Dirac Operator

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Five lectures:

1. Dirac operator
2. Atiyah-Singer revisited
3. What is K-homology?
4. The Riemann-Roch theorem
5. K-theory for group C^* algebras (BC conjecture)

DIRAC OPERATOR

The Dirac operator of $\mathbb{R}^n$ will be defined. This is a first order elliptic differential operator with constant coefficients. Next, the class of differentiable manifolds which come equipped with an order one differential operator which at the symbol level is locally isomorphic to the Dirac operator of $\mathbb{R}^n$ will be considered. These are the Spin^c manifolds. Spin^c is slightly stronger than oriented, so Spin^c can be viewed as "oriented plus epsilon". Most of the oriented manifolds that occur in practice are Spin^c. The Dirac operator of a closed Spin^c manifold is the basic example for the Hirzebruch-Riemann-Roch theorem and the Atiyah-Singer index theorem.

What is the Dirac operator of $\mathbb{R}^n$?

To answer this, shall construct matrices $E_1, E_2, \ldots, E_n$ with the following properties :

Properties of $E_1, E_2, \ldots, E_n$

- Each E_j is a $2^r \times 2^r$ matrix of complex numbers, where r is the largest integer $\leq n/2$.
- Each E_j is skew adjoint, i.e. $E_j^* = -E_j$ (* $=$ conjugate transpose)
- $E_j^2 = -I \qquad j = 1, 2, \ldots, n$ (I is the $2^r \times 2^r$ identity matrix.)
- $E_j E_k + E_k E_j = 0$ whenever $j \neq k$.
- For n odd, $(n = 2r + 1)$ $i^{r+1} E_1 E_2 \cdots E_n = I \qquad i = \sqrt{-1}$
- For n even, $(n = 2r)$ each E_j is of the form

$$E_j = \begin{bmatrix} \mathbf{0} & * \\ * & \mathbf{0} \end{bmatrix} \quad \text{and} \quad i^r E_1 E_2 \cdots E_n = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & -I \end{bmatrix}$$

These matrices are constructed by a simple inductive procedure.

$n = 1$, $E_1 = [-i]$

$n \rightsquigarrow n+1$ with n odd $\qquad (r \rightsquigarrow r+1)$

The new matrices $\widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{n+1}$ are

$$\widetilde{E}_j = \begin{bmatrix} \mathbf{0} & E_j \\ E_j & \mathbf{0} \end{bmatrix} \quad \text{for } j = 1, \ldots, n \quad \text{and} \quad \widetilde{E}_{n+1} = \begin{bmatrix} \mathbf{0} & -I \\ I & \mathbf{0} \end{bmatrix}$$

where $E_1, E_2, \ldots, E_n$ are the old matrices.

$n \rightsquigarrow n+1$ with n even $\qquad$ (r does not change)

The new matrices $\widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{n+1}$ are

$$\widetilde{E}_j = E_j \quad \text{for } j = 1, \ldots, n \quad \text{and} \quad \widetilde{E}_{n+1} = \begin{bmatrix} -iI & \mathbf{0} \\ \mathbf{0} & iI \end{bmatrix}$$

where $E_1, E_2, \ldots, E_n$ are the old matrices.

Example

$n = 1$: $E_1 = [-i]$

$n = 2$: $E_1 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$n = 3$: $E_1 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$

Example

$n = 4$: $E_1 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$ $\quad E_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$E_3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}$ $\quad E_4 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$D =$ Dirac operator of $\mathbb{R}^n$

$$\begin{cases} n = 2r & n \text{ even} \\ n = 2r+1 & n \text{ odd} \end{cases}$$

$$D = \sum_{j=1}^{n} E_j \frac{\partial}{\partial x_j}$$

D is an unbounded symmetric operator on the Hilbert space
$L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n)$ (2^r times)

To begin, the domain of D is
$C_c^\infty(\mathbb{R}^n) \oplus C_c^\infty(\mathbb{R}^n) \oplus \ldots \oplus C_c^\infty(\mathbb{R}^n)$ (2^r times)

D is essentially self-adjoint
(i.e. D has a unique self-adjoint extension)
so it is natural to view D as an unbounded self-adjoint operator on the Hilbert space
$L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n)$ (2^r times)

QUESTION : Let M be a C^∞ manifold of dimension n.
Does M admit a differential operator which (at the symbol level) is locally isomorphic to the Dirac operator of $\mathbb{R}^n$?

To answer this question, will define Spin^c vector bundle.

What is a Spin^c vector bundle?

Let X be a paracompact Hausdorff topological space.
On X let E be an $\mathbb{R}$ vector bundle which has been oriented.
i.e. the structure group of E has been reduced from $GL(n, \mathbb{R})$ to $GL^+(n, \mathbb{R})$

$$GL^+(n, \mathbb{R}) = \{[a_{ij}] \in GL(n, \mathbb{R}) \mid \det[a_{ij}] > 0\}$$

$n=$ fiber dimension (E)

Assume $n \geq 3$ and recall that for $n \geq 3$

$$H^2(GL^+(n, \mathbb{R}); \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

Denote by $\mathcal{F}^+(E)$ the principal $GL^+(n, \mathbb{R})$ bundle on X consisting of all positively oriented frames.

A point of $\mathcal{F}^+(E)$ is a pair $\big(x, (v_1, v_2, \ldots, v_n)\big)$ where $x \in X$ and $(v_1, v_2, \ldots, v_n)$ is a positively oriented basis of E_x. The projection $\mathcal{F}^+(E) \to X$ is

$$\big(x, (v_1, v_2, \ldots, v_n)\big) \mapsto x$$

For $x \in X$, denote by

$$\iota_x \colon \mathcal{F}^+_x(E) \hookrightarrow \mathcal{F}^+(E)$$

the inclusion of the fiber at x into $\mathcal{F}^+(E)$.

Note that (with $n \geq 3$)

$$H^2(\mathcal{F}^+_x(E); \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

A Spin^c vector bundle on X is an $\mathbb{R}$ vector bundle E on X (fiber dimension $E \geq 3$) with

1. E has been oriented.
2. $\alpha \in H^2(\mathcal{F}^+(E); \mathbb{Z})$ has been chosen such that $\forall x \in X$

$$\iota_x^*(\alpha) \in H^2(\mathcal{F}_x^+(E); \mathbb{Z}) \text{ is non-zero.}$$

Remarks

1.For $n = 1, 2$ "E is a Spin^c vector bundle" $=$"E has been oriented and an element $\alpha \in H^2(X; \mathbb{Z})$" has been chosen. ($\alpha$ can be zero.)

2. For all values of $n =$ fiber dimension(E), E is a Spin^c vector bundle iff the structure group of E has been changed from $GL(n, \mathbb{R})$ to Spin$^c(n)$.
i.e. Such a change of structure group is equivalent to the above definition of Spin^c vector bundle.

Topological obstruction to Spin^c-able

E an $\mathbb{R}$ vector bundle on X.
$w_1(E), w_2(E), \ldots, w_n(E)$ Stiefel-Whitney classes of E
$w_j(E) \in H^j(X; \mathbb{Z}/2\mathbb{Z})$
E is Spin^c-able iff:
(i) $w_1(E) = 0$ (i.e. E is orientable).
and
(ii) $w_2(E)$ is in the image of the mod 2 reduction map

$$H^2(X; \mathbb{Z}) \longrightarrow H^2(X; \mathbb{Z}/2\mathbb{Z})$$

By forgetting some structure a complex vector bundle or a Spin vector bundle canonically becomes a Spin^c vector bundle

$$
\begin{array}{ccc}
 & & \text{complex} \\
 & & \Downarrow \\
\text{Spin} & \Rightarrow & \text{Spin}^c \\
 & & \Downarrow \\
 & & \text{oriented}
\end{array}
$$

A Spin^c structure for an $\mathbb{R}$ vector bundle E can be thought of as an orientation for E plus a slight extra bit of structure. Spin^c structures behave very much like orientations. For example, an orientation on two out of three $\mathbb{R}$ vector bundles in a short exact sequence determines an orientation on the third vector bundle. An analogous assertion is true for Spin^c structures.

Two Out Of Three Lemma

Lemma

Let

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be a short exact sequence of $\mathbb{R}$*-vector bundles on* X*. If two out of three are Spin*^{c} *vector bundles, then so is the third.*

Definition

Let M be a C^{∞} manifold (with or without boundary). M is a Spin^c manifold iff the tangent bundle TM of M is a Spin^c vector bundle on M.

The Two Out Of Three Lemma implies that the boundary ∂M of a Spin^c manifold M with boundary is again a Spin^c manifold.

Various well-known structures on a manifold M make M into a Spin^c manifold.

$$
\begin{array}{ccc}
 & & \text{(complex-analytic)} \\
 & & \Downarrow \\
\text{(symplectic)} & \Rightarrow & \text{(almost complex)} \\
 & & \Downarrow \\
\text{(contact)} & \Rightarrow & \text{(stably almost complex)} \\
 & & \Downarrow \\
\text{Spin} & \Rightarrow & \text{Spin}^c \\
 & & \Downarrow \\
 & & \text{(oriented)}
\end{array}
$$

A Spin^c manifold can be thought of as an oriented manifold with a slight extra bit of structure. Most of the oriented manifolds which occur in practice are Spin^c manifolds.

A Spin^c manifold comes equipped with a first-order elliptic differential operator known as its Dirac operator. This operator is locally isomorphic (at the symbol level) to the Dirac operator of $\mathbb{R}^n$.

EXAMPLE. Let M be a compact complex-analytic manifold.
Set $\Omega^{p,q} = C^\infty(M, \Lambda^{p,q} T^*_{\mathbb{C}} M)$
$\Omega^{p,q}$ is the $\mathbb{C}$ vector space of all C^∞ differential forms of type (p, q)
Dolbeault complex

$$0 \longrightarrow \Omega^{0,0} \longrightarrow \Omega^{0,1} \longrightarrow \Omega^{0,2} \longrightarrow \cdots \longrightarrow \Omega^{0,n} \longrightarrow 0$$

The Dirac operator (of the underlying Spin^c manifold) is the assembled Dolbeault complex

$$\bar\partial + \bar\partial^* \colon \bigoplus_j \Omega^{0,\,2j} \longrightarrow \bigoplus_j \Omega^{0,\,2j+1}$$

The index of this operator is the arithmetic genus of M — i.e. is the Euler number of the Dolbeault complex.

TWO POINTS OF VIEW ON SPIN^c MANIFOLDS

1. Spin^c is a slight strengthening of oriented. Most of the oriented manifolds that occur in practice are Spin^c.

2. Spin^c is much weaker than complex-analytic. BUT the assempled Dolbeault complex survives (as the Dirac operator). AND the Todd class survives.

$$M \quad \mathrm{Spin}^c \Longrightarrow \quad \exists \quad Td(M) \in H^*(M;\mathbb{Q})$$

If M is a Spin^c manifold, then $Td(M)$ is

$$Td(M) := \exp^{c_1(M)/2} \widehat{A}(M) \qquad\qquad Td(M) \in H^*(M;\mathbb{Q})$$

If M is a complex-analyic manifold, then M has Chern classes $c_1, c_2, \ldots, c_n$ and

$$\exp^{c_1(M)/2} \widehat{A}(M) = P_{Todd}(c_1, c_2, \ldots, c_n)$$

WARNING!!!

The Todd class of a Spin^c manifold is not obtained by complexifying the tangent bundle TM of M and then applying the Todd polynomial to the Chern classes of $T_{\mathbb{C}}M$.

$$Td(T_{\mathbb{C}}M) = \widehat{A}(M)^2 = \widehat{A}(M) \cup \widehat{A}(M)$$

Correct formula for the Todd class of a Spin^c manifold M is:

$$Td(M) := \exp^{c_1(M)/2} \widehat{A}(M) \qquad\qquad Td(M) \in H^*(M; \mathbb{Q})$$

SPECIAL CASE OF ATIYAH-SINGER

Let M be a compact even-dimensional Spin^c manifold without boundary. Let E be a $\mathbb{C}$ vector bundle on M.

D_E denotes the Dirac operator of M tensored with E.

$$D_E \colon C^\infty(M, \mathcal{S}^+ \otimes E) \longrightarrow C^\infty(M, \mathcal{S}^- \otimes E)$$

$\mathcal{S}^+, (\mathcal{S}^-)$ are the positive (negative) spinor bundles on M.

<u>THEOREM</u> Index$(D_E) = (ch(E) \cup Td(M))[M]$.

SPECIAL CASE OF ATIYAH-SINGER

Let M be a compact even-dimensional Spin^c manifold without boundary. Let E be a $\mathbb{C}$ vector bundle on M. D_E denotes the Dirac operator of M tensored with E.

THEOREM Index$(D_E) = (ch(E) \cup Td(M))[M]$.

This theorem will be proved in the next lecture as a corollary of Bott periodicity.

In particular, this will prove the Hirzebruch-Riemann-Roch theorem.

Also, this will prove (for closed even-dimensional Spin^c manifolds) the Hirzebruch signature theorem.

From $E_1, E_2, \ldots, E_n$ obtain :

1) The Dirac operator of $\mathbb{R}^n$
2) The Bott generator vector bundle on S^n (n even)
3) The spin representation of $\text{Spin}^c(n)$

W finite dimensional $\mathbb{C}$ vector space $\qquad \dim_{\mathbb{C}}(W) < \infty$

$T : W \to W \qquad T \in \mathrm{Hom}_{\mathbb{C}}(W, W) \qquad T^2 = I$

$\Longrightarrow$ The eigen-values of T are $\quad \pm 1$

$W = W_1 \oplus W_{-1}$

$W_1 = \{v \in W \mid Tv = v\} \qquad W_{-1} = \{v \in W \mid Tv = -v\}$

Bott generator vector bundle

n even $\qquad n = 2r \qquad S^n \subset \mathbb{R}^{n+1} \qquad S^n \to M(2^r, \mathbb{C})$

$$S^n = \{(a_1, a_2, \ldots, a_{n+1}) \in \mathbb{R}^{n+1} \mid a_1^2 + a_2^2 + \cdots + a_{n+1}^2 = 1\}$$

$$(a_1, a_2, \ldots, a_{n+1}) \mapsto i(a_1E_1 + a_2E_2 + \cdots + a_{n+1}E_{n+1})$$

$$i = \sqrt{-1}$$

$$\begin{aligned} & (i)^2(a_1E_1 + a_2E_2 + \cdots + a_{n+1}E_{n+1})^2 \\ = \; & (-1)(-a_1^2 - a_2^2 - \ldots - a_{n+1}^2)\, I \\ = \; & I \end{aligned}$$

$\Longrightarrow$ The eigenvalues of $i(a_1E_1 + a_2E_2 + \cdots + a_{n+1}E_{n+1})$ are ± 1.

Bott generator vector bundle β on S^n $\qquad n$ even $\qquad n = 2r$

$$\begin{aligned} & \beta_{(a_1,a_2,\dots,a_{n+1})} \\ = \; & (+1 \text{ eigenspace of } i(a_1E_1 + a_2E_2 + \cdots + a_{n+1}E_{n+1}))^* \\ = \; & \mathrm{Hom}_{\mathbb{C}}(\{v \in \mathbb{C}^{2^r} \mid i(a_1E_1 + a_2E_2 + \cdots + a_{n+1}E_{n+1})\, v = v\}\,,\, \mathbb{C}) \end{aligned}$$

$$\begin{array}{ccccc} K^0(S^n) & = & \mathbb{Z} & \oplus & \mathbb{Z} \\ & & \mathbf{1} & & \beta \end{array}$$

$\mathbf{1} = S^n \times \mathbb{C}$

Bott generator vector bundle β on S^n n even $n = 2r$
β is determined by:

1. $\forall p \in S^n \, , \, \dim_{\mathbb{C}}(\beta_p) = 2^{r-1}$
2. $\mathrm{ch}(\beta)[S^n] = 1$

n even $\qquad n = 2r \qquad S^n \subset \mathbb{R}^{n+1}$

With the Spin (or Spin^c) structure S^n has as the boundary of the unit ball B^{n+1} of $\mathbb{R}^{n+1}$, the Spinor bundle $\mathcal{S}$ of S^n is:

$$\mathcal{S} = S^n \times \mathbb{C}^{2^r}$$

The positive (negative) Spinor bundles $\mathcal{S}^+(\mathcal{S}^-)$ are defined by :

$$\mathcal{S}^+_{(a_1,a_2,\ldots,a_{n+1})} = +1 \quad \text{eigenspace of } i(a_1E_1 + a_2E_2 + \cdots + a_{n+1}E_{n+1})$$

$$\mathcal{S}^-_{(a_1,a_2,\ldots,a_{n+1})} = -1 \quad \text{eigenspace of } i(a_1E_1 + a_2E_2 + \cdots + a_{n+1}E_{n+1})$$

$$\mathcal{S} = S^n \times \mathbb{C}^{2^r} = \mathcal{S}^+ \oplus \mathcal{S}^-$$

$$\beta = (\mathcal{S}^+)^*$$

M Spin^c manifold

∂M might be non-empty

$TM =$ the tangent bundle of M

Dirac operator

$$D : C_c^\infty(M, \mathcal{S}) \to C_c^\infty(M, \mathcal{S})$$

$\mathcal{S}$ is the Spinor bundle

$C_c^\infty(M, \mathcal{S}) = \{C^\infty$ sections with compact support of $\mathcal{S}\}$

$$D : C_c^\infty(M, \mathcal{S}) \to C_c^\infty(M, \mathcal{S})$$

such that

(1) D is $\mathbb{C}$-linear
$D(s_1 + s_2) = Ds_1 + Ds_2 \qquad s_j \in C_c^\infty(M, \mathcal{S})$
$D(\lambda x) = \lambda Ds \qquad \lambda \in \mathbb{C}$

(2) If $f : M \to \mathbb{C}$ is a C^∞ function, then

$$D(fs) = (df)s + f(Ds)$$

(3) If $s_j \in C_c^\infty(M, \mathcal{S})$ then

$$\int_M (Ds_1x, s_2x) = \int_M (s_1x, Ds_2x)dx$$

(4) If dim M is even, then D is off-diagonal $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$$

$D : C_c^\infty(M, \mathcal{S}) \to C_c^\infty(M, \mathcal{S})$ is an elliptic first-order differential operator.

D can be viewed as an unbounded operator on the Hilbert space $L^2(M, \mathcal{S})$

$$(s_1, s_2) = \int_M (s_1 x, s_2 x) dx$$

$$D : C_c^\infty(M, \mathcal{S}) \to C_c^\infty(M, \mathcal{S})$$

is a symmetric operator

EXAMPLE. Let M be a compact complex-analytic manifold. The positive (negative) Spinor bundles of the underlying Spin^c manifold are :

$$\mathcal{S}^+ = \bigoplus_j \Lambda^{0,\,2j}\, T^*_{\mathbb{C}} M$$

$$\mathcal{S}^- = \bigoplus_j \Lambda^{0,\,2j+1}\, T^*_{\mathbb{C}} M$$

$$D^+ : C^\infty(M, \mathcal{S}^+) \to C^\infty(M, \mathcal{S}^-) \;\; \text{is}$$

$$\bar\partial + \bar\partial^* \colon C^\infty(M,\, \bigoplus_j \Lambda^{0,\,2j}\, T^*_{\mathbb{C}} M) \longrightarrow C^\infty(M,\, \bigoplus_j \Lambda^{0,\,2j+1}\, T^*_{\mathbb{C}} M)$$

The index of this operator is the arithmetic genus of M — i.e. is the Euler number of the Dolbeault complex.

EXAMPLE. Let M be a compact even-dimensional Spin^c manifold without boundary.

$$D^+_{\mathcal{S}^*} \colon C^\infty(M, \mathcal{S}^+ \otimes \mathcal{S}^*) \longrightarrow C^\infty(M, \mathcal{S}^- \otimes \mathcal{S}^*)$$

is the Hirzebruch signature operator of M.
If the dimension of M is divisible by 4, the index of this operator is the signature of the quadratic form

$$H^r(M;\mathbb{R}) \otimes_{\mathbb{R}} H^r(M,\mathbb{R}) \longrightarrow \mathbb{R} \qquad n = 2r \quad r \text{ even}$$

$$a \otimes b \mapsto (a \cup b)[M]$$

Example.

$$n \text{ even} \qquad n = 2r \qquad S^n \subset \mathbb{R}^{n+1}$$

$D =$ Dirac operator of S^n

$\mathcal{S} =$ Spinor bundle of $S^n = S^n \times \mathbb{C}^{2^r}$

$\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$

$D : C^\infty(S^n, \mathcal{S}) \to C^\infty(S^n, \mathcal{S})$

$$D = \begin{bmatrix} 0 & D^- \\ & \\ D^+ & 0 \end{bmatrix}$$

$D^+ : C^\infty(S^n, \mathcal{S}^+) \to C^\infty(S^n, \mathcal{S}^-)$

$\mathrm{Index}\,(D^+) := \dim_{\mathbb{C}}(\mathrm{Kernel}\, D^+) - \dim_{\mathbb{C}}(\mathrm{Cokernel}\, D^+) = 0$

Theorem. $\mathrm{Index}\,(D^+) = 0$

Tensor D^+ with the Bott generator vector bundle β

$$D^+_\beta : C^\infty(S^n, \mathcal{S}^+ \otimes \beta) \to C^\infty(S^n, \mathcal{S}^- \otimes \beta)$$

Theorem. On S^n, with n even,
Index$(D^+) = 0$ and $\mathrm{Index}\,(D^+_\beta) = 1$.

BOTT PERIODICITY

$$\pi_j GL(n,\mathbb{C}) = \begin{cases} \mathbb{Z} & \text{j odd} \\ \\ 0 & \text{j even} \end{cases}$$

$$j = 0, 1, 2, \ldots, 2n-1$$

Why ???? does Bott periodicity imply

SPECIAL CASE OF ATIYAH-SINGER

Let M be a compact even-dimensional Spin^c manifold without boundary. Let E be a $\mathbb{C}$ vector bundle on M. D_E denotes the Dirac operator of M tensored with E.

THEOREM Index$(D_E) = (ch(E) \cup Td(M))[M]$.

This will be explained in the next lecture — tomorrow.