ζ-functions of Fourier Integral Operators: Integration Theory

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Bochner Integral

Definition

Let (Ω,Σ,μ) be a measure space and E a topological vector space.

- $f\in E^\Omega$ is called simple ($f\in\mathcal{S}(\mu;E)$) if and only if $f[\Omega]\subseteq_{\text{finite}}E$ and
$$\forall e\in f[\Omega]\setminus\{0\}:\ [\{e\}]f\in\Sigma\ \wedge\ \mu([\{e\}]f)<\infty.$$
- Let $f=\sum_{e\in f[\Omega]\setminus\{0\}}e1_{[\{e\}]f}\in\mathcal{S}(\mu;E)$. Then, we define the Bochner integral
$$\int_\Omega f\,d\mu:=\sum_{e\in f[\Omega]\setminus\{0\}}\mu([\{e\}]f)\ e.$$

Measurability

Definition (continued)

- $f \in E^{\Omega}$ is called measurable ($f \in \mathcal{M}(\mu; E)$) if and only if

$$\forall S \subseteq_{\text{open}} E: \ [S]f \in \Sigma.$$

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- $f \in E^\Omega$ is called strongly measurable ($f \in \mathcal{SM}(\mu; E)$) if and only if

$$\exists s \in \mathcal{S}(\mu; E)^{\mathbb{N}} : \ s_n \to f \ (n \to \infty) \ \mu\text{-almost everywhere.}$$

Note $\mathcal{SM}(\mu; E) \subseteq \mathcal{M}(\mu; E)$.

Pettis Integral

Definition (continued)

- Let $f \in \mathcal{M}(\mu;E)$, $\forall x' \in E':\ x' \circ f \in L_1(\mu)$, and $I \in (E')^*$ such that

$$\forall x' \in E':\ I(x') = \int_\Omega x' \circ f \ d\mu.$$

f is called μ-Dunford-Pettis-integrable if and only if I is unique. Then, we will use the notation $\int_\Omega f d\mu := I$.

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- $f \in \mathcal{M}(\mu;E)$ is called μ-Pettis-integrable if and only if f is μ-Dunford-Pettis-integrable and $I \in E$.

$L_p(\mu;E)$

Definition (continued)

Let E be locally convex with semi-norms $(p_\iota)_{\iota\in I}$. Then, we define

$$\begin{aligned}(\mathcal{S})\mathcal{L}_p(\mu;E) &:= \{f\in(\mathcal{S})\mathcal{M}(\mu;E);\ \forall\iota\in I:\ p_\iota\circ f\in L_p(\mu)\},\\ (\mathcal{S})\mathcal{N}_p(\mu;E) &:= \{f\in(\mathcal{S})\mathcal{L}_p(\mu;E);\ \forall\iota\in I:\ \|p_\iota\circ f\|_{L_p(\mu)}=0\}\end{aligned}$$

and

$$(\mathcal{S})L_p(\mu;E) := (\mathcal{S})\mathcal{L}_p(\mu;E)/_{(\mathcal{S})\mathcal{N}_p(\mu;E)}.$$

Bochner Integral

From now on: E locally convex and Hausdorff.

Theorem

$L_p(\mu;E)$ is locally convex and Hausdorff.

Theorem

The Bochner integral

$$\int:\ \mathcal{S}(\mu;E)\subseteq L_1(\mu;E)\to E$$

is a continuous linear operator and extends uniquely to

$$\int:\ \bar{\mathcal{S}}L_1(\mu;E):=\overline{\mathcal{S}L_1(\mu;E)}^{L_1(\mu;E)}\to\tilde{E}.$$

convex compactness

Definition

E has the convex compactness property if and only if

$$\forall C \subseteq_{\text{compact}} E : \ \overline{\operatorname{conv} C} \subseteq_{\text{compact}} E.$$

E has the metric convex compactness property if and only if

$$\forall C \subseteq_{\text{compact,metrizable}} E : \ \overline{\operatorname{conv} C} \subseteq_{\text{compact}} E.$$

Pettis integral

Theorem (Pfister; 1981)

Let E be a locally convex topological vector space and a Hausdorff space. Then, the following are equivalent.

(i) *E has the (metric) convex compactness property.*

(ii) *Let Ω be a compact (metric) space, μ a (positive) Borel measure on Ω, and $f \in C(\Omega, E)$. Then, f is μ-Pettis integrable.*

Let E be a topological vector space and $A \subseteq E$.

- A filter $\mathcal{F}$ on A is a family of subsets of A such that
  - $\varnothing \notin \mathcal{F}$
  - $X \in \mathcal{F} \ \wedge \ X \subseteq Y \subseteq A \Rightarrow Y \in \mathcal{F}$
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- A filter $\mathcal{F}$ is convergent to $x\in E$ if and only if $\mathcal{F}$ contains the neighborhood filter $\mathcal{U}_x$ of x;

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- A is complete if and only if every Cauchy filter on A is convergent in A.
- E is quasi-complete if and only if every closed and bounded subset of E is complete.

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$\Downarrow$

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metric convex
compactness property

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Lemma (Sombrero Lemma)

Let E *be metrizable and* (Ω, Σ, μ) *a compact Borel measure space. Then,* $C(\Omega, E) \subseteq \mathcal{SM}(\mu; E)$.

Lemma (generalized Sombrero Lemma)

Let E *be a separable metric space and* (Ω, Σ, μ) *a Radon measure space. Then,* $\mathcal{SM}(\mu; E) = \mathcal{M}(\mu; E)$.

Lemma

Let (Ω, Σ, μ) be σ-finite, F another Hausdorffian locally convex topological vector space, and $f \in \bar{\mathcal{S}}L_1(\mu; E)$.

(i) *Let $B \in L(\tilde{E}, \tilde{F})$. Then, $B \circ f \in \bar{\mathcal{S}}L_1(\mu; F)$ and*

$$B \int_\Omega f \, d\mu = \int_\Omega B \circ f \, d\mu.$$

(ii) *Let $E_0 \subseteq E$ be a closed subspace and $f(\omega) \in E_0$ for μ-almost every $\omega \in \Omega$. Then, $\int_\Omega f \, d\mu \in E_0$.*

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- $f\in\mathcal{S}(\mu;E)\Rightarrow B\int_\Omega f d\mu = \int_\Omega B\circ f d\mu$ trivial
- $f\mapsto B\int_\Omega f d\mu, f\mapsto\int_\Omega B\circ f d\mu\in L(L_1(\mu;E),\tilde{F})$
- $\Rightarrow B\int_\Omega f d\mu = \int_\Omega B\circ f d\mu$ on L_1 closure of $\mathcal{S}(\mu;E)$ by unique extension theorem

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- Let $\varphi\in E'$ with $\varphi|_{E_0}=0$
- $\varphi\int f\,d\mu=\int\varphi\circ f\,d\mu=0$
- Hahn-Banach: $\int f\,d\mu\in E_0$ (otherwise, $\exists\varphi\in E':\ \varphi|_{E_0}=0\ \wedge\ \varphi\int f\,d\mu=1$)

Theorem (“Hille”)

Let $f \in \bar{\mathcal{S}}L_1(\mu;E)$, F *another Hausdorffian locally convex topological vector space, and* $A: D(A) \subseteq E \to F$ *a closed linear operator. Let* $f(\omega) \in D(A)$ *for* μ*-almost every* $\omega \in \Omega$ *and* $A \circ f \in \bar{\mathcal{S}}L_1(\mu;F)$.

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Let $f \in \bar{\mathcal{S}}L_1(\mu; E)$*,* F *another Hausdorffian locally convex topological vector space, and* $A\colon D(A) \subseteq E \to F$ *a closed linear operator. Let* $f(\omega) \in D(A)$ *for* μ*-almost every* $\omega \in \Omega$ *and* $A \circ f \in \bar{\mathcal{S}}L_1(\mu; F)$*.*
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- Similarly: A sequentially-complete ⇒ need to approximate f and $A \circ f$ with sequences of simple functions
- Pettis integral:
  $\forall \varphi \in F'\ \forall B \in L(\tilde{E}, \tilde{F})\colon\ \varphi \circ B \in E' \ \wedge\ \varphi B \int f d\mu = \int \varphi \circ B \circ f d\mu = \varphi \int B \circ f d\mu$

Fourier Integral Operators on a manifold X

- Fourier Integral Operators are integral operators $A\colon\ C_c^\infty(X)\to C_c^\infty(X)'$ of the form

$$\forall\ \varphi\in C_c^\infty(X):\ \ A\varphi(x)=\int_X k_A(x,y)\varphi(y)d\mathrm{vol}_X(y)$$

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  where k_A is a Lagrangian distribution.
- Lagrangian distributions are classified by their wave front sets. The set of all Lagrangian distributions with wave front set in a suitable cone Γ of the co-tangent bundle $T^*X \setminus 0$ is the Hörmander space $\mathcal{D}'_\Gamma$.

$\mathcal{A}_\Gamma$ vs $\mathcal{D}'_\Gamma$

For an algebra $\mathcal{A}$, consider $\int_\Omega A d\mu$. Then,

$$\begin{aligned}\left\langle \int_\Omega A d\mu \ \varphi, \psi \right\rangle &= \int_\Omega \langle A(\omega)\varphi, \psi\rangle d\mu(\omega)\\ &= \int_\Omega \int_{X^2} k_A(\omega)(x,y)\varphi(y)\psi(x) d\mathrm{vol}_{X^2}(x,y) d\mu(\omega)\\ &= \int_\Omega \langle k_A(\omega), \psi \otimes \varphi\rangle d\mu(\omega)\\ &= \left\langle \int_\Omega k_A d\mu, \psi \otimes \varphi \right\rangle\end{aligned}$$

implies that the kernel $\int_\Omega k_A d\mu$ of $\int_\Omega A d\mu$ is given by an integral in $\mathcal{D}'_\Gamma$.

Hörmander spaces

Theorem (Dabrowski-Brouder; 2014)

In its normal topology, $\mathcal{D}'_\Gamma$ is a nuclear, semi-reflexive, semi-Montel, complete normal space of distributions.

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Theorem (Dabrowski-Brouder; 2014)

Bounded subsets are the same for the normal and Hörmander topology. Furthermore, closed bounded sets are compact and metrizable.

Fourier Integral Operator ζ-functions

- $\zeta(A)$ is meromorphic with isolated poles of finite order.
- $\exists r \in \mathbb{R}: \ \zeta(A)|_{\mathbb{C}_{\Re(\cdot)<r}}$ is holomorphic.

Definition ($\mathcal{D}'_{\Gamma,R,\Omega,\mathrm{plh}}$)

For $R \in \mathbb{R}$ and $\Omega \subseteq_{\text{open,connected}} \mathbb{C}$ such that $\forall r \in \mathbb{R}: \ \{z \in \Omega; \ \Re(z) < r\} \neq \emptyset$, we define $\mathcal{D}'_{\Gamma,R,\Omega,\mathrm{plh}} \subseteq C^\omega(\mathbb{C}, \mathcal{D}'_\Gamma)$ to be the set of gauged poly-log-homogeneous kernels in $\mathcal{D}'_\Gamma$ whose ζ-functions are holomorphic in Ω and none of the degrees of homogeneity at zero have real part greater than R.

Theorem

Let $R \in \mathbb{R}$ *and* $\Omega \subseteq \mathbb{C}$ *be open and connected such that* $\forall r \in \mathbb{R}: \ \{z \in \Omega; \ \Re(z) < r\} \neq \varnothing$. *Then,*

$$\zeta|_{\mathcal{D}'_{\Gamma,R,\Omega,\mathrm{plh}}} : \ \mathcal{D}'_{\Gamma,R,\Omega,\mathrm{plh}} \to C^{\omega}(\Omega)$$

has a quasi-complete extension $\zeta_{R,\Omega} \subseteq C^{\omega}(\mathbb{C}, \mathcal{D}'_{\Gamma}) \oplus C^{\omega}(\Omega)$.

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- Let $(v_\alpha, \zeta(v_\alpha))_{\alpha \in A}$ be a bounded net in $\mathcal{D}'_{\Gamma,R,\Omega,\mathrm{plh}} \oplus C^\omega(\Omega)$, $v_\alpha \to 0$, $\zeta(v_\alpha) \to: v \in C^\omega(\Omega)$, and $z \in \Omega$
- $V := \{v_\alpha(z);\ \alpha \in A\}$ bounded in $\mathcal{D}'_\Gamma \Rightarrow$ metrizable
- $Z := \{\zeta(v_\alpha)(z);\ \alpha \in A\} \cup \{v(z)\}$ bounded in $\mathbb{C} \Rightarrow$ metrizable
- $\{(v_\alpha(z), \zeta(v_\alpha)(z); \alpha \in A\} \subseteq V \times Z$ (metrizable set)$\Rightarrow$ suffices to use sequences

Theorem

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- Let $(u_n(z))_n \in V^{\mathbb{N}}$, $u_n(z) \to 0$, $\zeta(u_n)(z) \to v(z)$, $(f_m)_m$ δ-sequence $\to$ δ_{diag}
- then $\forall m :\ \lim_n \langle u_n, f_m \rangle = 0$ compactly
- for $\Re(z) \ll 0 :\ \lim_m \langle u_n(z), f_m \rangle = \zeta(u_n)(z)$
- $\frac{\varepsilon}{3}$ argument:
  $|v(z)| \le |v(z) - \zeta(u_n)(z)| + |\zeta(u_n)(z) - \langle u_n(z), f_m \rangle| + |\langle u_n(z), f_m \rangle| \le \varepsilon$
- $(\Re(z) \ll 0 \Rightarrow v(z) = 0) \Rightarrow v = 0$

Theorem

Let $R\in\mathbb{R}$ *and* $\Omega\subseteq\mathbb{C}$ *be open and connected such that* $\forall r\in\mathbb{R}:\ \{z\in\Omega;\ \Re(z)<r\}\neq\varnothing$. *Then,*

$$\zeta|_{\mathcal{D}'_{\Gamma,R,\Omega,\mathrm{plh}}}:\ \mathcal{D}'_{\Gamma,R,\Omega,\mathrm{plh}}\to C^\omega(\Omega)$$

has a quasi-complete extension $\zeta_{R,\Omega}\subseteq C^\omega(\mathbb{C},\mathcal{D}'_\Gamma)\oplus C^\omega(\Omega)$.

Corollary

Let $R_1,R_2\in\mathbb{R}$, $R_1\leq R_2$, *and* $\Omega\subseteq\mathbb{C}$ *be open and connected such that* $\forall r\in\mathbb{R}:\ \{z\in\Omega;\ \Re(z)<r\}\neq\varnothing$. *Then, it is possible to choose* $\zeta_{R_1,\Omega}$ *and* $\zeta_{R_2,\Omega}$ *such that* $\zeta_{R_1,\Omega}\subseteq\zeta_{R_2,\Omega}$.

ζ-extensions on joint holomorphic domains

Theorem

Let $\Omega \subseteq \mathbb{C}$ *be open and connected such that* $\forall r \in \mathbb{R}: \ \{z \in \Omega;\ \Re(z) < r\} \neq \varnothing$. *Then,*

$$\zeta_\Omega := \bigcup_{N \in \mathbb{N}} \zeta_{N,\Omega} \subseteq C^\omega(\mathbb{C}, \mathcal{D}'_\Gamma) \oplus C^\omega(\Omega)$$

is a quasi-complete operator.

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- Strict inductive limits of quasi-complete spaces are quasi-complete.

Let E be a vector space, (X_ι, τ_ι) a family of locally convex topological vector spaces, and $(f_\iota)_{\iota\in I}$ a family of linear maps $f_\iota: X_\iota \to E$.

(i) Then there exists a finest linear, locally convex topology τ on E for which all $f_\iota: (X_\iota, \tau_\iota) \to (E, \tau)$ are continuous. τ is called the final topology of E with respect to $(X_\iota, \tau_\iota, f_\iota)_{\iota\in I}$.
The final topology is called a locally convex inductive limit if an only if I is directed and $E = \bigcup_{\iota\in I} X_\iota$. Furthermore, the inductive limit is strict if and only if $X_\iota \subseteq X_\kappa \Rightarrow \tau_\iota = \tau_\kappa \cap X_\iota$.

(ii) Let (F, σ) be another locally convex topological vector space and $g: E \to F$ linear. Then $g: (E, \tau) \to (F, \sigma)$ is continuous if and only if $\forall \iota \in I: \ g \circ f_\iota \in C((X_\iota, \tau_\iota), (F, \sigma))$.

Let $\leq$ be a pre-order (reflexive and transitive binary relation) on the set A. Then, $(A, \leq)$ is called directed if and only $\forall a, b \in A \ \exists c \in A: \ a \leq c \ \wedge \ b \leq c$.

Example

Consider $\Omega \subseteq \mathbb{R}^n$ open and let

$$\mathcal{D}_K(\Omega) := \{ f \in C_c^\infty(\Omega);\ \operatorname{spt} f \subseteq K \}.$$

Then $\mathcal{D}_K(\Omega)$ is a Fréchet space with the seminorms $\left(\left\| \partial^k f \right\|_{L_\infty(K)} \right)_{k \in \mathbb{N}_0}$.
Since $K \subseteq K'$ implies $\mathcal{D}_K(\Omega) \subseteq \mathcal{D}_{K'}(\Omega)$, and $\tau_{\mathcal{D}_K(\Omega)} = \tau_{\mathcal{D}_{K'}(\Omega)} \cap \mathcal{D}_K(\Omega)$, we observe that $C_c^\infty(\Omega)$ is a strict inductive limit $\bigcup_{K \subseteq_{\text{compact}} \Omega} \mathcal{D}_K(\Omega)$. A strict inductive limit of Fréchet spaces is also known as LF-space.

Pettis integration in ζ_Ω

Theorem

Let (K,Σ,μ) *be a measure space, and* $f: K\to D(\zeta_\Omega)$ *and* $\zeta_\Omega\circ f$ *be* μ*-Pettis integrable (e.g.,* f *continuous,* K *compact, and* μ *a Borel measure). Then,*

$$\int_K f d\mu \in D\left(\zeta_\Omega\right)$$

and

$$\zeta_\Omega\left(\int_K f d\mu\right) = \int_K \zeta_\Omega\circ f d\mu.$$

Space of ζ-functions M_ζ

Definition

- $f \in M_\zeta$ if and only if $f \in L_{1,\mathrm{loc}}(\mathbb{C}) \cap W^1_{1,\mathrm{loc}}(\mathbb{R}^2)$ and there exists $r \in \mathbb{R}$ such that $f|_{\mathbb{C}_{\Re(\cdot)<r}}$ is holomorphic.

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  $f, g \in M_\zeta,\ f \sim g$ if and only if $\exists r \in \mathbb{R}:\ f|_{\mathbb{C}_{\Re(\cdot)<r}} = g|_{\mathbb{C}_{\Re(\cdot)<r}}$.

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- $D := \left\{\Omega \subseteq_{\mathrm{open,connected}} \mathbb{C};\ \exists r \in \mathbb{R}:\ \mathbb{C}_{\Re(\cdot)<r} \subseteq \Omega\right\}$

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- $D := \left\{\Omega \subseteq_{\mathrm{open,connected}} \mathbb{C};\ \exists r \in \mathbb{R}:\ \mathbb{C}_{\Re(\cdot)<r} \subseteq \Omega\right\}$
- for $\Omega \in D$, $H_\zeta(\Omega) := \left\{f \in M_\zeta;\ f|_\Omega \text{ holomorphic}\right\}$

Space of ζ-functions M_ζ

Corollary

$(D, \supseteq)$ *is a directed set.*

Corollary

$$M_\zeta = \bigcup_{\Omega \in D} H_\zeta(\Omega)$$

Laurent coefficients in M_ζ

Theorem (Trace Operator)

Let $G\subseteq\mathbb{R}^2$ *be open, bounded, connected, and have Lipschitz boundary. Then, there exists* $T\in L(W_1^1(G),L_1(\partial G))$ *such that* $\forall u\in C(\bar{G}):\ Tu=u|_{\partial G}$.

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- For $\zeta(A)$, consider

$$I_A := \{\Omega \in D;\ \exists f_\Omega \in H_\zeta(\Omega):\ f_\Omega|_\Omega = \zeta(A)|_\Omega\}.$$

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$$I_A := \{\Omega \in D; \ \exists f_\Omega \in H_\zeta(\Omega) : \ f_\Omega|_\Omega = \zeta(A)|_\Omega\}.$$
- $(I_A, \subseteq)$ is directed.
- Let $G \subseteq \mathbb{C}$ be open, bounded, connected, and with ∂G Lipschitz such that $\zeta(A)$ is continuous on ∂G, $z_0 \in G$, and $\zeta(A) \simeq [f_\Omega] \in M_\zeta$. Then,
$$\lim \left(\frac{1}{2\pi i} \int_{\partial G} \frac{T f_\Omega(z)}{(z - z_0)^{n+1}} dz \right)_{\Omega \in I_A} = \frac{1}{2\pi i} \int_{\partial G} \frac{\zeta(A)(z)}{(z - z_0)^{n+1}} dz.$$

Topology of $H_\zeta(\Omega)$

Definition

On $H_\zeta(\Omega)$, we consider the semi-norms generated by the quotient of $W^1_{1,\mathrm{loc}}(\mathbb{R}^2) \cap C^\omega(\Omega)$ equipped with the semi-norms

$$\forall K \subseteq_{\mathrm{compact}} \Omega : \; p_K^H(f) := \left\| f|_K \right\|_{C(K,\mathbb{C})}$$
$$\forall K \subseteq_{\mathrm{compact}} \mathbb{R}^2 : \; p_K^W(f) := \left\| f|_K \right\|_{L_1(K,\mathbb{C})} + \left\| f'|_K \right\|_{L_1(K,\mathbb{R}^{2,2})}$$

with respect to $f \sim g \; \Leftrightarrow \; \exists r \in \mathbb{R} : \; f|_{\mathbb{C}_{\Re(\cdot)<r}} = g|_{\mathbb{C}_{\Re(\cdot)<r}}$.

Topology of $H_\zeta(\Omega)$ and the space of ζ-functions M_ζ

Theorem

- *$H_\zeta(\Omega)$ is a Fréchet space.*
- *Let $\Omega_0, \Omega_1 \in D$ and $\Omega_0 \supseteq \Omega_1$. Then, $H_\zeta(\Omega_0) \subseteq H_\zeta(\Omega_1)$ and the topology induced by $H_\zeta(\Omega_1)$ coincides with the topology in $H_\zeta(\Omega_0)$.*
  Furthermore, $H_\zeta(\Omega_0)$ is closed in $H_\zeta(\Omega_1)$.

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  Furthermore, $H_\zeta(\Omega_0)$ is closed in $H_\zeta(\Omega_1)$.

Corollary

$M_\zeta = \bigcup_{n\in\mathbb{N}} H_\zeta\left(\mathbb{C}_{\Re(\cdot)<-n}\right)$ endowed with the strict inductive limit topology is a complete Hausdorff LF-space.

Lemma

All $\zeta|_{\mathcal{D}'_{\Gamma,n,\mathbb{C}_{\Re(\cdot)<-n},\mathrm{plh}}}$ *have a quasi-complete extension* $\tilde{\zeta}_{n,\mathbb{C}_{\Re(\cdot)<-n}}$ *in* $C^{\omega}\left(\mathbb{C},\mathcal{D}'_{\Gamma}\right)\oplus H_{\zeta}\left(\mathbb{C}_{\Re(\cdot)<-n}\right)$ *satisfying* $\tilde{\zeta}_{n,\mathbb{C}_{\Re(\cdot)<-n}}\subseteq\tilde{\zeta}_{n+1,\mathbb{C}_{\Re(\cdot)<-n-1}}$.

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- $(v_\alpha,\zeta(v_\alpha))_{\alpha\in A}$ bounded net, $v_\alpha\to 0$ in $C^{\omega}\left(\mathbb{C},\mathcal{D}'_{\Gamma}\right)$, $\zeta(v_\alpha)\to: v$ in $H_{\zeta}(\mathbb{C}_{\Re(\cdot)<-n})$
- $\Rightarrow\zeta(v_\alpha)|_{\mathbb{C}_{\Re(\cdot)<-n}}\to v|_{\mathbb{C}_{\Re(\cdot)<-n}}$ in $C^{\omega}(\mathbb{C}_{\Re(\cdot)<-n})$
- same proof as before: $v|_{\mathbb{C}_{\Re(\cdot)<-n}}=0$
- hence $\zeta(v_\alpha)=0$

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All $\zeta|_{\mathcal{D}'_{\Gamma,n,\mathbb{C}_{\Re(\cdot)<-n},\mathrm{plh}}}$ *have a quasi-complete extension* $\tilde{\zeta}_{n,\mathbb{C}_{\Re(\cdot)<-n}}$ *in* $C^\omega\left(\mathbb{C},\mathcal{D}'_\Gamma\right)\oplus H_\zeta\left(\mathbb{C}_{\Re(\cdot)<-n}\right)$ *satisfying* $\tilde{\zeta}_{n,\mathbb{C}_{\Re(\cdot)<-n}}\subseteq\tilde{\zeta}_{n+1,\mathbb{C}_{\Re(\cdot)<-n-1}}$.

$$\mathcal{D}'_{\Gamma,\mathrm{plh}} := \overline{\bigcup_{n\in\mathbb{N}}\mathcal{D}'_{\Gamma,n,\mathbb{C}_{\Re(\cdot)<-n},\mathrm{plh}}}^{C^\omega(\mathbb{C},\mathcal{D}'_\Gamma)} \subseteq C^\omega(\mathbb{C},\mathcal{D}'_\Gamma)$$

$$\zeta_{\Gamma,\mathrm{plh}} := \bigcup_{n\in\mathbb{N}}\tilde{\zeta}_{n,\mathbb{C}_{\Re(\cdot)<-n}}\subseteq\mathcal{D}'_{\Gamma,\mathrm{plh}}\oplus M_\zeta$$

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All $\zeta|_{\mathcal{D}'_{\Gamma,n,\mathbb{C}_{\Re(\cdot)<-n},\mathrm{plh}}}$ *have a quasi-complete extension* $\tilde{\zeta}_{n,\mathbb{C}_{\Re(\cdot)<-n}}$ *in* $C^\omega\left(\mathbb{C},\mathcal{D}'_\Gamma\right)\oplus H_\zeta\left(\mathbb{C}_{\Re(\cdot)<-n}\right)$ *satisfying* $\tilde{\zeta}_{n,\mathbb{C}_{\Re(\cdot)<-n}}\subseteq\tilde{\zeta}_{n+1,\mathbb{C}_{\Re(\cdot)<-n-1}}$.

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$$\zeta_{\Gamma,\mathrm{plh}}:=\bigcup_{n\in\mathbb{N}}\tilde{\zeta}_{n,\mathbb{C}_{\Re(\cdot)<-n}}\subseteq\mathcal{D}'_{\Gamma,\mathrm{plh}}\oplus M_\zeta$$

Theorem

$\zeta_{\Gamma,\mathrm{plh}}\subseteq\mathcal{D}'_{\Gamma,\mathrm{plh}}\oplus M_\zeta$ *is a quasi-complete operator.*

Pettis integration in ζ

Theorem

Let (K,Σ,μ) *be a measure space, and* $f: K \to D(\zeta_{\Gamma,\mathrm{plh}})$ *and* $\zeta_{\Gamma,\mathrm{plh}} \circ f$ *be* μ*-Pettis integrable (e.g.,* f *continuous,* K *compact, and* μ *a Borel measure). Then,*

$$\int_K f d\mu \in D\left(\zeta_{\Gamma,\mathrm{plh}}\right)$$

and

$$\zeta_{\Gamma,\mathrm{plh}}\left(\int_K f d\mu\right) = \int_K \zeta_{\Gamma,\mathrm{plh}} \circ f d\mu.$$

Example (Heat trace)

Let

- M closed, compact C^∞-manifold,
- $|\Delta|$ be the (positive) Laplacian on M, and
- T the semi-group generated by $-|\Delta|$.

Then,

$$\forall t \in \mathbb{R}_{>0} : \operatorname{tr} T(t) = \frac{\mathrm{vol}(M)}{(4\pi t)^{\frac{\dim M}{2}}} + \frac{\text{total curvature}(M)}{3(4\pi)^{\frac{\dim M}{2}} t^{\frac{\dim M}{2}-1}} + \dots$$

Example (Heat trace)

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- M closed, compact C^∞-manifold,
- $|\Delta|$ be the (positive) Laplacian on M, and
- T the semi-group generated by $-|\Delta|$.
- W the (semi-)group generated by $i\sqrt{|\Delta|}$.

Then,

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$$\mathrm{tr}_{\mathrm{KV}} W(t) \rightsquigarrow \text{ wave trace invariants}$$

Example

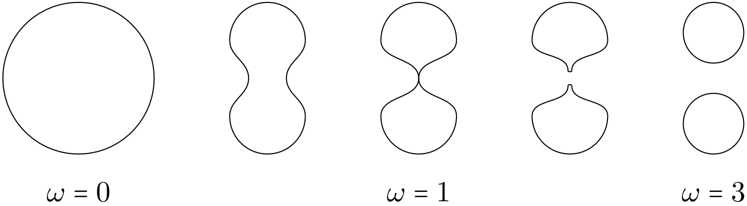
Let $(M,g(\omega))_{\omega\in\Omega}$ be a family of Riemannian C^∞-manifolds over a Radon measure probability space Ω such that $\Omega\ni\omega\mapsto T(\omega)=\left(t\mapsto e^{-|\Delta(\omega)|t}\right)$ and $\Omega\ni\omega\mapsto W(\omega)=\left(t\mapsto e^{i\sqrt{|\Delta(\omega)|}t}\right)$ bounded and almost separably valued. Then,

$$\mathbb{E}\zeta(T)=\zeta\left(\mathbb{E}T\right)\text{ and }\mathbb{E}\zeta(W)=\zeta\left(\mathbb{E}W\right).$$

In particular, if $\mathbb{E}T$ is the heat semi-group of some $(M,g_{\mathbb{E}})$ and $\mathbb{E}W$ the wave group of some $\left(M,g_{\mathbb{E},W}\right)$, then we the heat and wave invariants of $(M,g_{\mathbb{E}})$ and $\left(M,g_{\mathbb{E},W}\right)$ respectively coincide with the expected heat and wave invariants of $(M,g(\omega))_{\omega\in\Omega}$. E.g.,

$$\mathbb{E}\mathrm{vol}_g M=\mathrm{vol}_{g_{\mathbb{E}}}M.$$

Consider the probability space $\left([0,3],\mathcal{B}([0,3]),\frac{1}{3}\lambda\right)$ and the family of manifolds M_ω given by the following deformation from sphere to torus in $\mathbb{R}^3$.



Let W_ω be the wave group and T_ω the heat semi-group on M_ω. Then, $\omega \mapsto W_\omega$ and $\omega \mapsto T_\omega$ are bounded and almost separably valued, and $\mathbb{E}\zeta(W) = \zeta(\mathbb{E}W)$.

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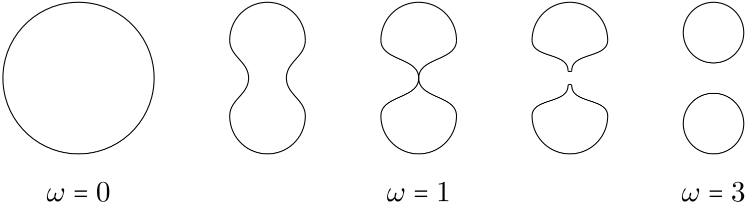

Let $D_\omega := d_\omega + d_\omega^*$ as a map from even to odd exterior powers of the cotangent bundle of M_ω. Then, $\operatorname{tr}\left(e^{-D_\omega^* D_\omega t} - e^{-D_\omega D_\omega^* t}\right) = \chi_{\text{Euler}}(M_\omega)$. Thus,

$$\mathbb{E}\chi_{\text{Euler}}(M) = 2\cdot\frac{1}{3} + 0\cdot\frac{2}{3} = \frac{2}{3}.$$

Theorem

$\operatorname{tr}:\ \Psi^{-\infty}\to\mathbb{C}$ *is continuous*

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$\mathrm{tr}: \Psi^{-\infty} \to \mathbb{C}$ *is continuous*

Theorem (Closed Graph Theorem)

Let X be an LF-space, Y a Fréchet space, and $T: X \to Y$ a linear operator (everywhere defined). Then, the following are equivalent.

(i) *T is continuous.*

(ii) *T is closed.*

(iii) *T is closable.*

Theorem

$\mathrm{tr} : \ \Psi^{-\infty} \to \mathbb{C}$ *is continuous*

- $(A_n)_n \in \mathbb{N} \in (\Psi^{-\infty})^{\mathbb{N}}$, $A_n \to 0$ in $\Psi^{-\infty}$, $\mathrm{tr} A_n \to: t$ in $\mathbb{C}$

Theorem

$\mathrm{tr}: \Psi^{-\infty} \to \mathbb{C}$ *is continuous*

- $(A_n)_n \in \mathbb{N} \in (\Psi^{-\infty})^{\mathbb{N}}$, $A_n \to 0$ in $\Psi^{-\infty}$, $\mathrm{tr} A_n \to: t$ in $\mathbb{C}$
- a_n symbol of $A_n \Rightarrow \forall m \in \mathbb{R}: a_n \to 0$ in $S^m(X \times X \times \mathbb{R}^{\dim X})$

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- Thus, $t \leftarrow \operatorname{tr} A_n = \tau(a_n) \to 0$, i.e., $t = 0$ and tr closable.

Index bundle

The index bundle of a family of operators $(f(\omega))_{\omega\in\Omega}$ is given by

$$\mathrm{IND}(f)(\omega) = \ker f(\omega) - \ker f(\omega)^*$$

as interpreted in the K-theory of isomorphism classes of vector bundles with the direct sum.
Here, we will consider the following construction. Let S be an abelian monoid. Then, we define

$$K(S) := S^2 /_{\{(x,y)\in S^2;\ x=y\}}$$

with the canonical injection $S \ni s \mapsto (s,0) \in K(S)$ and $\forall s \in S:\ -s = (0,s)$.

Index bundle

$$\begin{aligned}\mathrm{IND}(f)(\omega) &= \ker f(\omega) - \ker f(\omega)^* \\ &= (\ker f(\omega), 0) - (\ker f(\omega)^*, 0) \\ &= (\ker f(\omega), 0) + (0, \ker f(\omega)^*) \\ &= (\ker f(\omega), \ker f(\omega)^*)\end{aligned}$$

can be interpreted as $\ker f(\omega) \oplus \ker f(\omega)^*$ and, if each $f(\omega)$ is a closed Fredholm operator between Hilbert spaces H_0 and H_1, we obtain

$$\mathrm{IND}(f)(\omega) = \underbrace{\ker f(\omega) \oplus \ker f(\omega)^*}_{\subseteq H_0 \oplus H_1} \in \mathrm{CLR}(H_0, H_1).$$

Gap topology

Definition

Let H be a Hilbert space and $U, V \subseteq H$ closed linear (non-empty) subspaces. Then, we define

$$\delta_H(U,V) := \begin{cases} 0 & , \ U = \{0\} \\ \sup\{\operatorname{dist}_H(u,V);\ u \in U \cap \partial B_H\} & , \ U \neq \{0\} \end{cases}$$

and

$$\hat{\delta}_H(U,V) := \max\{\delta_H(U,V), \delta_H(V,U)\} = \|\operatorname{pr}_U - \operatorname{pr}_V\|_{L(H)} .$$

Then, $\left(CLR(H_0,H_1), \hat{\delta}_{H_0 \oplus H_1}\right)$ is a complete metric space.

Back to the index bundle

- Let $F(H_0,H_1):=\{f\in\mathrm{CLR}(H_0,H_1);\ f \text{ is a closed Fredholm operator}\}$.
- Let $\mathcal{P}(\mathrm{CLR}(H_0,H_1))$ be the power set of $\mathrm{CLR}(H_0,H_1)$.

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$$[\ker f(\omega) \oplus V_0] - [\ker f(\omega)^* \oplus V_1] = [\ker f(\omega)] - [\ker f(\omega)^*]$$

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  provided $\dim V_0 = \dim V_1$.
- Then, $\mathrm{IND}: \ F(H_0, H_1) \to \mathcal{P}(\mathrm{CLR}(H_0, H_1))$.

A topology on the set of index bundles

Definition

Let $x = \ker f - \ker f^* \in \mathrm{IND}[F(H_0, H_1)]$ and $\varepsilon \in \mathbb{R}_{>0}$. Then, we define $B_{\mathrm{IND}}(x, \varepsilon)$ as the set of $\ker g - \ker g^* \in \mathrm{IND}[F(H_0, H_1)]$ such that there exist $V_0 \subseteq_{\mathrm{lin}} (\ker g)^{\perp_{H_0}}$ and $V_1 \subseteq_{\mathrm{lin}} (\ker g^*)^{\perp_{H_1}}$ with

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The family

$$\{B_{\mathrm{IND}}(x, \varepsilon) \subseteq \mathrm{IND}[F(H_0, H_1)];\ x \in \mathrm{IND}[F(H_0, H_1)],\ \varepsilon \in \mathbb{R}_{>0}\}$$

defines a subbasis of the topology $\mathcal{T}_{\mathrm{IND}}$ of $\mathrm{IND}[F(H_0, H_1)]$.

Theorem

Let H_0 and H_1 be Hilbert spaces. Then,

$$\mathrm{IND} \in C\Big(\big(F(H_0,H_1), \hat{\delta}_{H_0\oplus H_1}\big) \ , \ (\mathrm{IND}[F(H_0,H_1)], \mathcal{T}_{\mathrm{IND}}) \Big)$$

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Corollary

Let H_0 and H_1 be Hilbert spaces, Ω a topological space, $f \in C(\Omega, F(H_0,H_1))$, and $g \in \mathcal{M}(\Omega, F(H_0,H_1))$. Then,

$$\begin{aligned} & \mathrm{IND} \circ f \in C\left(\Omega, (\mathrm{IND}[F(H_0,H_1)], \mathcal{T}_{\mathrm{IND}})\right) \\ \textit{and} \quad & \mathrm{IND} \circ g \in \mathcal{M}\left(\Omega, (\mathrm{IND}[F(H_0,H_1)], \mathcal{T}_{\mathrm{IND}})\right). \end{aligned}$$

Example

Let H_0 and H_1 be Hilbert spaces, and

$$\mathrm{DIM}: \ \mathrm{IND}[F(H_0,H_1)] \to \mathbb{Z}; \ \ker f - \ker f^* \mapsto \operatorname{ind} f.$$

Then,

$$\mathrm{DIM} \in C\left((\mathrm{IND}[F(H_0,H_1)], \mathcal{T}_{\mathrm{IND}}), \mathbb{Z} \right).$$

Furthermore, let Ω a topological space, $f \in C(\Omega, F(H_0,H_1))$, and $g \in \mathcal{M}(\Omega, F(H_0,H_1))$. Then,

$$\begin{aligned} \operatorname{ind} \circ f &= \mathrm{DIM} \circ \mathrm{IND} \circ f \in C(\Omega, \mathbb{Z}) \\ \text{and} \quad \operatorname{ind} \circ g &= \mathrm{DIM} \circ \mathrm{IND} \circ g \in \mathcal{M}(\Omega, \mathbb{Z}). \end{aligned}$$

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- Atiyah, Patodi, Singer defined spectral flow of paths of bounded self-adjoint Fredholm operators as the first Chern number of the “self-adjoint index bundle”
- Is the first Chern number continuous/measurable with respect to $\mathcal{T}_{\mathrm{IND}}$?
- If so ⇒ stochastic versions of the spectral flow.