Introduction to Microlocal Analysis

Third lecture: Distributions on manifolds

Ingo Witt (Göttingen)

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1. Pseudodifferential analysis on manifolds

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Densities

To define distributions on a $\mathscr{C}^\infty$ manifold X, one needs an invariant integration on X. This is achieved with the help of densities bundles.

Definition

The 1-density bundle is the line bundle $\Omega^1 = \Omega^1_X$ over X with transition function $|\det(\partial x/\partial y)|$ for coordinate changes $x \mapsto y$ on X.

Corollary

There is an invariant integration $\displaystyle\int_X : L^1_{\mathrm{c}}(X;\Omega^1) \to \mathbb{C}$.

Indeed, on local coordinates,

$$\int_X u = \int_U u(x)\,\mathrm{d}x = \int_V u(x)\,|\det(\partial x/\partial y)|\,\mathrm{d}y$$

for coordinate charts $U, V \subseteq X$, and $\operatorname{supp} u \subseteq U \cap V$.

Distributional sections

Definition

For a $\mathscr{C}^\infty$ $\mathbb{C}$-vector bundle E over X, $\mathscr{D}'(X;E)$ is defined as the topological dual of $\mathscr{C}_c^\infty(X;E'\otimes\Omega^1)$.

Remark (Justification)

It holds $L^1_{loc}(X;E)\subset\mathscr{D}'(X;E)$ canonically. For $f\in L^1_{loc}(X;E)$, the action on $\mathscr{C}_c^\infty(X;E'\otimes\Omega^1)$ is given by $\phi\mapsto\int_X\langle f,\phi\rangle_{E,E'}$.

Notation

The dual pairing between $\mathscr{D}'(X;E)$ and $\mathscr{C}_c^\infty(X;E'\otimes\Omega^1)$ will be denoted by

$$\langle u,\phi\rangle=\int_X\langle u,\phi\rangle_{E,E'}.$$

The Schwartz kernel theorem

Theorem (Schwartz kernel theorem, revised)

There is an one-to-one correspondence between linear continuous operators $A\colon \mathscr{C}_c^\infty(Y;E) \to \mathcal{D}'(X;F)$ *and their kernels* $K \in \mathscr{D}'(X \times Y; F \boxtimes (E' \otimes \Omega_Y^1))$*. For* $u \in \mathscr{C}_c^\infty(X; F' \otimes \Omega_X^1)$*,* $v \in \mathscr{C}_c^\infty(Y;E)$*,*

$$\langle Av, u\rangle = \int_{X\times Y} \langle K, u \otimes v\rangle_{F\boxtimes E', F'\boxtimes E}.$$

Note that $\Omega_x^1 \boxtimes \Omega_Y^1 \cong \Omega_{X\times Y}^1$, and then the necessary identifications are easily made.

Remark

It is a common strategy in modern (harmonic, microlocal, etc.) analysis to study linear operators through their kernels.

Pseudodifferential operators, I

Definition

A linear operator $A\colon \mathscr{C}_c^\infty(X;E) \to \mathscr{C}^\infty(X;F)$ is said to be a pseudodifferential operator of order $m \in \mathbb{R}$ and type $1,0$ if

- $\operatorname{sing\,supp} K \subseteq \Delta_X$,
- $A \in \Psi^m_{1,0}(U;\mathbb{C}^K,\mathbb{C}^L)$ in any coordinate patch $U \subseteq X$ over which $E \cong U \times \mathbb{C}^K$, $F \cong U \times \mathbb{C}^L$ are trivial.

The space $\Psi^\mu_{\mathrm{cl}}(X;E,F)$ of classical pseudodifferential operators of order $\mu \in \mathbb{C}$ is similarly defined.

- $\Psi^\mu_{\mathrm{cl}}(X;E,F) \subset \Psi^m_{1,0}(X;E,F)$, where $m = \Re\mu$.
- $\operatorname{Diff}^m(X;E,F) \subset \Psi^m_{\mathrm{cl}}(X;E,F)$.

- If $A \in \Psi^m_{1,0}(X;E,F)$, then $A\colon \mathscr{E}'(X;E) \to \mathscr{D}'(X;F)$.

Pseudodifferential operators, II

- $A \in \Psi^m_{1,0}(X; E, F)$ is said to be properly supported if both projections $\pi_L \colon \operatorname{supp} K \to X$ and $\pi_R \colon \operatorname{supp} K \to X$ are proper. In this case, A respects compact supports. Moreover, $A \colon \mathscr{C}^\infty(X; E) \to \mathscr{C}^\infty(X; F)$ and $A \colon \mathscr{D}'(X; E) \to \mathscr{D}'(X; F)$.
- (Composition) If $A \in \Psi^m_{1,0}(X; E, F)$, $B \in \Psi^{m'}_{1,0}(X; F, G)$, and one of them is properly supported, then $B \circ A \in \Psi^{m+m'}(X; E, F)$. If both A, B are classical, then $B \circ A$ is classical.
- (Adjoints) If $A \in \Psi^m_{1,0}(X; E, F)$, then $A^* \in \Psi^m_{1,0}(X; F^* \otimes \Omega^1, E^* \otimes \Omega^1)$. If A is classical, then A^* is classical.
- (Mapping properties) If $A \in \Psi^m_{1,0}(X; E, F)$, then $A \colon H^{\sigma+m}_{\mathrm{c}}(X; E) \to H^\sigma_{\mathrm{loc}}(X; F)$ for any $\sigma \in \mathbb{R}$.

Principal symbol

- $A \in \Psi^\mu_{\mathrm{cl}}(X; E, F)$ possesses a well-defined principal symbol $\sigma^\mu(A) \in S^{(\mu)}(\dot{T}^*X; \mathrm{Hom}(\pi^*E, \pi^*F))$, where $\pi\colon \dot{T}^*X \to X$ is the canonical projection.
- If $Au(x) = \int e^{i(x-y)\xi} a(x, y, \xi) u(y)\, dy đ\xi$ in local coordinates, then

$$\sigma^\mu(A)(x, \xi) = a_{(\mu)}(x, x, \xi).$$

- $\sigma^{\mu+\mu'}(B \circ A) = \sigma^{\mu'}(B)\, \sigma^\mu(A)$.

Theorem

The principal symbol map fits into a short exact sequence

$$0 \to \Psi^{\mu-1}_{\mathrm{cl}}(X; E, F) \to \Psi^\mu_{\mathrm{cl}}(X; E, F) \xrightarrow{\sigma^\mu} S^{(\mu)}(\dot{T}^*X; \mathrm{Hom}(\pi^*E, \pi^*F)) \to 0.$$

Ellipticity

Definition

$A \in \Psi^{\mu}_{\mathrm{cl}}(X; E, F)$ is said to be elliptic if $\sigma^{\mu}(A)$ is pointwise a linear isomorphism.

Recall that $\sigma^{\mu}(A)(x, \xi)\colon E_x \to F_x$ is a linear map for $(x, \xi) \in \dot{T}^* X$.

Theorem

Let $A \in \Psi^{\mu}_{\mathrm{cl}}(X; E, F)$ be elliptic. Then there is a parametrix, i.e., a properly supported pseudodifferential operator $B \in \Psi^{-\mu}_{\mathrm{cl}}(X; F, E)$ such that both $A \circ B - 1$, $B \circ A - 1$ have a $\mathscr{C}^{\infty}$ kernel.

Corollary (Elliptic regularity)

Let $A \in \Psi^{\mu}_{\mathrm{cl}}(X; E, F)$ be elliptic, $u \in \mathscr{E}'(X; E)$ and $Au \in H^{\sigma}_{\mathrm{loc}}(X; F)$ for some $\sigma \in \mathbb{R}$. Then $u \in H^{\sigma+m}_{\mathrm{c}}(X; E)$, where $m = \Re\mu$.

Basics on symplectic geometry

- $\dot{T}^*X$ is a homogeneous symplectic manifold with symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}\xi = \mathrm{d}x_1 \wedge \mathrm{d}\xi_1 + \ldots + \mathrm{d}x_n \wedge \mathrm{d}\xi_n$. It holds $\omega = -\mathrm{d}\alpha$, where $\alpha = \omega(\cdot, R) = \xi\,\mathrm{d}x = \xi_1\mathrm{d}x_1 + \ldots + \xi_n\mathrm{d}x_n$ is the canonical 1-form and $R = \xi\partial/\partial\xi$ is the radial vector field.
- Given $p \in \mathscr{C}^\infty(\dot{T}^*X; \mathbb{R})$, we define the Hamilton vector field H_p by $\omega(\cdot, H_p) = \mathrm{d}p$, i.e.,

$$H_p = \frac{\partial p}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial}{\partial \xi}.$$

- In particular, the integral curves $(x(t), \xi(t))$ of H_p are given as solutions of the Hamilton equations

$$\dot{x}(t) = \frac{\partial p}{\partial \xi}(x(t), \xi(t)), \quad \dot{\xi}(t) = -\frac{\partial p}{\partial x}(x(t), \xi(t)).$$

Note that p is constant entlang the integral curves of H_p, for

$$\frac{\mathrm{d}}{\mathrm{d}t}p(x(t), \xi(t)) = p_x\dot{x} + p_\xi\dot{\xi} = p_x p_\xi - p_\xi p_x = 0.$$

First-order hyperbolic systems

Now let $X = \mathbb{R}^n$ or X be compact.

We consider the Cauchy problem

$$(1) \qquad D_t u = a(t, x, D_x)u + f(t, x), \quad u\big|_{t=0} = g(x)$$

where $a \in \mathscr{C}^\infty([0, T]; \Psi^1_{\mathrm{cl}}(X; \mathbb{C}^N))$. We assume that $\sigma^1(a)(t, x, \xi)$ is real-valued.

Proposition

Under these condition, Eq. (1) *possesses a unique solution $u \in \mathscr{C}^k([0, T]; H^\sigma(X; \mathbb{C}^N))$ provided that $f \in W^{k,1}((0, T); H^\sigma(X; \mathbb{C}^N))$ and $g \in H^\sigma(X; \mathbb{C}^N)$.*

Proof. Energy estimates combined with functional-analytic arguments. □

Propagator and fundamental solution

It follows that there are invertible linear operators (propagator)

$$U(t,s)\colon H^{\sigma}(X;\mathbb{C}^N) \to H^{\sigma}(X;\mathbb{C}^N)$$

for $0 \le s,\ t \le T$ with $U(t,t) = 1$ and $U(t,r) \circ U(r,s) = U(t,s)$ for $0 \le s,\ r,\ t \le T$ and a linear operator (forward fundamental solution)

$$E\colon W^{1,k}((0,T);H^{\sigma}(X;\mathbb{C}^N)) \to \mathscr{C}^k([0,T];H^{\sigma}(X;\mathbb{C}^N))$$

with the property that the solution of Eq. (1) equals

$$u = U(\cdot,0)g + Ef.$$

Proposition

It holds

$$Ef(t,\cdot) = \mathrm{i}\int_0^t U(t,s)f(s,\cdot)\,\mathrm{d}s, \quad 0 \le t \le T.$$

Egorov's theorem

We shall see later that $U(t, s)$ is a Fourier integral operator.

Remark

The fundamental solution E is more complicated. In fact, its kernel is a so-called one-sided paired Lagrangian distribution.

Theorem (Egorov)

Let $P \in \Psi^{\mu}_{\mathrm{cl}}(X; \mathbb{C}^N)$*. Then* $P(t) = U(t, 0)PU(0, t) \in \Psi^{\mu}_{\mathrm{cl}}(X; \mathbb{C}^N)$ *and*

$$\sigma^{\mu}(P(t)) = \sigma^{\mu}(P) \circ \chi_{t,0},$$

where $\chi_{t,0}$ *is the flow of the time-dependent Hamilton vector field* $H_{\sigma^1(a)}$ *from time* t *to time* 0.

Propagation of singularities

Let $P \in \Psi^{\mu}_{\mathrm{cl}}(X; E)$. Recall that

$$\mathrm{WF}(Pu) \subseteq \mathrm{WF}(u) \subseteq \mathrm{WF}(Pu) \cup \mathrm{Char}\, P.$$

Theorem

Let $q = \det \sigma^m(P)$ be real-valued. Then $\mathrm{WF}(u) \setminus \mathrm{WF}(Pu)$ is invariant under the flow of the Hamilton vector field H_q.

Proof. There are several known proofs. One employs Egorov's theorem. □

Remark

This statement on the propagation of singularities is a microlocalized version of the basic energy inequalities.

Lagrangian distributions

Let $\Lambda \subset \dot{T}^*X$ be a conic Lagrangian submanifold. In particular, $\alpha\big|_\Lambda = 0$ and $\dim\Lambda = \dim X$.

Definition

$u \in \mathscr{D}'(X)$ is said to be a classical Lagrangian distribution with respect to Λ, of order $\mu \in \mathbb{C}$, if

- $\mathrm{WF}(u) \subseteq \Lambda$,
- microlocally near any $\lambda \in \Lambda$,

$$u(x) = (2\pi)^{-(n+2N)/4} \int e^{i\varphi(x,\theta)} a(x,\theta)\, d\theta,$$

  where φ is a non-degenerate phase function parametrizing Λ $(= \Lambda_\varphi)$ near λ and $a \in S_{\mathrm{cl}}^{\mu+(n-2N)/4}(\mathbb{R}^n \times \mathbb{R}^N)$.

- We write $u \in I_{\mathrm{cl}}^{\mu}(X,\Lambda)$.

Examples of Lagrangian distributions

- Conormal distributions: $I^\mu_{\mathrm{cl}}(X, Y) = I^\mu_{\mathrm{cl}}(X, \Lambda)$, where $Y \subset X$ is a submanifold and $\Lambda = \dot{N}^* Y$ is the conormal bundle of Y in X.
- In particular, $\delta(x) = \int e^{ix\cdot\xi}\, đ\xi$ belongs to $I^{n/4}_{\mathrm{cl}}(\mathbb{R}^n, \{0\})$.
- $A \in \Psi^\mu_{\mathrm{cl}}(X)$ means exactly that the kernel of A belongs to $I^\mu_{\mathrm{cl}}(X \times X, \Delta_X; \mathbb{C} \boxtimes \Omega^1)$.

Theorem

The kernel of the operator $U(\cdot, 0)$ *belongs to* $I^{-1/4}_{\mathrm{cl}}((0, T) \times X \times X, \Lambda)$, *where*

$$\Lambda' = \{(t, \tau, x, \xi, y, \eta) \mid \tau = \sigma^1(a)(t, x, \xi),\ \chi_{t,0}(x, \xi) = (y, \eta)\}$$

Proof. Geometric optics construction. □