Granted that

\[ K_0(\cdot) \rightarrow \mathbb{Z} \]

\[ (M, E) \mapsto \text{Index}(D_E) \]

is an isomorphism, to prove that these two homomorphisms are equal, it suffices to check one example \((M, E)\) with \(\text{Index}(D_E) = 1\).
Reference. P. F. Baum and E. van Erp, $K$-homology and Fredholm Operators I: Dirac Operators, on arXiv.
Symbol of a differential operator

Let $Y$ be a $C^\infty$ manifold (possibly with boundary). $Y$ is not required to be oriented. $Y$ is not required to be even dimensional. On $Y$ let

$$\delta : C^\infty(Y, E^0) \longrightarrow C^\infty(Y, E^1)$$

be a differential operator of order $k$. Denote by $\pi : T^*Y \to Y$ the projection $T^*Y \to Y$. The symbol (or principal symbol) of $\delta$ is for each $\xi \in T^*Y$ a $\mathbb{C}$-linear map

$$\sigma(\xi) : E^0_\pi(\xi) \longrightarrow E^1_\pi(\xi)$$

defined as follows:
Symbol of a differential operator

\[
\delta : C^\infty(Y, E^0) \longrightarrow C^\infty(Y, E^1) \quad k = \text{order } (\delta)
\]

Given \( \xi \in T^*Y \) and \( u \in E^0_{\pi(\xi)} \), set \( p = \pi(\xi) \), and choose:

(i) \( s \in C^\infty(Y, E^0) \) with \( s(p) = u \).
(ii) a \( C^\infty \) function \( f : Y \to \mathbb{R} \) with \( f(p) = 0 \) and \( df(p) = \xi \).

Then:

\[
\sigma(\xi)(u) := \left( \frac{1}{k!} \right) \delta(f^k s)(p)
\]

\( \sigma(\xi) : E^0_p \to E^1_p \) does not depend on the choices (i) (ii).
Symbol of a differential operator

\[ \delta : C^\infty(Y, E^0) \longrightarrow C^\infty(Y, E^1) \]

The differential operator \( \delta \) is elliptic if for every non-zero \( \xi \in T^*Y \)

\[ \sigma(\xi) : E^0_{\pi(\xi)} \rightarrow E^1_{\pi(\xi)} \]

is an isomorphism.

The symbol \( \sigma \) of \( \delta \) can be viewed as a vector bundle map

\[ \sigma : \pi^* E^0 \rightarrow \pi^* E^1 \]

This basic theory (i.e. symbol, elliptic etc.) extends to pseudo-differential operators.
Let $X$ be a compact $C^\infty$ manifold without boundary. $X$ is not required to be oriented. $X$ is not required to be even dimensional.

On $X$ let

$$
\delta : C^\infty(X, E^0) \rightarrow C^\infty(X, E^1)
$$

be an elliptic differential (or elliptic pseudo-differential) operator.

$$(S(T^*X \oplus 1_\mathbb{R}), E_\sigma) \in K_0(\cdot), \text{ and}$$

$$\text{Index}(D_{E_\sigma}) = \text{Index}(\delta).$$
\[(S(T^*X \oplus 1_\mathbb{R}), E_\sigma)\]

\[\downarrow\]

Index(δ) = \((ch(E_\sigma) \cup Td((S(T^*X \oplus 1_\mathbb{R}))))[(S(T^*X \oplus 1_\mathbb{R})]\]

and this is the general Atiyah-Singer formula.

\(S(T^*X \oplus 1_\mathbb{R})\) is the unit sphere bundle of \(T^*X \oplus 1_\mathbb{R}\).

\(S(T^*X \oplus 1_\mathbb{R})\) is even dimensional and is — in a canonical way — a Spin\(^c\) manifold.

\(E_\sigma\) is the \(\mathbb{C}\) vector bundle on \(S(T^*X \oplus 1_\mathbb{R})\) obtained by doing a clutching construction using the symbol \(\sigma\) of \(\delta\).
Construction of $E_\sigma$

upper hemisphere  lower hemisphere

$$S(T^*X \oplus 1_\mathbb{R}) = B_+(T^*X \oplus 1_\mathbb{R}) \cup S(T^*X) B_-(T^*X \oplus 1_\mathbb{R})$$

$$E_\sigma := \pi^*(E^0) \cup_\sigma \pi^*(E^1)$$
\[(S(T^* X \oplus 1_{\mathbb{R}}), E_\sigma) \in K_0(\cdot)\]

\[\text{Index}(D_{E_\sigma}) = \text{Index}(\delta)\]

**Proof.** Show that can go from $\delta$ to $D_{E_\sigma}$ by an explicit finite sequence of index-preserving moves. This uses pseudo-differential operators.

Five lectures:

1. Dirac operator✓
2. Atiyah-Singer revisited✓
3. What is K-homology?
4. The Riemann-Roch theorem
5. K-theory for group $C^*$ algebras (BC Conjecture)
Let $X$ be a (possibly singular) projective algebraic variety over $\mathbb{C}$.

Grothendieck defined two abelian groups:

$K^0_{\text{alg}}(X) = \text{Grothendieck group of algebraic vector bundles on } X.$

$K^0_{\text{alg}}(X) = \text{the algebraic geometry } K\text{-theory of } X \text{ (contravariant).}$

$K^0_{\text{alg}}(X) = \text{Grothendieck group of coherent algebraic sheaves on } X.$

$K^0_{\text{alg}}(X) = \text{the algebraic geometry } K\text{-homology of } X \text{ (covariant).}$
Problem

How can $K$-homology be taken from algebraic geometry to topology?
$K$-homology is the dual theory to $K$-theory. There are three ways in which $K$-homology in topology has been defined:

**Homotopy Theory** $K$-theory is the cohomology theory and $K$-homology is the homology theory determined by the Bott (i.e. $K$-theory) spectrum. This is the spectrum $\ldots, \mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U, \ldots$

**$K$-Cycles** $K$-homology is the group of $K$-cycles.

**$C^*$-algebras** $K$-homology is the Atiyah-BDF-Kasparov group $KK^*(A, \mathbb{C})$. 
Let $X$ be a finite CW complex.
The three versions of $K$-homology are isomorphic.

$$K_{j}^{\text{homotopy}}(X) \xrightarrow{\sim} K_{j}(X) \to KK_{j}(C(X), \mathbb{C})$$

homotopy theory $K$-cycles Atiyah-BDF-Kasparov

$j = 0, 1$
Cycles for $K$-homology

Let $X$ be a CW complex.

**Definition**

A $K$-cycle on $X$ is a triple $(M, E, \varphi)$ such that:

1. $M$ is a compact Spin$^c$ manifold without boundary.
2. $E$ is a $\mathbb{C}$ vector bundle on $M$.
3. $\varphi: M \to X$ is a continuous map from $M$ to $X$. 
Set $K_*(X) = \{(M, E, \varphi)\}/\sim$ where the equivalence relation $\sim$ is generated by the three elementary steps:

- Bordism
- Direct sum - disjoint union
- Vector bundle modification
**Isomorphism** \((M, E, \varphi)\) is isomorphic to \((M', E', \varphi')\) iff \(\exists\) a diffeomorphism

\[
\psi: M \to M'
\]

preserving the Spin\(^c\)-structures on \(M, M'\) and with

\[
\psi^*(E') \cong E
\]

and commutativity in the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\psi} & M' \\
\downarrow{\varphi} & & \downarrow{\varphi'} \\
X & & \\
\end{array}
\]
Bordism \((M_0, E_0, \varphi_0)\) is **bordant** to \((M_1, E_1, \varphi_1)\) iff \(\exists (\Omega, E, \varphi)\) such that:

1. \(\Omega\) is a compact \(\text{Spin}^c\) manifold with boundary.
2. \(E\) is \(\mathbb{C}\) vector bundle on \(\Omega\).
3. \((\partial \Omega, E|_{\partial \Omega}, \varphi|_{\partial \Omega}) \cong (M_0, E_0, \varphi_0) \sqcup (-M_1, E_1, \varphi_1)\)

\(-M_1\) is \(M_1\) with the \(\text{Spin}^c\) structure reversed.
\[(M_0, E_0, \varphi_0) \quad \downarrow \quad X \quad \downarrow \quad (-M_1, E_1, \varphi_1)\]
Direct sum - disjoint union

Let $E, E'$ be two $\mathbb{C}$ vector bundles on $M$

$$(M, E, \varphi) \sqcup (M, E', \varphi) \sim (M, E \oplus E', \varphi)$$
Vector bundle modification

\((M, E, \varphi)\)

Let \(F\) be a Spin\(^c\) vector bundle on \(M\)

Assume that

\[
\dim_{\mathbb{R}}(F_p) \equiv 0 \mod 2 \quad p \in M
\]

for every fiber \(F_p\) of \(F\)

\[
1_{\mathbb{R}} = M \times \mathbb{R}
\]

\[
S(F \oplus 1_{\mathbb{R}}) := \text{unit sphere bundle of } F \oplus 1_{\mathbb{R}}
\]

\[
(M, E, \varphi) \sim (S(F \oplus 1_{\mathbb{R}}), \beta \otimes \pi^* E, \varphi \circ \pi)
\]
This is a fibration with even-dimensional spheres as fibers.

\[(M, E, \varphi) \sim (S(F \oplus 1_{\mathbb{R}}), \beta \otimes \pi^* E, \varphi \circ \pi)\]

The restriction of \(\beta\) to each fiber of \(\pi\) is the Bott generator vector bundle of that oriented sphere.
\{(M, E, \varphi)\} / \sim = K_0(X) \oplus K_1(X)

\[ K_j(X) = \text{subgroup of } \{(M, E, \varphi)\} / \sim \]

\text{consisting of all } (M, E, \varphi) \text{ such that every connected component of } M \text{ has dimension } \equiv j \mod 2 \quad j = 0, 1
Addition in $K_j(X)$ is disjoint union.

$$(M, E, \varphi) + (M', E', \varphi') = (M \sqcup M', E \sqcup E', \varphi \sqcup \varphi')$$

Additive inverse of $(M, E, \varphi)$ is obtained by reversing the Spin$^c$ structure of $M$.

$- (M, E, \varphi) = (-M, E, \varphi)$
Let $X, Y$ be CW complexes and let $f: X \rightarrow Y$ be a continuous map.

Then $f_*: K_j(X) \rightarrow K_j(Y)$ is

$$f_*(M, E, \varphi) := (M, E, f \circ \varphi)$$
Let $X$ be a finite CW complex.

$C(X) = \{ \alpha : X \to \mathbb{C} \mid \alpha \text{ is continuous} \}$

$L(\mathcal{H}) = \{ \text{bounded operators } T : \mathcal{H} \to \mathcal{H} \}$

Any element in the Atiyah-BDF-Kasparov $K$-homology group $KK^0(C(X), \mathbb{C})$ is given by a 5-tuple $(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T)$ such that:
\begin{itemize}
\item $\mathcal{H}_0$ and $\mathcal{H}_1$ are separable Hilbert spaces.
\item $\psi_0 : C(X) \longrightarrow \mathcal{L}(\mathcal{H}_0)$ and $\psi_1 : C(X) \longrightarrow \mathcal{L}(\mathcal{H}_1)$ are unital $^*$-homomorphisms.
\item $T : \mathcal{H}_0 \longrightarrow \mathcal{H}_1$ is a (bounded) Fredholm operator.
\item For every $\alpha \in C(X)$ the commutator $T \circ \psi_0(\alpha) - \psi_1(\alpha) \circ T \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is compact.
\end{itemize}

$$KK^0(C(X), \mathbb{C}) := \{ (\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) \}/ \sim$$
\[ KK^0(C(X), \mathbb{C}) := \{(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T)\}/ \sim \]

\[(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) + (\mathcal{H}_0', \psi_0', \mathcal{H}_1', \psi_1', T') = (\mathcal{H}_0 \oplus \mathcal{H}_0', \psi_0 \oplus \psi_0', \mathcal{H}_1 \oplus \mathcal{H}_1', \psi_1 \oplus \psi_1', T \oplus T') \]

\[-(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) = (\mathcal{H}_1, \psi_1, \mathcal{H}_0, \psi_0, T^*) \]
Let $X$ be a finite CW complex. Any element in the Atiyah-BDF-Kasparov $K$-homology group $KK^1(C(X), \mathbb{C})$ is given by a 3-tuple $(\mathcal{H}, \psi, T)$ such that:

- $\mathcal{H}$ is a separable Hilbert space.
- $\psi : C(X) \rightarrow \mathcal{L}(\mathcal{H})$ is a unital $\ast$-homomorphism.
- $T : \mathcal{H} \rightarrow \mathcal{H}$ is a (bounded) self-adjoint Fredholm operator.
- For every $\alpha \in C(X)$ the commutator $T \circ \psi(\alpha) - \psi(\alpha) \circ T \in \mathcal{L}(\mathcal{H})$ is compact.
\[ KK^1(C(X), \mathbb{C}) := \{(\mathcal{H}, \psi, T)\}/\sim \]

\[ (\mathcal{H}, \psi, T) + (\mathcal{H}', \psi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', T \oplus T') \]

\[ -(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi, -T) \]
Let $X, Y$ be CW complexes and let $f: X \to Y$ be a continuous map.

Denote by $f^\sharp: C(X) \leftarrow C(Y)$ the $\ast$-homomorphism

$$f^\sharp(\alpha) := \alpha \circ f \quad \alpha \in C(Y)$$

Then $f_*: KK^j(C(X), \mathbb{C}) \to KK^j(C(Y), \mathbb{C})$ is

$$f_*(\mathcal{H}, \psi, T) := (\mathcal{H}, \psi \circ f^\sharp, T) \quad j = 1$$

$$f_*(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) := (\mathcal{H}_0, \psi_0 \circ f^\sharp, \mathcal{H}_1, \psi_1 \circ f^\sharp, T) \quad j = 0$$
Theorem (PB and R. Douglas and M. Taylor, PB and N. Higson and T. Schick)

Let $X$ be a finite CW complex.

Then for $j = 0, 1$ the natural map of abelian groups

$$K_j(X) \rightarrow KK^j(C(X), \mathbb{C})$$

is an isomorphism.
For \( j = 0, 1 \) the natural map of abelian groups

\[
K_j(X) \to KK^j(C(X), \mathbb{C})
\]

is \((M, E, \varphi) \mapsto \varphi^* [D_E] \)

where

1. \( D_E \) is the Dirac operator of \( M \) tensored with \( E \).
2. \([D_E] \in KK^j(C(M), \mathbb{C})\) is the element in the Kasparov \( K \)-homology of \( M \) determined by \( D_E \).
3. \( \varphi_* : KK^j(C(M), \mathbb{C}) \to KK^j(C(X), \mathbb{C}) \) is the homomorphism of abelian groups determined by \( \varphi : M \to X \).
Let \((M, E, \varphi)\) be a \(K\)-cycle on \(X\), with \(M\) even-dimensional.

\[
D_E : C^\infty(M, S^+ \otimes E) \longrightarrow C^\infty(M, S^- \otimes E)
\]

Set \(\mathcal{H}_0 = L^2(M, S^+ \otimes E)\) \(\mathcal{H}_1 = L^2(M, S^- \otimes E)\)

For \(j = 0, 1\) define \(\psi_j : C(M) \rightarrow \mathcal{L}(\mathcal{H}_j)\) by:

\[
\alpha \mapsto M_\alpha \quad \alpha \in C(M)
\]

where \(M_\alpha\) is the multiplication operator

\[
M_\alpha(u) = \alpha u \quad (\alpha u)(p) = \alpha(p)u(p) \quad \alpha \in C(M), \, u \in \mathcal{H}_j, \, p \in M
\]
Set \( T = D_E (I + D_E^* D_E)^{-1/2} \) \quad \text{Then}:

\( (\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) \in KK^0 (C(M), \mathbb{C}) \)

and

\( \varphi_* (\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) \in KK^0 (C(X), \mathbb{C}) \)

\[ \varphi_* (\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) := (\mathcal{H}_0, \psi_0 \circ \varphi^\dag, \mathcal{H}_1, \psi_1 \circ \varphi^\dag, T) \]

\( \varphi^\dag : C(M) \leftarrow C(X) \quad \varphi^\dag (\gamma) := \gamma \circ \varphi \quad \gamma \in C(X) \)
Let \((M, E, \varphi)\) be a \(K\)-cycle on \(X\), with \(M\) odd-dimensional.

\[D_E : C^\infty(M, S \otimes E) \longrightarrow C^\infty(M, S \otimes E)\]

Set \(\mathcal{H} = L^2(M, S \otimes E)\)

Define \(\psi : C(M) \rightarrow \mathcal{L}(\mathcal{H})\) by :

\[\alpha \mapsto M_\alpha \quad \alpha \in C(M)\]

where \(M_\alpha\) is the multiplication operator

\[M_\alpha(u) = \alpha u \quad (\alpha u)(p) = \alpha(p)u(p) \quad \alpha \in C(M), u \in \mathcal{H}, p \in M\]
Set \( T = D_E(I + D_E^* D_E)^{-1/2} \) Then:

\[
(\mathcal{H}, \psi, T) \in KK^1(C(M), \mathbb{C})
\]

and

\[
\varphi_*(\mathcal{H}, \psi, T) \in KK^1(C(X), \mathbb{C})
\]

\[
\varphi_*(\mathcal{H}, \psi, T) := (\mathcal{H}, \psi \circ \varphi^\flat, T)
\]

\[
\varphi^\flat : C(M) \leftarrow C(X) \quad \varphi^\flat(\gamma) := \gamma \circ \varphi \quad \gamma \in C(X)
\]
EXAMPLE. \( S^1 \subset \mathbb{R}^2 \)

\( S^1 \) with its usual \( \text{Spin}^c \) structure has \( S = S^1 \times \mathbb{C} \).

The Dirac operator \( D : L^2(S^1) \rightarrow L^2(S^1) \) is:

\[
D = -i \frac{\partial}{\partial \theta}
\]

The functions \( e^{in\theta} \) are an orthonormal basis for \( L^2(S^1) \).

Each \( e^{in\theta} \) is an eigenvector of \( D \):

\[
-i \frac{\partial}{\partial \theta}(e^{in\theta}) = ne^{in\theta} \quad n \in \mathbb{Z}
\]

\( D \) is an unbounded self-adjoint operator. \( D^* = D \).

The bounded operator \( T := D(I + D^*D)^{-1/2} \) is

\[
T(e^{in\theta}) = \frac{n}{\sqrt{1 + n^2}}e^{in\theta} \quad n \in \mathbb{Z}
\]
$K$-cycles are very closely connected to the $D$-branes of string theory. A $D$-brane is a $K$-cycle for the twisted $K$-homology of space-time.

In some models, the D-branes are allowed to evolve with time. This evolution is achieved by permitting the D-branes to change by the three elementary steps. Thus the underlying charge of a $D$-brane (i.e. the element in the twisted $K$-homology of space-time determined by the $D$-brane) remains unchanged as the $D$-brane evolves.

For more, see Jonathan Rosenberg’s CBMS string theory lectures. Also, see Baum-Carey-Wang paper $K$-cycles for twisted $K$-homology Journal of $K$-theory 12, 69-98, 2013. Also, lectures (at Erwin Schrodinger Institute) by Bai-Ling Wang.
Comparison of $K_*(X)$ and $KK^*(C(X), \mathbb{C})$

Given some analytic data on $X$ (i.e. an index problem) it is usually easy to construct an element in $KK^*(C(X), \mathbb{C})$. This does not solve the given index problem. $KK^*(C(X), \mathbb{C})$ does not have a simple explicitly defined chern character mapping it to $H_*(X; \mathbb{Q})$.

$K_*(X)$ does have a simple explicitly defined chern character mapping it to $H_*(X; \mathbb{Q})$.

\[
ch: K_j(X) \longrightarrow \bigoplus_l H_{j+2l}(X; \mathbb{Q})
\]

\[
(M, E, \varphi) \mapsto \varphi_*(ch(E) \cup Td(M) \cap [M])
\]
With $X$ a finite CW complex, suppose a datum (i.e. some analytical information) is given which then determines an element
\[ \xi \in KK^j(C(X), \mathbb{C}). \]

**QUESTION :** What does it mean to solve the index problem for $\xi$?

**ANSWER :** It means to explicitly construct the $K$-cycle $(M, E, \varphi)$ such that
\[ \mu(M, E, \varphi) = \xi \]
where $\mu : K_j(X) \rightarrow KK^j(C(X), \mathbb{C})$ is the natural map of abelian groups.
Suppose that $j = 0$ and that a $K$-cycle $(M, E, \varphi)$ with

$$\mu(M, E, \varphi) = \xi$$

has been constructed. It then follows that for any $\mathbb{C}$ vector bundle $F$ on $X$

$$\text{Index}(F \otimes \xi) = \epsilon_*(ch(F) \cap ch(M, E, \varphi))$$

$\epsilon: X \longrightarrow \cdot$. $\epsilon$ is the map of $X$ to a point.

$$ch(M, E, \varphi) := \varphi_*(ch(E) \cup Td(M) \cap [M])$$
REMARK. If the construction of the $K$-cycle $(M, E, \varphi)$ with

$$\mu(M, E, \varphi) = \xi$$

has been done correctly, then it will work in the equivariant case and in the case of families of operators.
Example

General case of the Atiyah-Singer index theorem

Let $X$ be a compact $C^\infty$ manifold without boundary. $X$ is not required to be oriented. $X$ is not required to be even dimensional.

On $X$ let

$$\delta : C^\infty(X, E_0) \longrightarrow C^\infty(X, E_1)$$

be an elliptic differential (or pseudo-differential) operator.

Then $\delta$ determines an element

$$[\delta] \in KK^0(C(X), \mathbb{C})$$

The $K$-cycle on $X$ – which solves the index problem for $\delta$ – is:

$$(S(TX \oplus 1_{\mathbb{R}}), E_\sigma, \pi).$$
\[(S(TX \oplus 1_\mathbb{R}), E_\sigma, \pi)\]

\(S(TX \oplus 1_\mathbb{R})\) is the unit sphere bundle of \(TX \oplus 1_\mathbb{R}\).

\(\pi : S(TX \oplus 1_\mathbb{R}) \rightarrow X\) is the projection of \(S(TX \oplus 1_\mathbb{R})\) onto \(X\).

\(S(TX \oplus 1_\mathbb{R})\) is even-dimensional and is a Spin\(^c\) manifold.

\(E_\sigma\) is the \(\mathbb{C}\) vector bundle on \(S(TX \oplus 1_\mathbb{R})\) obtained by doing a clutching construction using the symbol \(\sigma\) of \(\delta\).

\[\mu((S(TX \oplus 1_\mathbb{R}), E_\sigma, \pi)) = [\delta]\]

\[\downarrow\]

\[\text{Index}(\delta) = (ch(E_\sigma) \cup Td(S(TX \oplus 1_\mathbb{R})))[(S(TX \oplus 1_\mathbb{R})]\]

which is the general Atiyah-Singer formula.
Next lecture — Tomorrow Thursday, 9 February:
The Riemann-Roch Theorem.
A contact manifold is an odd dimensional $C^\infty$ manifold $X$ with dimension$(X) = 2n + 1$ with a given $C^\infty$ 1-form $\theta$ such that

$$\theta(d\theta)^n$$

is non zero at every $x \in X$ – i.e. $\theta(d\theta)^n$ is a volume form for $X$. 
Let $X$ be a compact connected contact manifold without boundary ($\partial X = \emptyset$).

Set dimension$(X) = 2n + 1$.

Let $r$ be a positive integer and let $\gamma: X \to M(r, \mathbb{C})$ be a $C^\infty$ map. 

$M(r, \mathbb{C}) := \{r \times r \text{ matrices of complex numbers}\}$.

Assume: For each $x \in X$,

$\{\text{Eigenvalues of } \gamma(x)\} \cap \{\ldots, -n - 4, -n - 2, -n, n, n + 2, n + 4, \ldots\} = \emptyset$

i.e. $\forall x \in X$,

$\lambda \in \{\ldots -n - 4, -n - 2, -n, n, n + 2, n + 4, \ldots\} \implies \lambda I_r - \gamma(x) \in GL(r, \mathbb{C})$
\( \gamma : X \rightarrow M(r, \mathbb{C}) \)

Are assuming: \( \forall x \in X, \lambda \in \{ \ldots -n-4, -n-2, -n, n, n+2, n+4, \ldots \} \implies \lambda I_r - \gamma(x) \in GL(r, \mathbb{C}) \)

Associated to \( \gamma \) is a differential operator \( P_\gamma \) which is hypoelliptic and Fredholm.

\[
P_\gamma : C^\infty (X, X \times \mathbb{C}^r) \rightarrow C^\infty (X, X \times \mathbb{C}^r)
\]

\( P_\gamma \) is constructed as follows.
The sub-Laplacian $\Delta_H$

Let $H$ be the null-space of $\theta$.

$$H = \{ v \in TX \mid \theta(v) = 0 \}$$

$H$ is a $C^\infty$ sub vector bundle of $TX$ with

For all $x \in X$, $\dim_{\mathbb{R}}(H_x) = 2n$

The sub-Laplacian

$$\Delta_H : C^\infty(X) \rightarrow C^\infty(X)$$

is locally $-W_1^2 - W_2^2 - \cdots - W_{2n}^2$

where $W_1, W_2, \ldots, W_{2n}$ is a locally defined $C^\infty$ orthonormal frame for $H$.

These locally defined operators are then patched together using a $C^\infty$ partition of unity to give the sub-Laplacian $\Delta_H$. 
The Reeb vector field is the unique $C^\infty$ vector field $W$ on $X$ with:

$$\theta(W) = 1 \text{ and } \forall v \in TX, \ d\theta(W, v) = 0$$

Let

$$\gamma : X \rightarrow M(r, \mathbb{C})$$

be as above, $P_\gamma : C^\infty(X, X \times \mathbb{C}^r) \rightarrow C^\infty(X, X \times \mathbb{C}^r)$ is defined:

$$P_\gamma = i\gamma(W \otimes I_r) + (\Delta_H) \otimes I_r \quad I_r = r \times r \text{ identity matrix} \quad i = \sqrt{-1}$$

$P_\gamma$ is a differential operator (of order 2) and is hypoelliptic but not elliptic.
These operators $P_\gamma$ have been studied by:

- C. Epstein and R. Melrose.

A class of operators with somewhat similar analytic and topological properties has been studied by A. Connes and H. Moscovici.
M. Hilsum and G. Skandalis.
Set $T_\gamma = P_\gamma(I + P_\gamma^* P_\gamma)^{-1/2}$. 

Let $\psi : C(X) \to \mathcal{L}(L^2(X) \otimes \mathbb{C} \mathbb{C}^r)$ be 

$$\psi(\alpha)(u_1, u_2, \ldots, u_r) = (\alpha u_1, \alpha u_2, \ldots, \alpha u_r)$$

where for $x \in X$ and $u \in L^2(X)$, $(\alpha u)(x) = \alpha(x) u(x)$

$$\alpha \in C(X) \quad u \in L^2(X)$$

Then

$$(L^2(X) \otimes \mathbb{C} \mathbb{C}^r, \psi, L^2(X) \otimes \mathbb{C} \mathbb{C}^r, \psi, T_\gamma) \in KK^0(C(X), \mathbb{C})$$

Denote this element of $KK^0(C(X), \mathbb{C})$ by $[P_\gamma]$.

$$[P_\gamma] \in KK^0(C(X), \mathbb{C})$$
\[ [P_\gamma] \in KK^0(C(X), \mathbb{C}) \]

**QUESTION.** What is the K-cycle that solves the index problem for \([P_\gamma]\)?

**ANSWER.** To construct this K-cycle, first recall that the given 1-form \(\theta\) which makes \(X\) a contact manifold also makes \(X\) a stably almost complex manifold:

\[
\text{(contact)} \implies \text{(stably almost complex)}
\]
(contact) \iff (stably almost complex)

Let $\theta$, $H$, and $W$ be as above. Then:

- $TX = H \oplus 1_\mathbb{R}$ where $1_\mathbb{R}$ is the (trivial) $\mathbb{R}$ line bundle spanned by $W$.

- A morphism of $C^\infty$ $\mathbb{R}$ vector bundles $J : H \to H$ can be chosen with $J^2 = -I$ and $\forall x \in X$ and $u, v \in H_x$

\[ d\theta(Ju, Jv) = d\theta(u, v) \quad d\theta(Ju, u) \geq 0 \]

- $J$ is unique up to homotopy.
(contact) $\iff$ (stably almost complex)

$J : H \to H$ is unique up to homotopy.
Once $J$ has been chosen:

$$H \text{ is a } C^\infty \mathbb{C} \text{ vector bundle on } X.$$ 

$\Downarrow$

$$TX \oplus 1_\mathbb{R} = H \oplus 1_\mathbb{R} \oplus 1_\mathbb{R} = H \oplus 1_\mathbb{C} \text{ is a } C^\infty \mathbb{C} \text{ vector bundle on } X.$$ 

$\Downarrow$

$$X \times S^1 \text{ is an almost complex manifold.}$$
REMARK. An almost complex manifold is a \( \mathbb{C}^\infty \) manifold \( \Omega \) with a given morphism \( \zeta : T\Omega \to T\Omega \) of \( \mathbb{C}^\infty \) \( \mathbb{R} \) vector bundles on \( \Omega \) such that

\[
\zeta \circ \zeta = -I
\]

The conjugate almost complex manifold is \( \Omega \) with \( \zeta \) replaced by \(-\zeta\).

NOTATION. As above \( X \times S^1 \) is an almost complex manifold, \( \overline{X \times S^1} \) denotes the conjugate almost complex manifold.

Since (almost complex) \(\to\) (Spin\(^c\)), the disjoint union \( X \times S^1 \sqcup \overline{X \times S^1} \) can be viewed as a Spin\(^c\) manifold.
Let

$$\pi: X \times S^1 \sqcup X \times S^1 \rightarrow X$$

be the evident projection of $X \times S^1 \sqcup X \times S^1$ onto $X$.

i.e.

$$\pi(x, \lambda) = x \quad (x, \lambda) \in X \times S^1 \sqcup X \times S^1$$

The solution $K$-cycle for $[P_\gamma]$ is $(X \times S^1 \sqcup X \times S^1, E_\gamma, \pi)$.
$$E_\gamma = \left( \bigoplus_{j=0}^{j=N} L(\gamma, n+2j) \otimes \pi^* \text{Sym}^j(H) \right) \bigcup \left( \bigoplus_{j=0}^{j=N} L(\gamma, -n-2j) \otimes \pi^* \text{Sym}^j(H^*) \right)$$

1. “Sym\(^j\)” is “j-th symmetric power”.
2. \(H^*\) is the dual vector bundle of \(H\).
3. \(N\) is any positive integer such that : \(n + 2N > \sup\{||\gamma(x)||, x \in X\}\).
4. \(L(\gamma, n + 2j)\) is the \(\mathbb{C}\) vector bundle on \(X \times S^1\) obtained by doing a clutching construction using \((n + 2j)I_r - \gamma: X \to GL(r, \mathbb{C})\).
5. Similarly, \(L(\gamma, -n - 2j)\) is obtained by doing a clutching construction using \((-n - 2j)I_r - \gamma: X \to GL(r, \mathbb{C})\).
Restriction of $E_\gamma$ to $X \times S^1$

Let $N$ be any positive integer such that:

$$n + 2N > \sup_{x \in X} \{ ||\gamma(x)||, x \in X \}$$

The restriction of $E_\gamma$ to $X \times S^1$ is:

$$E_\gamma \mid X \times S^1 = \bigoplus_{j=0}^{j=N} L(\gamma, n + 2j) \otimes \pi^* \text{Sym}^j(H)$$
The restriction of $E_{\gamma}$ to $X \times S^1$ is:

$$E_{\gamma} \mid X \times S^1 = \bigoplus_{j=0}^{j=N} L(\gamma, -n - 2j) \otimes \pi^*\text{Sym}^j(H^*)$$

Here $H^*$ is the dual vector bundle of $H$:

$$H_x^* = \text{Hom}_\mathbb{C}(H_x, \mathbb{C}) \quad x \in X$$
\[ E_\gamma = \left( \bigoplus_{j=0}^{j=N} L(\gamma, n+2j) \otimes \pi^* \text{Sym}^j(H) \right) \uplus \left( \bigoplus_{j=0}^{j=N} L(\gamma, -n-2j) \otimes \pi^* \text{Sym}^j(H^*) \right) \]

**Theorem (PB and Erik van Erp)**

\[ \mu(X \times S^1 \sqcup \overline{X \times S^1}, \ E_\gamma, \pi) = [P_\gamma] \]