Summerschool on microlocal methods in Global analysis

Dynamical Resonances via microlocal analysis

1. Dynamical resonances and decay of correlations

Global Analysis (geodesic flows) $\longrightarrow$ Dynamical System

$\rightsquigarrow$

Linear Operator

Microlocal Analysis (Section 2) $\longrightarrow$ $\rightsquigarrow$

discrete Spectrum
(meromorphic resolvent

$\rightsquigarrow$

decay of correlation

From now on: (M,g) C^∞-Riem. Mfld

Interpretation: $m \in M$ describes the "state of the dynamical system" at a given time. e.g $M = \mathbb{R}^6 \simeq$ Ort Impuls eines klass. Teilchens

Def 1.1

Let $\varphi: M \to M$ be a C^∞-Diffeomorphism then (φ, M) form a smooth (C^∞) discrete time dynamical system

Time evolution / trajectories

$$(\ldots \underset{t=-1}{\varphi^{-1}(m)}, \underset{t=0}{m}, \underset{t=1}{\varphi(m)}, \underset{t=2}{\varphi^2(m)}, \ldots)$$

Example 1.2

$M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

$$\varphi: \begin{cases} \mathbb{T}^2 \longrightarrow \mathbb{T}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \bmod \mathbb{Z}^2 \end{cases}$$

Vague Definition

A dynamical system is chaotic if trajectories depend „sensitively" on initial conditions

$\rightsquigarrow$ Individual trajectories are useless for long time behaviour

$\rightsquigarrow$ study propagation of probability measures

$$\varphi_* \begin{cases} P(M) \longrightarrow P(M) \\ d\mu \longmapsto \varphi_* d\mu \end{cases}$$ $\leftarrow$ space of probability measures

if we restrict to regular measures

$$d\mu = \rho \, d\lambda \quad , \; d\lambda = \text{Lebesgue measure on } M$$

then $$\varphi_* \rho \, d\lambda = \frac{1}{|\mathrm{Jac}\,\varphi|} (\rho \circ \varphi^{-1}) \, d\lambda$$ Exercise?

Definition 1.3 (Transfer Operator / Perron Frobenius Oper)

$$\mathcal{L} : \begin{cases} C^\infty(M) \longrightarrow C^\infty(M) \\ \rho \longmapsto \frac{1}{|\mathrm{Jac}\,\varphi|} (\rho \circ \varphi^{-1}) \end{cases}$$

Interpretation: ρ : Prob density at $t=0$

$\rightsquigarrow \mathcal{L}^n \rho$ Prob density at $t=n$

Def 1.4 A smooth Observable is given by $f \in C^\infty(M)$

e.g. Observe if Particle is in a subset $U \subset M$

$\rightsquigarrow f = \mathbb{1}_U * \underbrace{\chi_\varepsilon}$ ← slight smoothing to assure C^∞

Def 1.5 (Correlation function)

Given $\rho, f \in C^\infty(M)$ define

$$C_{f,\rho}(n) := \int_M f \cdot (\mathcal{L}^n \rho)\, d\lambda$$

Interpretation: If ρ: Prob density, f Observable

$$\rightsquigarrow C_{f,\rho}(n) = \langle f \rangle_{\mathcal{L}^n \rho}$$

↑ prob at time n

= „Expectation value of f at time n if initial prob density at $t=0$ was ρ"

Ex: $C_{f,\rho}(n)$ is bounded in $n \in \mathbb{N}$

$\rightsquigarrow$ no hope to obtain exact expressions of $C_{f,\rho}(n)$ for a complicated dyn System

Def 1.6 Discrete Laplace transform

$$\hat{C}_{f,g}(z) := \sum_{n=0}^{\infty} C_{f,g}(n)\, z^{-n-1}$$

Properties: • holomorphic for $|z|>1$

• Inversion formula

$$C_{f,g}(n) = \frac{1}{2\pi i} \int_{\partial B_{1+\varepsilon}(0)} \hat{C}_{f,g}(z)\, z^n \, dz$$

Assumption:

$\exists\, r>0$ s.t. $\hat{C}_{f,g}(z)$ has a meromorphic continuation to $|z|>r$ with poles $\lambda_1, \dots \lambda_k$

Residue Thm, $r'>r$

$$C_{f,g}(n) = \underbrace{\sum_{i=1}^{k} \operatorname{Res}_{z=\lambda_i}\left(z_i^n\, \hat{C}_{f,g}(z)\right)}_{} + \underbrace{\frac{1}{2\pi i}\int_{\partial B_{r'}} \hat{C}_{f,g}(z)\, z^n\, dz}_{\mathcal{O}(r'^n)}$$

if Poles = Order 1

$\to \lambda_i^n \operatorname{Res}\left(\hat{C}_{f,g}(z)\right)$

Up to an exponential remainder term

Dynamics governed by Poles λ_i

Def 1.7 Poles of $\hat{C}_{f,g}(z)$ are dynamical resonances (Ruelle resonances, Ruelle-Pollicott resonances)

Problems 1. How to prove meromorphic continuation of $\hat{C}_{f,g}(z)$

2. Are poles intrinsic to dynamical system or also dependent on f, g

Consider: $\hat{C}_{g,f}(z) = \sum_{n=0}^{\infty} \left(\int f \cdot (\mathcal{L}^n g) \, d\lambda \right) z^{-n-1}$

$$\overset{|z|>>1}{=} \int f \left(\frac{1}{z} \sum_{n\geq 0} \frac{\mathcal{L}^n}{z^n} \right) g \, d\lambda$$

$$= -\int f \, \underbrace{(\mathcal{L} - z)^{-1}}_{=:R(z) \text{ resolvent of } \mathcal{L}} g \, d\lambda$$

$g, f \in C^\infty \rightsquigarrow$ need meromorphic continuation of

$$R(z) : C^\infty(M) \to \mathcal{D}'(M)$$

Summary

Asymptotics of $C_{f,g}(n)$ $\rightsquigarrow$ Poles of $\hat{C}_{f,g}(z)$ $\rightsquigarrow$ Poles of $R(z) = (\mathcal{L} - z)^{-1}$

- $C_{f,g}(n) = \sum_{i=0}^{k} \lambda_i^n \langle f, \Pi_{\lambda_i} g \rangle + O(\varepsilon^n)$

  Poles (discrete Spectrum)

  Residue Operators (spectral projectors!)

- If 1 is a simple eigenvalue, $|\lambda_i| < 1$ for $i = 1, \dots$

  $$C_{f,g}(n) = \langle f, \Pi_0 g \rangle + O(|\lambda_1|^n)$$

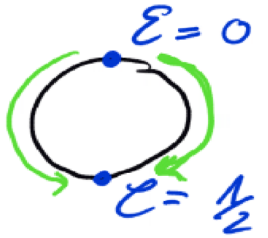
  exponential decay towards a constant!

  „exponential Mixing"

2. A Toy model

$\mathcal{M} = S^1 = \mathbb{R}/\mathbb{Z}$

$\varphi : S^1 \to S^1$ s.t.



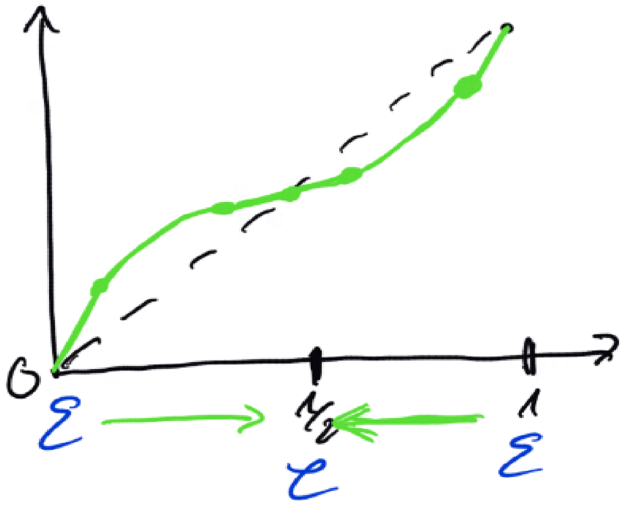
(i) Two fixpoints $\varphi(\mathcal{C}) = \mathcal{C}$, $\varphi(\mathcal{E}) = \mathcal{E}$

(ii) $\varphi'(x) > 0$

(iii) $\exists\ \varepsilon > 0\ \lambda_{\mathcal{C}}, \lambda_{\mathcal{E}} < 1$ s.t. $\varphi'(x) = \lambda_{\mathcal{C}} < 1$ on $B_\varepsilon(\mathcal{C})$

$\varphi'(x) = \lambda_{\mathcal{E}}^{-1} > 1$ on $B_\varepsilon(\mathcal{E})$

(iv) $\dfrac{\ln(\lambda_{\mathcal{E}})}{\ln(\lambda_{\mathcal{C}})} \notin \mathbb{Q}$



Exercise 2.1

Let $1 > \vartheta > \max(\sqrt{\lambda_{\mathcal{E}}}, \sqrt{\lambda_{\mathcal{C}}})$, then for $n > N_0$

$$\varphi^n(S^1 \setminus B_{\vartheta^n}(\mathcal{E})) \subset B_{\vartheta^n}(\mathcal{C})$$

Lemma 2.2

If $f, g \in C^\infty(S^1)$, $\quad f^{(k)}(\mathcal{C}) = 0$, $\quad g^{(k)}(\mathcal{E}) = 0$ f.a. $k = 0, \dots, N-1$

then $|C_{f,g}(n)| \leq C(\vartheta^N)^n$

$\hat{C}_{f,g}(z)$ holomorphic for $|z| > \vartheta^N$

proof

$$C_{f,g}(n) = \int f(\varphi^n(x))\, g(x)\, dx$$

$$= \int_{S^1} \underbrace{(\mathbb{1}_{B_{\vartheta^n}(\mathcal{E})} f)}_{\| \|_\infty \leq C\vartheta^{nN}} (\varphi^n(x))\, g(x)\, dx + \int \underbrace{\underbrace{(\mathbb{1}_{S^1 \setminus B_{\vartheta^n}(\mathcal{E})} f)(\varphi^n(x)) \cdot g(x)\, dx}_{\operatorname{supp} \subset B_{\vartheta^n}(\Sigma)}}_{\| \|_\infty < C\vartheta^{nN}}$$

f vanishes at $\mathcal{E}$

Ex. 2.1

g vanishes at Σ

$\square$

Exercise 2.3

The same statement holds for $f, g \in \mathcal{D}'(S^1)$ with

f smooth near S and vanishing to Order N

g smooth near N and vanishing to Order N

Theorem 2.4

Let $f, g \in C^\infty(S^1)$, then $\hat{C}_{f,g}(z)$ has a meromorphic extension to $\mathbb{C} \setminus \{0\}$ with possible poles at $1, \lambda_S^k, \lambda_N^k, k \in \mathbb{N}$

proof

Taylor expansion of f on $B_\varepsilon(\mathcal{E})$

$$f(x) = \sum_{k=0}^{N-1} \frac{f^{(k)}(\mathcal{E})}{k!} (x - \mathcal{E})^k + \tilde{f}(x)$$

vanish at order N

on $B_\varepsilon(\mathcal{E})$ $\varphi(x) = \mathcal{E} + \lambda_\mathcal{E}(x - \mathcal{E})$

$\rightsquigarrow (\varphi(x) - \mathcal{E})^k = \lambda_\mathcal{E}^k (x - \mathcal{E})^k$ locally an eigenfunction of $\mathcal{L}^*$

Extend this to an eigenfunction on $S^1 \setminus \{\Sigma\}$

for $x \in S^1 \setminus \Sigma$ let N_x s.t. $\varphi^{N_x}(x) \in B_\varepsilon(\mathcal{E})$

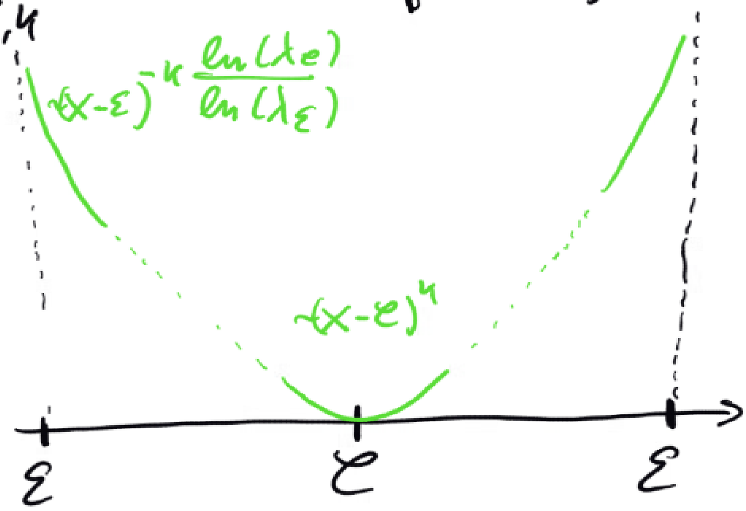
and set

$$\psi_{\mathcal{E},k}(x) := \lambda_\mathcal{E}^{-k N_x} (\varphi^{N_x}(x) - \mathcal{E})^k$$

well defined, $\psi_{\mathcal{E},k} \in C^\infty(S^1 \setminus \Sigma)$, $\mathcal{L}^* \psi_{\mathcal{E},k} = \lambda_\mathcal{E}^k \psi_{\mathcal{E},k}$

on $B_\varepsilon(\mathcal{E})$ $\varphi'(x) = \lambda_\mathcal{E}^{-1}$

$$\Rightarrow \psi_{e,k}(x) \overset{x \approx \mathcal{E}}{=} c \cdot (x-\mathcal{E})^{-k \frac{\ln(\lambda_e)}{\ln(\lambda_\mathcal{E})}}$$



If $k \gg 0 \rightsquigarrow$ nonintegrable singularity

<u>But</u> Hadamard/Riesz regularization

$\Rightarrow \psi_{e,k}$ extends to $\mathcal{D}'(S^1)$ and $\mathcal{L}^* \psi_{e,k} = \lambda_e^k \psi_{e,k}$

write $f = \sum_{k=0}^{N-1} \frac{f^{(k)}(e)}{k!} \psi_{e,k} + \tilde{f}$ $\in \mathcal{D}'(S^1)$ smoothly vanishing at e

Similarly construct

$$\psi_{\mathcal{E},k} \in \mathcal{D}'(S^1), \psi_{\mathcal{E},k}(x) = (x-\mathcal{E})^k, \; \mathcal{L}\psi_{\mathcal{E},k} = \lambda_\mathcal{E}^{k+1} \psi_{\mathcal{E}k}$$

write $g = \sum_{l=0}^{N-1} \frac{g^{(k)}(\mathcal{E})}{k!} \psi_{e,l} + \tilde{g}$ $\in \mathcal{D}'(S^1)$, smoothly vanishing at $\mathcal{E}$

note

$$0 = \langle \mathcal{L}^* \psi_{e,k}, \psi_{\mathcal{E},l} \rangle - \langle \psi_{e,k}, \mathcal{L}\psi_{\mathcal{E},l} \rangle = \underbrace{(\lambda_e^k - \lambda_\mathcal{E}^{l+1})}_{\neq 0} \underbrace{\langle \psi_{e,k}, \psi_{\mathcal{E},l} \rangle}_{=0}$$

Thus

$$C_{f,g}(n) = \sum_{k=0}^{N-1} \frac{f^{(k)}(e)}{k!} \langle \psi_{e,k}, g \rangle \cdot (\lambda_e^k)^n$$

$$+ \langle f, \psi_{\mathcal{E},k} \rangle \frac{g^{(k)}(\mathcal{E})}{k!} (\lambda_\mathcal{E}^{k+1})^n$$

$$+ C_{\tilde{f}\tilde{g}}(n)$$

$\xrightarrow{\text{Laplace Trafo}}$ Poles at λ_e^k

$\rightsquigarrow$ Poles at $\lambda_\mathcal{E}^{k+1}$

$\rightsquigarrow$ holom on $|z| > \vartheta^N$

Remark 2.5

The result suggest, that

$$\mathcal{L} = \sum_{k=0}^{N} \underbrace{\lambda_e^k}_{\text{Eigenvalues}} \underbrace{\left|\tfrac{1}{k!}\delta_e^{(k)}\right\rangle\left\langle \psi_{e,k}\right|}_{\text{Spectral projectors}} + \lambda_\varepsilon^{k+1} \left|\psi_{\varepsilon,k}\right\rangle\left\langle \tfrac{1}{k!}\delta_\varepsilon^{(k)}\right| + \mathcal{O}(\vartheta^N)$$

But on which spaces?

Need to allow singularities of order k @ e

assure k-times differentiable @ ε

In fact construction of $\psi_{e/\varepsilon,\alpha}(x)$ with eigenvalue $\lambda_{e/\varepsilon}^{\alpha}$ works for any $\alpha \in \mathbb{R}_{\geq 0}$, but not smooth at e/ε

Exercise 2.6

Proof $\hat{C}_{f,g}(z)$ has no meromorphic continuation for some $f,g \in C(S^1)$.

Exercise 2.7

Consider $\varphi: \begin{cases} S^1 \to S^1 \\ x \mapsto x+\sqrt{2} \end{cases}$

Show that there are $f,g \in C^\infty(S^1)$, such that $\hat{C}_{f,g}(z)$ has no meromorphic continuation

Proof: $f_k(x) = e^{2\pi i k x}$

$$f_k \circ \varphi = e^{2\pi i k\sqrt{2}} f_k \rightsquigarrow C_{f_k,\bar{f}_k}(n) = \left(e^{2\pi i k\cdot\sqrt{2}}\right)^n$$

$$\rightsquigarrow \hat{C}_{f_k,\bar{f}_k}(z) = \frac{1}{z - e^{2\pi i k\sqrt{2}}}$$

Consider $f = \sum_{k>0} e^{-k^2} f_k \in C^\infty$

$$\hat{C}_{f,\bar{f}}(z) = \sum_{k\geq 0} \frac{e^{-k^2}}{z - e^{2\pi i k\sqrt{2}}}$$

Poles are dense on unit circle

3. Toy model via microlocal analysis

Theorem 3.1

Let $\varphi : S^1 \to S^1$ be as in section 2, recall $\mathcal{L}^* g = g \circ \varphi$

Then there is $\vartheta < 1$, and for any $N \in \mathbb{N}$ Hilbert spaces $C^\infty(S^1) \subset \mathcal{H}^N \subset \mathcal{D}'(S^1)$ s.t.

$\mathcal{L}^* = K + R$ where $K : \mathcal{H}^N \to \mathcal{H}^N$ cpt

$$\|R\|_{\mathcal{H}^N \to \mathcal{H}^N} < C \vartheta^N$$

Corollary 3.2

$R(z) = (\mathcal{L}^* - z)^{-1}$ has a meromorphic extension to $\mathbb{C} \setminus \{0\}$

proof

$$\mathcal{L}^* - z = \underset{\text{cpt}}{K} + \underbrace{R - z}_{\text{invertible for } |z| > C\vartheta^N}$$

$\Rightarrow (\mathcal{L}^* - z) : \mathcal{H}^N \to \mathcal{H}^N$ analytic family of Fredholm operators for $|z| > C\vartheta^N$

$\Longrightarrow (\mathcal{L}^* - z)^{-1} : \mathcal{H}^N \circlearrowleft$ meromorphic for $|z| > C\vartheta^N$

as $C^\infty \subset \mathcal{H}^N \subset \mathcal{D}'$ $\rightsquigarrow$ meromorphic extension of $R(z) : C^\infty \to \mathcal{D}'$ on $|z| > C\vartheta^N$

But N can be chosen arbitrarily large ∎

Idea $\mathcal{H}^N := \mathrm{Op}(A_N)^{-1} L^2(S^1)$

$$\begin{array}{ccc} \mathcal{H}^N & \xrightarrow{\mathcal{L}^*} & \mathcal{H}^N \\ \downarrow{\scriptstyle \mathrm{Op}(A_N)} & & \downarrow{\scriptstyle \mathrm{Op}(A_N)} \\ L^2 & \xrightarrow{Q} & L^2 \end{array}$$

$Q : \mathrm{Op}(A_N) \mathcal{L}^* \mathrm{Op}(A_N) : L^2 \circlearrowleft$

Note

$$\mathcal{L}^* f = \frac{1}{2\pi}\int e^{i(\varphi(x)\xi - y\xi)} f(y)\,dy\,d\xi$$ (in local

$\rightsquigarrow \mathcal{L}^*$ is a FIO

Exercise 3.3

Its canonical transformation is given by

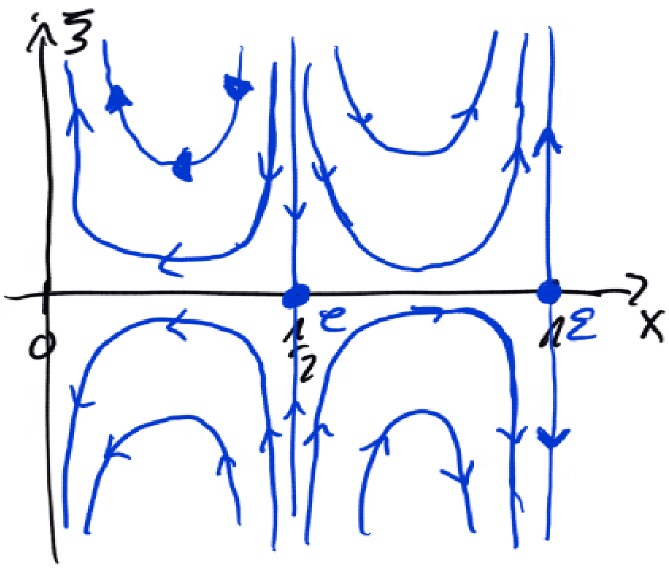
$$T : \begin{cases} T^*S^1 \to T^*S^1 \\ (x,\xi) \mapsto (\varphi^{-1}(x), \varphi'(\varphi^{-1}(x))^t \xi) \end{cases}$$

Consider $P := (\mathcal{L}^*)^{-1} Q = \underbrace{(\mathcal{L}^*)^{-1} Op(A_N) \mathcal{L}^*}_{\text{Egorov / Variable change}} A_N^{-1}$

$$= Op\left(\frac{A_N(T(x,\xi))}{A_N(x,\xi)}\right) + \underbrace{l.o.t.}_{\text{reg. cpt}}$$

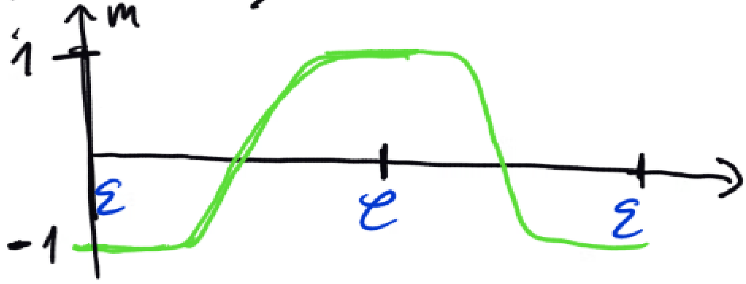
Want: $\dfrac{A_N(T(x,\xi))}{A_N(x,\xi)} < \vartheta^N$

Visualize Dynamics of T on T^*S^1



Choose

$$A_N(x,\xi) := \langle \xi \rangle^{N m(x)}$$



Lemma 3.4
There is $m \in C^\infty(S^n)$, $\vartheta < 1$ and $R > 0$ such that
$$\frac{A_N(T(x,\xi))}{A_N(x,\xi)} \leq \vartheta^N \quad \text{For } |\xi| > R$$
proof: Use properties of φ, $\vartheta = \min(\lambda_e, \lambda_\varepsilon) + \varepsilon$

What calculus to take?

Note standard calculus:
$$A_N \in S^N \text{ but } \frac{1}{A_N(x,\xi)} \in S^N$$
$\rightsquigarrow$ l.o.t $\in S^{2N-1}$ Bad

$\rightsquigarrow$ need fine symbol classes
$$a \in S^{m(x,\xi)} \quad |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m(x,\xi) - \beta - 0+}$$
Everything holds as in the standard calculus with slightly worse remainder e.g. $A_N(x,\xi) = \langle \xi \rangle^{Nm(x)} \in S^{Nm(x)}$, elliptic
$$\Rightarrow Op(A_N)^{-1} - Op\left(\frac{1}{A_N}\right) \in \Psi^{-Nm(x) - 1 + \varepsilon}$$
$$\rightsquigarrow P = \underbrace{(\mathcal{L}^*)^{-1} Op(A_N) \mathcal{L}^*}_{\Psi^{m(x)}} \underbrace{Op(A_N)^{-1}}_{\in \Psi^{-m(x)}} = Op\left(\frac{A_N(T(x,\xi))}{A_N(x,\xi)}\right) + \underbrace{O(\Psi^{Nm - Nm - 1 + \varepsilon})}_{\substack{\text{smoothing} \\ \sim \text{cpt}}} \checkmark$$
Lemma 3.4 + L^2-cont.
$$\Rightarrow Op\left(\frac{A_N(T(x,\xi))}{A_N(x,\xi)}\right) = B + K$$
$$\|B\|_{L^2} \leq \vartheta^N + \varepsilon \quad K: L^2 \to L^2 \text{ cpt}$$

Finally

$$Q = \mathcal{L}^* P = \mathcal{L}^* (B + K) = B' + K'$$

$$\|B'\|_{L^2} \leq \|\mathcal{L}^*\|_{L^2}\, \vartheta^N + \varepsilon$$

$$K' : L^2 \circlearrowleft \text{ cpt}$$

this proofs Thm 3.1 !

$\blacksquare$

4 Dynamical resonances for Anosov Diffeomorphisms

Def 4.1.

Let (M, g) be a cpt. Riem. Manifold. A C^∞ diffeomorphism $\varphi: M \to M$ is called uniformly hyperbolic (or „Anosov") if there exists a continuous splitting $TM = E_s \oplus E_u$ and $0 < \lambda < 1$

- Splitting $D\varphi$-invariant
- for $v \in E_s$: $\|D\varphi^n v\| \leq C \lambda^n \|v\|$
- for $v \in E_u$: $\|D\varphi^{-n} v\| \leq C \lambda^n \|v\|$

Recall

$$T: \begin{cases} T^*M \longrightarrow T^*M \\ (x, \xi) \longmapsto (\varphi^{-1}(x), (D_{\varphi^{-1}(x)}\varphi)^t \xi) \end{cases}$$

Is canonical relation of the FIO $\mathcal{L}^*: f \mapsto f \circ \varphi$

Lemma 4.2 (Faure-Roy-Sjöstrand)

There exists $m(x,\xi) \in S^0_{1,0}(M)$, $R > 0$, $0 < \vartheta < 1$ s.t.

(i) $m(x,\xi) = \begin{cases} 1 & \text{in a cone near } E_s^* \text{ for } |\xi| > R \\ -1 & \text{in a cone near } E_u^* \text{ for } |\xi| > R \end{cases}$

(ii) $m \circ T \leq m$

(iii) For $A_N := \langle \xi \rangle^{N m(x,\xi)} \in S^{N m(x,\xi)}(M)$

one has for $|\xi| > R$ $\dfrac{A_N(T(x,\xi))}{A_N(x,\xi)} \leq \vartheta^N$

proof ~ 3 pages in Faure Roy - Sjöstrand

Idea:

Dynamics of T is complicated on M but simple in the Fibres of T^*M

Dynamics of hyperbolic fixed point

slight difficulty: $E_s \oplus E_u$ only Hölder continuous!

Theorem 4.3

Let $\mathcal{H}^N := Op(A_N)^{-1} L^2(M)$

Then $\mathcal{L}^* = K + R$ where $K : \mathcal{H}^N \circlearrowleft$ cpt

$\|R\|_{\mathcal{H}^N \to \mathcal{H}^N} \leq C \vartheta^N$

- Blank-Keller-Liverani 02 } Banach spaces
  Gouezel-Liverani 06
- Baladi-Tsujii 07 Hilbert spaces
- Faure Roy Sjöstrand 08 — " — via ΨDOs

5. Flows

Let (M, g) be a Riemannian manifold,
$\varphi_t \curvearrowright M$ a flow, $d\mu$ a smooth preserved measure

$f, g \in C_c^\infty(M)$

$$C_{f,g}(t) = \int_M (f \circ \varphi_t) \cdot g \, d\mu$$

Laplace Trafo

$$\hat{C}_{f,g}(s) = \int_0^\infty e^{-st} \int (e^{tX} f) \, g \, d\mu \quad \text{where } X = \frac{d}{dt}\Big|_{t=0} \varphi_t$$

generating vector field of φ_t

$$= - \int g \underbrace{(X-s)^{-1}}_{=:R(s)} f \, d\mu$$

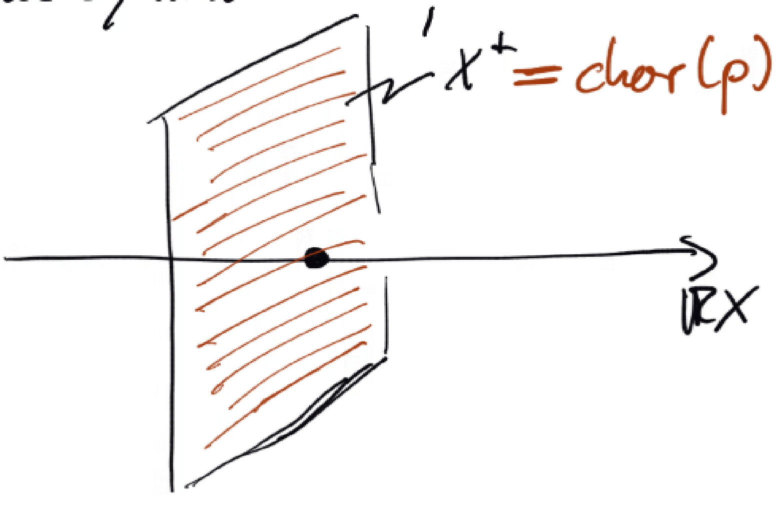
Note: On $L^2(M)$, $X^* = -X \implies (X-s)^{-1} : L^2 \to L^2$ holomorphic for $\mathrm{Re}(s) > 0$

We want $C^\infty \subset \mathcal{H} \subset \mathcal{D}'$, $P(s) : \mathcal{H} \to \mathcal{Y}$ analytic family of op
s.t. $(X-s) P(s) = 1 + K(s)$ where $K(s) : \mathcal{H} \to \mathcal{H}$ cpt

$$\underbrace{(X-s) \, P(s) \underbrace{(1+K(s))^{-1}}_{\text{merom. by analytic Fredholm Thm}}}_{R(s) : \mathcal{H} \to \mathcal{H}} = 1$$

X: first order differential operator
principal symbol $\sigma(X)(x,\xi) = i\xi(X_x) =: i p(x,\xi)$

$T^*M \simeq TM$



Let $\Phi_t : \begin{cases} T^*M \longrightarrow T^*M \\ (x,\xi) \mapsto (\varphi_t(x), (D_x\varphi_t^*)^{-1}\xi) \end{cases}$

Hamiltonian flow of $p(x,\xi)$

φ_t^* is FIO with canonical relation Φ_{-t}

Let $H^r = Op(e^{-rG})L^2$

then $X-s : H^r \longrightarrow H^r$ is unitary equivalent to

$$Op(e^{-rG})^{-1}(X-s)\,Op(e^{-rG}) : L^2 \to L^2$$

$$= X - s + Op(e^{-rG})^{-1}[X, Op(e^{-rG})]$$

$$= X - s + Op\left(e^{rG}\tfrac{1}{i}\{ip, e^{-rG}\}\right) + l.o.t.$$

$$= \underbrace{X}_{\in \Psi^1} \underbrace{-s - r\,Op(\{p,G\})}_{\in \Psi^{0^+}} + \underbrace{l.o.t.}_{\in \Psi^{-1^+}}$$

<u>Def</u> <u>(Anosov flows)</u>

Let (M,g) be cpt Riem. Mfld. A flow $\varphi_t \curvearrowright M$ is called Anosov if $\exists\, C,\lambda>0$, a continuous $D\varphi_t$ invariant splitting $TM = E_0 \oplus E_s \oplus E_u$

where $E_0 = \mathbb{R}X$

$$v \in E_s \Rightarrow \|D\varphi_t v\| \leq Ce^{-\lambda t}\|v\|$$
$$v \in E_u \Rightarrow \|D\varphi_{-t} v\| \leq Ce^{-\lambda t}\|v\| \quad \text{for } t>0$$

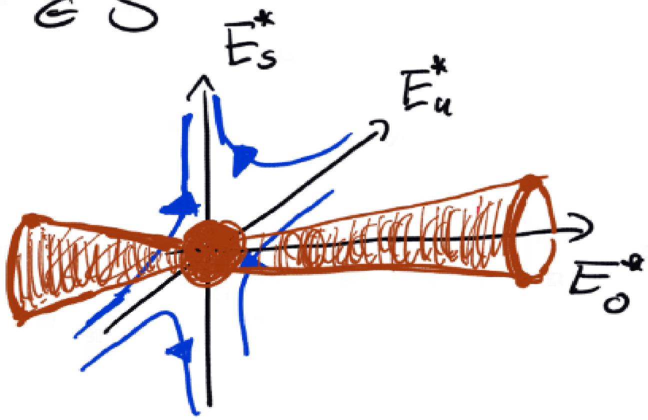
Lemma

For $\varphi_t \curvearrowright M$ Anosov there are cones N_0, N_u, N_s around E_0, E_s, E_u $R \geq 0$ and $G \in C^\infty(T^*M)$ s.t.

(i) $\{p, G\} > 1$ outside $\{|\xi| < R\} \cup N_0$

(ii) $\{p, G\} \geq 0$ everywhere

(iii) $e^G \in S^m$



Use h-calculus!

Let $q \in C_c^\infty(T^*M)$ $q = 1$ on $\{|\xi| < R\}$, $q \geq 0$

consider $X_Q(s) := hX - \underbrace{Op_h(q)}_{Q} - hs$

Proposition

For $Re(s) > C - r$, $h < h_0$, $|Im(s)| < h^{-1/2}$

$X_Q(s) : \mathcal{H}^r \to \mathcal{H}^r$ is invertible

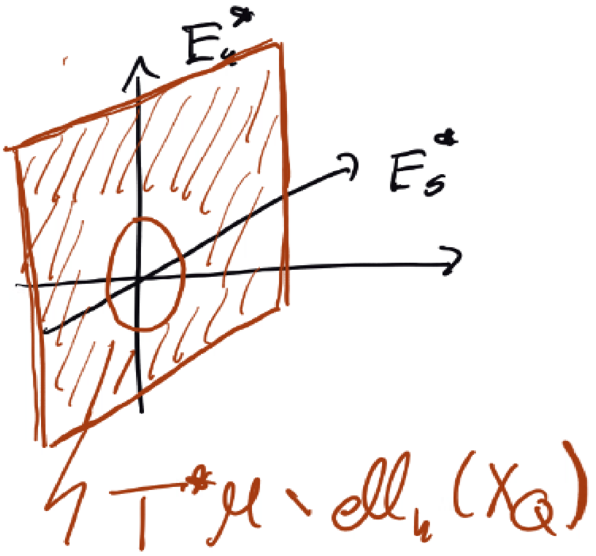
with $\|X_Q(s)^{-1}\| = O(h^{-1})$

proof $X_Q(s)$ unit. equivalent to

$$\tilde{X}_Q = hX - Q - ihr\,Op(\{q,G\}) - h(r\,Op\{p,G\}+s) + O(h^2\Psi^{-1})$$

Ψ^1 Ψ^0 $h\Psi^{-\infty}$ $h\Psi^{0+}$

- $\sigma_{princ}(\tilde{X}_Q) = ip - q$



h-ellipt-estimate

$A_1 \in \Psi^0$ $\quad WF_h(A_2) \subset ell_h(X_Q)$

$$\Rightarrow \|Au\|_{L^2} \le C\|X_Q u\|_{L^2} + O(h^\infty)\|u\|_{L^2}$$

- $Re(X_Q(s)) = \frac{1}{2}\left(X_Q(s) + X_Q(s)^*\right)$

$$= -Q - h(r\,Op(\{p,G\} + Re(s)) + O(h^2\Psi^{-1})$$

if $R(s) > -r$ $\quad \sigma_{princ}(Re(X_Q(s))) > 0$

on $\Omega_{Gaard} \supset T^*M \setminus ell_h(X_Q(s))$

Sharp Gaarding + calculations: $\quad A_2 \in S^0$ $\quad WF_h(A_2) \subset \Omega_{Gaard}$

$$\Rightarrow \|A_2 u\| \le Ch^{-1}\|X_Q(s)u\| + O(h^\infty)\|u\|_{L^2}$$

This proves injectivity of $X_Q(s)$

Surjectivity: Consider $X_Q(s)^*$ $\blacksquare$

<u>Thm</u> $(X-s)^{-1} : \mathcal{H}^r \to \mathcal{H}^r$ meromorphic for $\mathrm{Re}(s) > C - r$

<u>proof</u>:

$$\underbrace{(X-s)}_{(X-Q-s)+Q}(X-Q-s)^{-1} = \mathbb{1} + Q\underbrace{(X-Q-s)^{-1}}_{\text{bounded holom}}$$

$Q \in \Psi^{-\infty} \Rightarrow$ cpt

$\boxed{\text{IFM cpt}}$

$$(X-s)^{-1} = \underbrace{(X-Q-s)^{-1}}_{\text{holom}}\underbrace{\left(1+Q(X-Q-s)^{-1}\right)^{-1}}_{\text{meromorphic by analytic Fredholm}}$$

$\blacksquare$