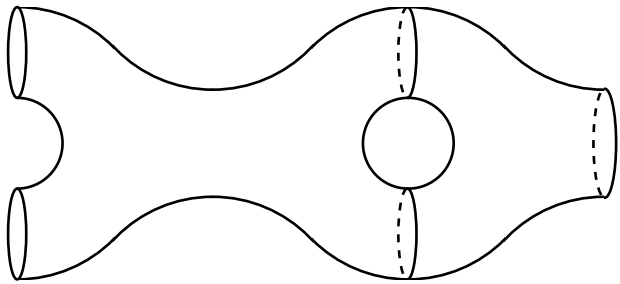
preserving the Spin^c -structures on M, M' and with

$$\psi^*(E') \cong E.$$

Bordism (M_0, E_0) is **bordant** to (M_1, E_1) iff $\exists (\Omega, E)$ such that:

- 1 Ω is a compact odd-dimensional Spin^c manifold with boundary.
- 2 E is a \mathbb{C} vector bundle on Ω .
- 3 $(\partial\Omega, E|_{\partial\Omega}) \cong (M_0, E_0) \sqcup (-M_1, E_1)$

$-M_1$ is M_1 with the Spin^c structure reversed.



(M_0, E_0)

$(-M_1, E_1)$

Direct sum - disjoint union

Let E, E' be two \mathbb{C} vector bundles on M

$$(M, E) \sqcup (M, E') \sim (M, E \oplus E')$$

Vector bundle modification

(M, E)

Let F be a Spin^c vector bundle on M

Assume that

$$\dim_{\mathbb{R}}(F_p) \equiv 0 \pmod{2} \quad p \in M$$

for every fiber F_p of F

$$\mathbf{1}_{\mathbb{R}} = M \times \mathbb{R}$$

$S(F \oplus \mathbf{1}_{\mathbb{R}}) :=$ unit sphere bundle of $F \oplus \mathbf{1}_{\mathbb{R}}$

$$(M, E) \sim (S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^* E)$$

$$\begin{array}{c} S(F \oplus \mathbf{1}_{\mathbb{R}}) \\ \downarrow \pi \\ M \end{array}$$

This is a fibration with even-dimensional spheres as fibers.

$F \oplus \mathbf{1}_{\mathbb{R}}$ is a Spin^c vector bundle on M with odd-dimensional fibers.

The Spin^c structure for F causes there to appear on $S(F \oplus \mathbf{1}_{\mathbb{R}})$ a \mathbb{C} -vector bundle β whose restriction to each fiber of π is the Bott generator vector bundle of that oriented even-dimensional sphere.

$$(M, E) \sim (S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^* E)$$

Addition in $K_0(\cdot)$ is disjoint union.

$$(M, E) + (M', E') = (M \sqcup M', E \sqcup E')$$

In $K_0(\cdot)$ the additive inverse of (M, E) is $(-M, E)$ where $-M$ denotes M with the Spin^c structure reversed.

$$-(M, E) = (-M, E)$$

DEFINITION. (M, E) bounds $\iff \exists (\Omega, \tilde{E})$ with :

- 1 Ω is a compact odd-dimensional Spin^c manifold with boundary.
- 2 \tilde{E} is a \mathbb{C} vector bundle on Ω .
- 3 $(\partial\Omega, \tilde{E}|_{\partial\Omega}) \cong (M, E)$

REMARK. $(M, E) = 0$ in $K_0(\cdot)$ $\iff (M, E) \sim (M', E')$ where (M', E') bounds.

Consider the homomorphism of abelian groups

$$\begin{aligned} K_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \text{Index}(D_E) \end{aligned}$$

Notation

D_E is the Dirac operator of M tensored with E .

$$\begin{aligned} K_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \text{Index}(D_E) \end{aligned}$$

It is a corollary of Bott periodicity that this homomorphism of abelian groups is an isomorphism.

Equivalently, $\text{Index}(D_E)$ is a complete invariant for the equivalence relation generated by the three elementary steps; i.e.

$(M, E) \sim (M', E')$ if and only if $\text{Index}(D_E) = \text{Index}(D'_{E'})$.

Have three problems for :

$$\begin{aligned} K_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \text{Index}(D_E) \end{aligned}$$

- (i) well-defined
- (ii) surjective
- (iii) injective

In order to prove that the homomorphism of abelian groups

$$\begin{aligned} K_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \text{Index}(D_E) \end{aligned}$$

is well-defined, the three elementary moves (bordism, direct sum - disjoint union, vector bundle modification) must be proved to be index-preserving.

Proof that $K_0(\cdot) \rightarrow \mathbb{Z}$ is well-defined.

For bordism-invariance of the index have two proofs :

- 1 Proof in R. Palais book *Seminar on the Atiyah-Singer Index Theorem* (1965).
- 2 Proof (following N. Higson) in Baum - van Erp expository paper.

Proof that $K_0(\cdot) \rightarrow \mathbb{Z}$ is well-defined.

For vector bundle modification :

Start with (M, E) . Given a Spin^c vector bundle F on M with fiber dimension $\mathbb{R}(F)$ even, — form $(S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^* E)$

$$\begin{array}{c} S(F \oplus \mathbf{1}_{\mathbb{R}}) \\ \downarrow \pi \\ M \end{array}$$

π is a fibration with oriented even-dimensional spheres as fibers. Restriction of β to each fiber of π is the Bott generator vector bundle of that oriented even-dimensional sphere.

$$\begin{array}{c} S(F \oplus \mathbf{1}_{\mathbb{R}}) \\ \downarrow \pi \\ M \end{array}$$

For each fiber of π can form the elliptic operator
(Dirac of the fiber) \otimes (β restricted to the fiber).

Thus for each point $p \in M$ have an elliptic operator.

Hence have a family of elliptic operators over M .

Key point is that the index of this family is the trivial line bundle on M — i.e. is $M \times \mathbb{C}$.

This proof that vector bundle modification is index-preserving is in essence the same as :

- Proof by Atiyah and Singer of compatibility of index and Thom isomorphism.

BOTT PERIODICITY

$$\pi_j GL(n, \mathbb{C}) = \begin{cases} \mathbb{Z} & j \text{ odd} \\ 0 & j \text{ even} \end{cases}$$

$$j = 0, 1, 2, \dots, 2n - 1$$

Why does Bott periodicity imply that

$$\begin{aligned} K_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \text{Index}(D_E) \end{aligned}$$

is an isomorphism?

To prove surjectivity must find an (M, E) with $\text{Index}(D_E) = 1$.

e.g. Let $M = \mathbb{C}P^n$, and let E

be the trivial (complex) line bundle on $\mathbb{C}P^n$

$$E = 1_{\mathbb{C}} = \mathbb{C}P^n \times \mathbb{C}$$

$$\text{Index}(\mathbb{C}P^n, 1_{\mathbb{C}}) = 1$$

Thus Bott periodicity is not used in the proof of surjectivity.

Lemma used in the Proof of Injectivity

Given any (M, E) there exists an even-dimensional sphere S^{2n} and a \mathbb{C} -vector bundle F on S^{2n} with $(M, E) \sim (S^{2n}, F)$.

Bott periodicity is not used in the proof of this lemma.

The lemma is proved by a direct argument using the definition of the equivalence relation on the pairs (M, E) .

Let r be a positive integer, and let $\text{Vect}_{\mathbb{C}}(S^{2n}, r)$ be the set of isomorphism classes of \mathbb{C} vector bundles on S^{2n} of rank r , i.e. of fiber dimension r .

$$\text{Vect}_{\mathbb{C}}(S^{2n}, r) \longleftrightarrow \pi_{2n-1}GL(r, \mathbb{C})$$

PROOF OF INJECTIVITY

Let (M, E) have $\text{Index}(M, E) = 0$.

By the above lemma, we may assume that $(M, E) = (S^{2n}, F)$.

Using Bott periodicity plus the bijection

$$\text{Vect}_{\mathbb{C}}(S^{2n}, r) \longleftrightarrow \pi_{2n-1}GL(r, \mathbb{C})$$

we may assume that F is of the form

$$F = \theta^p \oplus q\beta$$

$\theta^p = S^{2n} \times \mathbb{C}^p$ and β is the Bott generator vector bundle on S^{2n} .

Convention. If $q < 0$, then $q\beta = |q|\beta^*$.

$$\text{Index}(S^{2n}, \beta) = 1 \quad \text{Index}(S^{2n}, \theta^p) = 0$$

Therefore

$$\text{Index}(S^{2n}, F) = 0 \implies q = 0$$

Hence $(S^{2n}, F) = (S^{2n}, \theta^p)$. This bounds

$$(S^{2n}, \theta^p) = \partial(B^{2n+1}, B^{2n+1} \times \mathbb{C}^p)$$

and so is zero in $K_0(\cdot)$.

QED

Define a homomorphism of abelian groups

$$\begin{aligned} K_0(\cdot) &\longrightarrow \mathbb{Q} \\ (M, E) &\longmapsto (ch(E) \cup Td(M))[M] \end{aligned}$$

where $ch(E)$ is the Chern character of E and $Td(M)$ is the Todd class of M .

$ch(E) \in H^*(M, \mathbb{Q})$ and $Td(M) \in H^*(M, \mathbb{Q})$.

$[M]$ is the orientation cycle of M . $[M] \in H_*(M, \mathbb{Z})$.

One problem for

$$\begin{aligned} K_0(\cdot) &\longrightarrow \mathbb{Q} \\ (M, E) &\longmapsto (ch(E) \cup Td(M))[M] \end{aligned}$$

(i) well-defined

Granted that

$$\begin{aligned} K_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \text{Index}(D_E) \end{aligned}$$

is an isomorphism, to prove that these two homomorphisms are equal, it suffices to check one example (M, E) with $\text{Index}(D_E) = 1$.

Reference. P. F. Baum and E. van Erp, *K-homology and Fredholm Operators I : Dirac Operators*, on arXiv.

Symbol of a differential operator

Let Y be a C^∞ manifold (possibly with boundary).

Y is not required to be oriented.

Y is not required to be even dimensional.

On Y let

$$\delta : C^\infty(Y, E^0) \longrightarrow C^\infty(Y, E^1)$$

be a differential operator of order k .

Denote by $\pi : T^*Y \rightarrow Y$ the projection $T^*Y \rightarrow Y$.

The **symbol** (or principal symbol) of δ is for each $\xi \in T^*Y$ a \mathbb{C} -linear map

$$\sigma(\xi) : E_{\pi(\xi)}^0 \longrightarrow E_{\pi(\xi)}^1$$

defined as follows :

Symbol of a differential operator

$$\delta : C^\infty(Y, E^0) \longrightarrow C^\infty(Y, E^1) \quad k = \text{order}(\delta)$$

Given $\xi \in T^*Y$ and $u \in E_{\pi(\xi)}^0$, set $p = \pi(\xi)$, and choose :

(i) $s \in C^\infty(Y, E^0)$ with $s(p) = u$.

(ii) a C^∞ function $f: Y \rightarrow \mathbb{R}$ with $f(p) = 0$ and $df(p) = \xi$.

Then:

$$\sigma(\xi)(u) := \left(\frac{1}{k!}\right)\delta(f^k s)(p)$$

$\sigma(\xi): E_p^0 \rightarrow E_p^1$ does not depend on the choices (i) (ii).

Symbol of a differential operator

$$\delta : C^\infty(Y, E^0) \longrightarrow C^\infty(Y, E^1)$$

The differential operator δ is **elliptic** if for every non-zero $\xi \in T^*Y$

$$\sigma(\xi) : E_{\pi(\xi)}^0 \rightarrow E_{\pi(\xi)}^1$$

is an isomorphism.

The symbol σ of δ can be viewed as a vector bundle map

$$\sigma : \pi^* E^0 \rightarrow \pi^* E^1$$

This basic theory (i.e. symbol, elliptic etc.) extends to pseudo-differential operators.

Let X be a compact C^∞ manifold without boundary.

X is not required to be oriented.

X is not required to be even dimensional.

On X let

$$\delta : C^\infty(X, E^0) \longrightarrow C^\infty(X, E^1)$$

be an elliptic differential (or elliptic pseudo-differential) operator.

$(S(T^*X \oplus 1_{\mathbb{R}}), E_\sigma) \in K_0(\cdot)$, and

$$\text{Index}(D_{E_\sigma}) = \text{Index}(\delta).$$

$$(S(T^*X \oplus 1_{\mathbb{R}}), E_{\sigma})$$



$$\text{Index}(\delta) = (ch(E_{\sigma}) \cup \text{Td}((S(T^*X \oplus 1_{\mathbb{R}})))[(S(T^*X \oplus 1_{\mathbb{R}}))])$$

and this is the general Atiyah-Singer formula.

$S(T^*X \oplus 1_{\mathbb{R}})$ is the unit sphere bundle of $T^*X \oplus 1_{\mathbb{R}}$.

$S(T^*X \oplus 1_{\mathbb{R}})$ is even dimensional and is — in a natural way — a Spin^c manifold.

E_{σ} is the \mathbb{C} vector bundle on $S(T^*X \oplus 1_{\mathbb{R}})$ obtained by doing a clutching construction using the symbol σ of δ .

Construction of E_σ

upper hemisphere

lower hemisphere

$$S(T^*X \oplus 1_{\mathbb{R}}) = B_+(T^*X \oplus 1_{\mathbb{R}}) \cup_{S(T^*X)} B_-(T^*X \oplus 1_{\mathbb{R}})$$

$$E_\sigma := \pi^*(E^0) \cup_\sigma \pi^*(E^1)$$

$$(S(T^*X \oplus 1_{\mathbb{R}}), E_{\sigma}) \in K_0(\cdot)$$

$$\text{Index}(D_{E_{\sigma}}) = \text{Index}(\delta)$$

Proof. Show that can go from δ to $D_{E_{\sigma}}$ by an explicit finite sequence of index-preserving moves. This uses pseudo-differential operators.

Reference. P. F. Baum and E. van Erp,
K-homology and Fredholm Operators II : Elliptic Operators,
to appear in Pure and Applied Mathematics Quarterly