QUANTUM HILBERT MATRICES AND
ORTHOGONAL POLYNOMIALS

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Abstract

In [2] it was noticed that the reciprocal Fibonacci numbers form a
moment sequence of a positive discrete measure, and the corresponding
orthogonal polynomials were identified as special little $q$-Jacobi polyno-
mials corresponding to the value $q = (1 - \sqrt{5})/(1 + \sqrt{5})$.

The Hankel matrix of reciprocal Fibonacci numbers, called the Fil-
bert matrix in [3],

$$F_n = \begin{pmatrix} 1 \\ F_{i+j+1} \end{pmatrix}$$

looks pretty much as the classical Hilbert matrix

$$H_n = \begin{pmatrix} 1 \\ i + j + 1 \end{pmatrix}$$

and shares with it the property that the inverse matrix consists of
integers. We will explain these things using the theory of orthogonal
polynomials.

Considering the quantum integers, defined by

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}, \quad n = 0, 1, \ldots,$$  \hspace{1cm} (1)

where $q \in \mathbb{C} \setminus \{0\}$, we can unify the two things: “Fibonacci is a quan-
tized version of Hilbert.”

References

[1] J. Ellegaard Andersen, Christian Berg, Quantum Hilbert matrices and orthog-
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