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Introductory notes on microlocal analysis in QFT on curved spacetimes

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1. QFT ON CURVED SPACETIMES

1.1. Lorentzian manifolds.

Definition. Lorentzian manifold (M, g): M smooth manifold, g Lorentzian metric, i.e. a smooth map $M \ni x \mapsto g(x)$, where $g(x)$ is a sym. bilinear form on T_xM of signature $(1, n-1)$.

Definition. A vector $v \in T_xM$ is time-like if $v \cdot g(x)v < 0$, null if $v \cdot g(x)v = 0$, space-like if $v \cdot g(x)v > 0$.

A spacetime is a Lorentzian manifold (M, g) equipped with a *time-orientation*, i.e. a continuous time-like Killing vector field. This splits cone of time-like vector fields $C(x) \subset T_xM$ into two components $C^{\pm}(x)$.

Definitions. A piecewise C^1 curve $\gamma : I \to M$ is causal if its tangent vectors are time-like or null. If $K \subset M$, its causal future/past is $J_{\pm}(K) := \bigcup_{x \in K} J_{\pm}(x)$, where

$$J_{\pm}(x) := \{\gamma(s) : \ \gamma \text{ causal future/past directed starting at } x, \ s \in \mathbb{R}\}.$$

We set $J(K) := J_+(K) \cup J_-(K)$. One says $K_1, K_2 \subset M$ are causally separated if $J(K_1) \cap K_2 = \varnothing$.

1.2. Introduction to QFT.

Let (M, g) be a spacetime. Let $m \in \mathbb{R}$, and

$$P = -\Box_g + m^2 = -|g|^{-\frac{1}{2}}\partial_a |g|^{\frac{1}{2}} g^{ab}\partial_b + m^2 \ \text{ (the } \textit{Klein-Gordon operator}\text{)}.$$

Linear quantum fields: $\phi \in \mathcal{D}'(M; \mathcal{H})$ with values in Hilbert space $\mathcal{H}$ s.t. $P\phi = 0$ and:

(1) $\phi(v)^* = \phi(v)$ for $v \in C^\infty_{\rm c}(M; \mathbb{R})$ (where $\phi(v) = \int_M \phi(x) d\mathrm{vol}_g(x)$)
(2) $\exists \Omega \in \mathcal{H}$ s.t.

$$\{\phi(v_1) \dots \phi(v_i)\Omega : \ v_1, \dots, v_i \in C^\infty_{\rm c}(M), \ i \in \mathbb{N}\}$$

is dense in $\mathcal{H}$
(3) $[\phi(x), \phi(x')] = 0$ if $x, x' \in M$ are space-like separated
(canonical choice: $[\phi(x), \phi(x')] = \mathrm{i}G(x, x')\mathbf{1}$)

If $(M, g) = \mathbb{R}^{1,d}$ and $m > 0$, $\phi_{\mathrm{vac}}(x)$ is the reference dynamics for non-interacting (non-linear) fields.

In general, no canonical choice of $\phi(x)$: we can probe quantum effects induced by the geometry.

Difficulties:

- $\mathcal{H}$ not a priori given!
- $\phi(x)$ very singular, $\phi(x)^2$ does not exist
- locally, $\phi(x)$ should ressemble $\phi_{\mathrm{vac}}(x)$

This boils down to **two-point functions**

$$\Lambda^+(x, x') := (\Omega|\phi(x)\phi(x')\Omega).$$

The program is to construct first $\Lambda^+(x, x')$.

Remark 1. Formally, $(\Omega|\phi^2(x)\Omega) = \lim_{x\to x'} \Lambda^+(x, x') = \infty$

Remark 2. Necessarily, $\Lambda^+ \geqslant 0$. Other global or asymptotic conditions often imposed on physical grounds.

1.3. **Quantization.**

Remark. Commutation relations encoded by choice of real symplectic space.

Let $\mathfrak{h}$ a (complex) Hilbert space. The bosonic Fock space is

$$\Gamma_{\mathrm{s}}(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n} \mathfrak{h}.$$

Creation/annihilation operators:

$$a^*(h)\Psi_n := \sqrt{n+1}\, h \otimes_{\mathrm{s}} \Psi_n,$$
$$a(h)\Psi_n := \sqrt{n}\big((h| \otimes_{\mathrm{s}} \mathbf{1}_{n-1}\big)\Psi_n,$$

for $h \in \mathfrak{h}$ and $\Psi_n \in \otimes_{\mathrm{s}}^{n}\mathfrak{h}$, where $(h|$ is the map $\mathfrak{h} \ni u \mapsto (h|u) \in \mathbb{C}$. As quadratic forms on a suitable domain,

$$[a(h_1), a(h_2)] = [a^*(h_1), a^*(h_2)] = 0,$$
$$[a(h_1), a^*(h_2)] = (h_1|h_2)\mathbf{1}, \quad h_1, h_2 \in \mathfrak{h}.$$

Therefore, if $\phi_{\mathrm{F}}(h) := \frac{1}{\sqrt{2}}(a(h) + a^*(h))$ then

$$\big[\phi_{\mathrm{F}}(h_1), \phi_{\mathrm{F}}(h_2)\big] = \mathrm{i}\,\mathrm{Im}(h_1|h_2)\mathbf{1} =: \mathrm{i}(h_1 \cdot \sigma h_2)\mathbf{1}.$$

The vacuum vector is $\Omega = (1, 0, \dots)$. Observe that we can modify the Hilbert space while keeping the above commutation relation unchanged. Indeed, for $(\mathfrak{h}_{\mathbb{R}}, \sigma)$ a fixed symplectic space, we can define using some operator j:

$$(h_1|h_2)_{\mathrm{F}} := h_1 \cdot \sigma \mathrm{j} h_2 + \mathrm{i} h_1 \cdot \sigma h_2.$$

This works provided $(\mathfrak{h}_{\mathbb{R}}, \sigma, \mathrm{j})$ is Kähler, i.e. $\mathrm{j}^2 = -\mathbf{1}$ and $\sigma \circ \mathrm{j} \geqslant 0$. A new Hilbert space is obtained by complexification $(\alpha + \mathrm{i}\beta)h := \alpha h + \mathrm{j}\beta h$ for $h \in \mathfrak{h}_{\mathbb{R}}$, $\alpha + \mathrm{i}\beta \in \mathbb{C}$, and by taking the completion. Thus, different choices of j give different Hilbert spaces and different fields (possibly non-unitarily equivalent).

In practice it is better to work with complex vector spaces exclusively, and encode the choice of j in terms of *two-point functions* $\Lambda^{\pm}$.

Proposition. Let q be a hermitian form on a complex vector space V. Suppose $\Lambda^{\pm}$ are two non-degenerate forms s.t.

$$(1)\ \ \Lambda^{\pm} \geqslant 0, \quad (2)\ \ \Lambda^{+} - \Lambda^{-} = q.$$

Let V^{cpl} be the completion w.r.t. $\frac{1}{2}(\Lambda^{+} + \Lambda^{-})$. Then there exists j such that $(V^{\mathrm{cpl}}_{\mathbb{R}}, \sigma, \mathrm{j})$ is Kähler and

$$\sigma \mathrm{j} = \frac{1}{2} \operatorname{Re}(\Lambda^{+} + \Lambda^{-}), \ \ \sigma = \operatorname{Im} q.$$

Consequently,

$$(v_1|v_2)_{\mathrm{F}} = \frac{1}{2} \left(\overline{v_1} \cdot \Lambda^{+} v_2 + \overline{v_1} \cdot \Lambda^{-} v_2 \right).$$

The proof is particularly easy if $\Lambda^{\pm} = \pm q c^{\pm}$, where $c^{\pm}$ are projections (note $c^{+} + c^{-} = \mathbf{1}$), i.e. $\mathrm{j} = \mathrm{i}(c^{+} - c^{-})$.

This gives $(\Omega|\phi(v_1)\phi(v_2)\Omega) = \overline{v_1} \cdot \Lambda^{+} v_2$, $\forall v_i \in V$ s.t. $\overline{v_i} = v_i$.

1.4. **Propagators.**

Assumption. (M, g) is globally hyperbolic, i.e., $J_{+}(K_1) \cap J_{-}(K_2)$ is compact for all K_1, K_2 compact.

Working assumption. We assume $M = \mathbb{R}_t \times \Sigma$ with Σ compact or $\Sigma = \mathbb{R}^d$, and

$$g = -dt^2 + h_t, \ \ t \mapsto h_t \text{ smooth with value in Riemannian metrics.}$$

In this setting, global hyperbolicity equivalent to: for fixed $t \in \mathbb{R}$, each maximally extended time-like geodesic hits $\mathbb{R}_t \times \Sigma$ once.

Then $P = \partial_t^2 + r(t)\partial_t + a(t, \mathrm{x}, \partial_{\mathrm{x}})$, where $r(t) = |h_t|^{-\frac{1}{2}} \partial_t |h_t|^{\frac{1}{2}}$ and

$$i) \quad \sigma_{\mathrm{pr}}(a)(t, \mathrm{x}, k) = k \cdot h_t^{-1}(\mathrm{x}) k,$$

$$ii) \quad a(t, \mathrm{x}, \partial_{\mathrm{x}}) = a^{*}(t, \mathrm{x}, \partial_{\mathrm{x}})$$

w.r.t. $(f_1|f_2)_t = \int_{\Sigma} \overline{f_1} f_2 |h_t|^{\frac{1}{2}} d\mathrm{x}$.

Remark. Considering $f_1 \circ P \circ f_2$ instead of P for $f_1, f_2 \in C^{\infty}(M)$, $f_1, f_2 > 0$ corresponds to more general g.

Terminology. One says $G : C_c^\infty(M) \to C^\infty(M)$ is a propagator if either

(1) $PG = \mathbf{1}$ and $GP = \mathbf{1}$ on $C_c^\infty(M)$ (inverse), or
(2) $PG = 0$ and $GP = 0$ on $C_c^\infty(M)$ (bi-solution).

Theorem. [goes back to Leray] There exist unique retarded/advanced inverses $G_\pm : C_c^\infty(M) \to C^\infty(M)$, i.e. $\forall v$, $\operatorname{supp} G_\pm v \subset J_\pm(\operatorname{supp} v)$.

Here, $(\operatorname{supp} G_\pm v) \cap \{t = s\}$ is compact for all s, and empty for large $\pm s$.

Definition. Pauli-Jordan bi-solution (or causal propagator) $G := G_+ - G_-$.

By $P = P^*$ and uniqueness of $G_\pm$, $G_\pm^* = G_\mp$. Hence $G^* = -G$ on $C_c^\infty(M)$.

The symplectic space for QFT is $C_c^\infty(M;\mathbb{R})/PC_c^\infty(M;\mathbb{R})$ equipped with G. Complex version: $C_c^\infty(M)/PC_c^\infty(M)$ equipped with $\mathrm{i}G$.

To quantize we need two-point functions $\Lambda^\pm : C_c^\infty(M) \to C^\infty(M)$ s.t.

$$(1)\ \Lambda^\pm \geqslant 0, \quad (2)\ \Lambda^+ - \Lambda^- = \mathrm{i}G, \quad (3)\ P\Lambda^\pm = \Lambda^\pm P = 0.$$

From this we get fields $\phi([v])$, $v \in C_c^\infty(M;\mathbb{R})$. Note that $P\phi = 0$.

Example. Suppose $P = \partial_t^2 - \Delta_{\mathrm{x}} + m^2$ and $m > 0$. Then

$$\big(\Lambda_{\mathrm{vac}}^\pm v\big)(t) = \int_{\mathbb{R}} \frac{\mathrm{e}^{\pm\mathrm{i}(t-s)\sqrt{-\Delta_{\mathrm{x}}+m^2}}}{\sqrt{-\Delta_{\mathrm{x}}+m^2}} v(s)ds$$

Characteristic feature: solves $\big(\mathrm{i}^{-1}\partial_t \pm \sqrt{-\Delta_{\mathrm{x}}+m^2}\big)u(t,\mathrm{x}) = 0$.

Physical principle. Admissible $\Lambda^\pm$ should have same short-distance behaviour as $\Lambda_{\mathrm{vac}}^\pm$ (Hadamard condition). Consequence (Radzikowski theorem): $\Lambda^\pm =$ *singular, geometric part + smooth part*.

2. Hadamard two-point functions

2.1. Cauchy problem.

We fix $s \in \mathbb{R}$.

Theorem. $\forall v \in C_c^\infty(\Sigma)^2$, $\exists! u \in C^\infty(M)$ (space-compact) solving

$$\begin{cases} Pu = 0 \\ \varrho(s)u = f \end{cases}$$

where $\varrho(s)u = (u(s), \mathrm{i}^{-1}\partial_t u(s))$.

The dual is $\varrho(s)^* f = f^0 \otimes \delta(s) - \mathrm{i} f^1 \otimes \delta'(s) : \mathcal{D}'(\Sigma)^2 \to \mathcal{D}'(M)$.

Let $q = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$.

Proposition. $U(s) = \mathrm{i}^{-1}(\varrho(s)G)^* q$ on $C_{\mathrm{c}}^\infty(\Sigma; \mathbb{C}^2)$.
Proof: Green's formula gives

$$\int_{J_\pm(\Sigma)} \big(\overline{u_1} P u_2 - \overline{P u_1} u_2\big) d\mathrm{vol}_g = \int_\Sigma \big(\overline{\partial_t u_1} u_2 - \overline{u_1} \partial_t u_2\big) d\mathrm{vol}_h.$$

Applied to $u_1 = G_\mp v$, $u_2 = u = U(s) f$, $v \in C_{\mathrm{c}}^\infty(M)$,

$$\int_{J_+(\Sigma)} \overline{v} u \, d\mathrm{vol}_g = \int_\Sigma \big(\overline{G_- v} \partial_t u - \overline{\partial_t G_- v} u\big) d\mathrm{vol}_h,$$
$$\int_{J_-(\Sigma)} \overline{v} u \, d\mathrm{vol}_g = \int_\Sigma \big(\overline{G_+ v} \partial_t u - \overline{\partial_t G_+ v} u\big) d\mathrm{vol}_h.$$

Since $J(\Sigma) = M$, adding the two we get

$$\int_M \overline{v} u \, d\mathrm{vol}_g = \int_\Sigma \big(\overline{\partial_t G v} u - \overline{G v} \partial_t u\big) d\mathrm{vol}_h.$$

Now use $G^* = -G$ and formula for $\varrho(s)^*$. □

Hence, continuous extension $U(s) : \mathcal{E}'(\Sigma)^2 \to \mathcal{D}'(M)$.

Proposition. Suppose $c^\pm(s) : C_{\mathrm{c}}^\infty(\Sigma)^2 \to C^\infty(\Sigma)^2$ satisfy

$$(1) \quad \pm\, q c^\pm(s) \geqslant 0, \quad (2) \;\; c^+(s) + c^-(s) = \mathbf{1}.$$

Then $\Lambda^\pm := \pm U(s)^* q c^\pm(s) U(s)$ are two-point functions.

We write $\big(\partial_t^2 + r(t)\partial_t + a(t)\big) u(t) = 0$ as

$$\mathrm{i}^{-1}\partial_t \psi(t) = H(t)\psi(t), \quad H(t) = \begin{pmatrix} 0 & \mathbf{1} \\ a(t) & \mathrm{i} r(t) \end{pmatrix},$$

by setting

$$\psi(t) = \begin{pmatrix} u(t) \\ \mathrm{i}^{-1}\partial_t u(t) \end{pmatrix} =: \varrho(t) u.$$

$U(t,s) := \varrho(t) U(s) \in B(H^1(\Sigma) \oplus L^2(\Sigma))$ evolution generated by $H(t)$. Then:

$$q = U^*(s,t) q U(s,t).$$

Example. If $a(t) = a \geqslant 0$, $r(t) = 0$ then $\Lambda^\pm_{\mathrm{vac}}$ has data

$$c^\pm_{\mathrm{vac}}(s) = c^\pm_{\mathrm{vac}} = \mathbf{1}_{\mathbb{R}^\pm}(H) = \frac{1}{2}\begin{pmatrix} \mathbf{1} & \pm a^{-\frac12} \\ \pm a^{\frac12} & \mathbf{1} \end{pmatrix}.$$

2.2. **Hadamard condition.**

The principal symbol of P is $p(t,\mathrm{x},\tau,k) = \tau^2 - k\cdot h_t(\mathrm{x})k$.

$$\mathrm{Char}(P) = \mathcal{N}^+ \cup \mathcal{N}^-, \ \ \mathcal{N}^\pm = \Big\{(t,\mathrm{x},\tau,k) : \tau = \pm(k\cdot h_t(\mathrm{x})k)^{\frac{1}{2}}, \ k \neq 0\Big\}$$

If $\Gamma \subset T^*M \times T^*M$,

$$\Gamma' := \big\{\big((x_1,\xi_1),(x_2,\xi_2)\big) : \big((x_1,\xi_1),(x_2,-\xi_2)\big) \in \Gamma\big\}\,.$$

Definition. $\Lambda^\pm$ is *Hadamard* if

(Had) $$\mathrm{WF}(\Lambda^\pm)' \subset \mathcal{N}^\pm \times \mathcal{N}^\pm.$$

Theorem. [Radzikowski] If $\Lambda^\pm$, $\widetilde{\Lambda}^\pm$ are Hadamard two-point functions then $\Lambda^\pm - \widetilde{\Lambda}^\pm$ has $C^\infty(M\times M)$ kernel.
Proof: $\Lambda^+ - \Lambda^- = \tilde{\Lambda}^+ - \tilde{\Lambda}^- = \mathrm{i}G$, hence $\Lambda^+ - \tilde{\Lambda}^+ = \Lambda^- - \tilde{\Lambda}^-$. These have disjoint wave front sets by (Had). Hence $\mathrm{WF}(\Lambda^\pm - \widetilde{\Lambda}^\pm)' = \varnothing$. □

Remark. We can deduce $\mathrm{WF}(\Lambda^\pm)'$ exactly.

Lemma. $\mathrm{WF}(\Lambda^\pm)' \subset \mathcal{N}^\pm \times T^*M$ implies (Had).
Proof: Use $\Lambda^\pm \geqslant 0$ to symmetrize $\mathrm{WF}(\Lambda^\pm)'$. Then eliminate singularities in $T^*M \times o$ using [Duistermaat, Hörmander]. □

Theorem. $\Lambda^\pm_{\mathrm{vac}}$ are Hadamard.
Proof: Use $(\mathrm{i}^{-1}\partial_t \pm \sqrt{-\Delta_{\mathrm{x}} + m^2})\Lambda^\pm_{\mathrm{vac}} = 0$.

Application. (Quantum Energy Inequalities, [Fewster]) For fixed $\mathrm{x} \in \Sigma$,

$$E_\varphi := \int_{\mathbb{R}} (\Lambda^+ - \widetilde{\Lambda}^+)(t,t,\mathrm{x},\mathrm{x})\varphi^2(t) \ \text{ exists}\,.$$

(Renormalized charge density, averaged along timelike curve). Setting $\Lambda^\pm_\varphi : C^\infty_{\mathrm{c}}(\mathbb{R}) \to C^\infty_{\mathrm{c}}(\mathbb{R})$ the op. with kernel $\varphi(t)\Lambda^\pm(t,t',\mathrm{x},\mathrm{x})\varphi(t')$,

$$\begin{aligned} E_\varphi = \mathrm{Tr}(\Lambda^+_\varphi - \widetilde{\Lambda}^+_\varphi) &= \mathrm{Tr}(\theta(D_t)(\Lambda^+_\varphi - \widetilde{\Lambda}^+_\varphi)\theta(D_t)) + \mathrm{Tr}(\theta(-D_t)(\Lambda^+_\varphi - \widetilde{\Lambda}^+_\varphi)\theta(-D_t)) \\ &= \mathrm{Tr}(\theta(D_t)(\Lambda^+_\varphi - \widetilde{\Lambda}^+_\varphi)\theta(D_t)) + \mathrm{Tr}(\theta(-D_t)(\Lambda^-_\varphi - \widetilde{\Lambda}^-_\varphi)\theta(-D_t)) \\ &\geqslant -\mathrm{Tr}(\theta(D_t)\widetilde{\Lambda}^+_\varphi\theta(D_t)) - \mathrm{Tr}(\theta(-D_t)\widetilde{\Lambda}^-_\varphi\theta(-D_t)) =: -C_\varphi. \end{aligned}$$

3. Construction by pseudo-differential calculus

3.1. **Uniform PDO calculus.**

In what follows $\Psi^\mu(\Sigma)$ is Hörmander's (uniform) calculus if $\Sigma = \mathbb{R}^d$ and the usual calculus on manifolds if Σ is compact. In more general non-compact cases one needs some global calculus that replaces $\Psi^\mu(\Sigma)$.

Let $b(t) = b_1(t) + b_0(t)$, s.t.:

$$\text{(E)}\quad \begin{array}{l} b_i(t) \in C^\infty(\mathbb{R}; \Psi^i(M)),\ i = 0, 1, \\ b_1(t) \text{ is elliptic, symmetric and bounded from below on } H^\infty(M). \end{array}$$

Define $U_b(t, s)$ by:

$$\begin{cases} \frac{\partial}{\partial t} U_b(t, s) = \mathrm{i} b(t) U_b(t, s),\ t, s \in \mathbb{R}, \\ \frac{\partial}{\partial s} U_b(t, s) = -\mathrm{i} U_b(t, s) b(s),\ t, s \in \mathbb{R}, \\ U_b(s, s) = \mathbf{1},\ s \in \mathbb{R}. \end{cases}$$

Here $U_b(t, s)$ is strongly continuous in (t, s) with values in $B(L^2(M))$ (one needs to work a bit and use perturbation theory, note that b is not necessarily self-adjoint).

Lemma.

(1) $U_b(t, s) \in B(H^m(M))$ for $m \in \mathbb{Z} \cup \{\pm\infty\}$, $\mathbb{R}^2 \ni (t, s) \mapsto U_b(t, s)$ is strongly continuous on $H^m(M)$,
(2) if $r_{-\infty} \in \Psi^{-\infty}(M)$ then $U_b(t, s) r_{-\infty},\ r_{-\infty} U_b(t, s) \in C^\infty(\mathbb{R}^2_{t,s}, \Psi^{-\infty}(M))$.

Theorem. [Egorov] Let $a \in \Psi^m(M)$ and $b(t)$ satisfying (E). Then

$$a(t, s) := U_b(t, s) a U_b(s, t) \in C^\infty(\mathbb{R}^2, \Psi^m(M)).$$

Moreover

$$\sigma_{\rm pr}(a)(t, s) = \sigma_{\rm pr}(a) \circ \Phi(s, t),$$

where $\Phi(t, s) : T^*M \to T^*M$ is the flow of the time-dependent Hamiltonian $\sigma_{\rm pr}(b)(t)$.

Theorem. [essentially Seeley] Let $a \in C^\infty(\mathbb{R}; \Psi^m(\Sigma))$ be elliptic, selfadjoint, $a(t) \geqslant c\mathbf{1}$ for $c > 0$, $t \in \mathbb{R}$. Then $a^s \in C^\infty(\mathbb{R}; \Psi^{ms}(\Sigma))$ for any $s \in \mathbb{R}$ and

$$\sigma_{\rm pr}(a^s)(t) = \sigma_{\rm pr}(a(t))^s.$$

3.2. **Approximate diagonalization of evolution.**

Method due to [Junker], [Junker, Schrohe], [Gérard, W.], [Gérard, Oulghazi, W.]

Suppose we have $b(t) \in C^\infty(\mathbb{R}; \Psi^1(\Sigma))$ elliptic s.t.

(J) $\quad (\partial_t + \mathrm{i} b^\pm(t) + r(t)) \circ (\partial_t - \mathrm{i} b^\pm(t)) = \partial_t^2 + r(t)\partial_t + a(t)$ mod smoothing

Set

$$\tilde{\psi}(t) := \begin{pmatrix} \partial_t - \mathrm{i} b^-(t) \\ \partial_t - \mathrm{i} b^+(t) \end{pmatrix} u(t).$$

Then $\tilde{\psi}(t) = S^{-1}(t)\psi(t)$ with

$$S^{-1}(t) = \mathrm{i} \begin{pmatrix} -b^-(t) & \mathbf{1} \\ -b^+(t) & \mathbf{1} \end{pmatrix}, \quad S(t) = \mathrm{i}^{-1} \begin{pmatrix} \mathbf{1} & -\mathbf{1} \\ b^+(t) & -b^-(t) \end{pmatrix} (b^+(t) - b^-(t))^{-1},$$

if $b^+(t) - b^-(t)$ invertible. We have

$$\begin{pmatrix} \partial_t + \mathrm{i}b^- + r & 0 \\ 0 & \partial_t + \mathrm{i}b^+ + r \end{pmatrix} \tilde{\psi}(t) = 0$$

modulo smoothing. Even better diagonalization:

$$T(t) := S(t)(b^+ - b^-)^{\frac{1}{2}}(t) = \mathrm{i}^{-1} \begin{pmatrix} \mathbf{1} & -\mathbf{1} \\ b^+ & -b^- \end{pmatrix} (b^+ - b^-)^{-\frac{1}{2}},$$

$$T^{-1}(t) = \mathrm{i}(b^+ - b^-)^{-\frac{1}{2}} \begin{pmatrix} -b^- & \mathbf{1} \\ -b^+ & \mathbf{1} \end{pmatrix},$$

gives

$$T^*(t) q T(t) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} =: q^{\mathrm{ad}}. \tag{3.1}$$

We get:

$$\begin{aligned} U(t,s) &= T(t) U^{\mathrm{ad}}(t,s) T(s)^{-1} \\ &= T(t) U^{\mathrm{d}}(t,s) T(s)^{-1} + C^\infty(\mathbb{R}^2; \Psi^{-\infty}(\Sigma)). \end{aligned}$$

Now: $c^{\pm}(t_0) := T(t_0)\pi^{\pm} T^{-1}(t_0)$,

$$\pi^+ = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi^- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

And $U^{\pm}(t,s) := U(t,t_0) c^{\pm}(t_0) U(t_0,s)$ propagates with correct wave front set!

3.3. **Riccati equation.**

Equation (J) is:

$$\mathrm{i}\partial_t b^{\pm} - b^{\pm 2} + a + \mathrm{i} r b^{\pm} = 0 \ \text{ mod smoothing }.$$

Without loss, assume $a(t)$ uniformly positive.

Theorem. $\exists\, b \in C^\infty(\mathbb{R}; \Psi^1(\Sigma))$ s.t.

i) $b = a^{\frac{1}{2}} + C^\infty(\mathbb{R}; \Psi^0(\Sigma))$,

ii) $(b + b^*)^{-1} = (2a)^{-\frac{1}{4}}(\mathbf{1} + r_{-1})(2a)^{-\frac{1}{4}}, \ r_{-1} \in C^\infty(\mathbb{R}; \Psi^{-1}(\Sigma))$,

iii) $(b + b^*)^{-1} \geqslant c a^{-\frac{1}{2}}$, for some $c \in C^\infty(\mathbb{R}; \mathbb{R}), \ c > 0$,

iv) $\mathrm{i}\partial_t b^{\pm} - b^{\pm 2} + a + \mathrm{i} r b^{\pm} = r^{\pm}_{-\infty} \in C^\infty(\mathbb{R}; \Psi^{-\infty}(\Sigma))$,

for $b^+ := b, \ b^- := -b^*$.

Proof: Modulo smoothing, $a = \mathrm{Op}(c)$, $c \in C^\infty(\mathbb{R}; S^1_{\mathrm{ph}}(T^*\Sigma))$, with $c_{\mathrm{pr}}(t,x,\xi) = (\xi \cdot h_t^{-1}(x)\xi)^{\frac{1}{2}}$. We look for b of the form $b = \mathrm{Op}(c) + \mathrm{Op}(d)$ for $d \in C^\infty(\mathbb{R}; S^0_{\mathrm{ph}}(T^*\Sigma))$. Since $\mathrm{Op}(c)$ is elliptic, we can fix a symbol $\hat{c} \in C^\infty(\mathbb{R}; S^{-1}_{\mathrm{ph}}(T^*\Sigma))$ s.t. $\mathrm{Op}(\hat{c})$ is a parametrix of $\mathrm{Op}(c)$.

Modulo error terms in $C^\infty(\mathbb{R}; \mathcal{W}^{-\infty}(\Sigma))$, (J) becomes:

$$\text{(3.2)} \qquad \mathrm{Op}(d) = \frac{\mathrm{i}}{2}(\mathrm{Op}(\hat{c})\mathrm{Op}(\partial_t c) + \mathrm{Op}(\hat{c}) r \mathrm{Op}(c)) + F(\mathrm{Op}(d)),$$

for:

$$F(\mathrm{Op}(d)) = \frac{1}{2}\mathrm{Op}(\hat{c})\left(\mathrm{i}\mathrm{Op}(\partial_t d) + [\mathrm{Op}(c), \mathrm{Op}(d)] + \mathrm{i}r\mathrm{Op}(d) - \mathrm{Op}(d)^2\right).$$

From symbolic calculus, we obtain that:

$$F(\mathrm{Op}(d)) = \mathrm{Op}(\tilde{F}(d)) + C^\infty(\mathbb{R}; \Psi^{-\infty}(\Sigma)),$$

for

$$\tilde{F}(d) = \frac{1}{2}\hat{c}* \left(\mathrm{i}\partial_t d + c*d - d*c + \mathrm{i}r*d - d*d\right),$$

The equation (3.2) becomes:

$$\text{(3.3)} \qquad d = a_0 + \tilde{F}(d),$$

for

$$a_0 = \frac{\mathrm{i}}{2}(\hat{c}*\partial_t c + \hat{c}*r*c) \in C^\infty(\mathbb{R}; S^0_{\mathrm{ph}}(T^*\Sigma)).$$

The map $\tilde{F}$ has the following property:

$$\text{(3.4)} \qquad \begin{array}{l} d_1, d_2 \in C^\infty(\mathbb{R}; S^0_{\mathrm{ph}}(T^*\Sigma)),\ d_1 - d_2 \in C^\infty(\mathbb{R}; S^{-j}_{\mathrm{ph}}(T^*\Sigma)) \\ \Rightarrow\ \tilde{F}(d_1) - \tilde{F}(d_2) \in C^\infty(\mathbb{R}; S^{-j-1}_{\mathrm{ph}}(T^*\Sigma)). \end{array}$$

This allows to solve symbolically (3.3) by setting

$$d_{-1} = 0,\ \ d_n := a_0 + \tilde{F}(d_{n-1}),$$

and

$$d \simeq \sum_{n \in \mathbb{N}} d_n - d_{n-1},$$

which is an asymptotic series since by (3.4) we see that

$$d_n - d_{n-1} \in C^\infty(\mathbb{R}; S^{-n}_{\mathrm{ph}}(T^*\Sigma)).$$

Hence $\mathrm{Op}(c+d)$ solves (J) modulo $C^\infty(\mathbb{R}; \Psi^{-\infty}(\Sigma))$.

We observe then that if $b \in C^\infty(\mathbb{R}; \Psi^\infty(\Sigma))$ we have:

$$(\partial_t b)^* = \partial_t(b^*) + rb^* - b^* r,$$

This implies that $-\mathrm{Op}(d)^*$ is also a solution modulo $C^\infty(\mathbb{R}; \Psi^{-\infty}(\Sigma))$.

To complete the construction of $b^\pm$, we consider

$$s = \mathrm{Op}(c+d) + \mathrm{Op}(c+d)^*,$$

which is selfadjoint, with principal symbol equal to $2(\xi\cdot h_t^{-1}(x)\xi)^{\frac12}$. There exists $r_{-\infty}\in C^\infty(\mathbb{R};\Psi^{-\infty}(\Sigma))$ such that

$$s + r_{-\infty} \sim a^{\frac12}. \tag{3.5}$$

We set now:

$$b := \mathrm{Op}(c+d) + \frac12 r_{-\infty}.$$

Properties *i)* and *iv)* follow from the same properties of $\mathrm{Op}(c+d)$. To prove property *ii)* we write

$$b + b^* = (2a)^{\frac14}(\mathbf{1}+\tilde r_{-1})(2a)^{\frac14},$$

where $\tilde r_{-1}\in C^\infty(\mathbb{R};\Psi^{-1}(\Sigma))$, by [Seeley]. Since $(\mathbf{1}+\tilde r_{-1})$ is boundedly invertible, we have again by [Seeley]

$$(\mathbf{1}+\tilde r_{-1})^{-1} = \mathbf{1} + r_{-1},\ r_{-1}\in C^\infty(\mathbb{R};\Psi^{-1}(\Sigma)),$$

which implies *ii)*. □

4. Scattering by geometry

4.1. Setup.

Definition. $\Psi^{m,\delta}_{\mathrm{td}}(\mathbb{R};\Sigma) :=$ Op of t-dependent symbols $a(t,\mathrm{x},k)\in S^{m,\delta}_{\mathrm{td}}(\mathbb{R};\Sigma)$, i.e.:

$$|\partial_t^\alpha\partial_{\mathrm{x}}^\beta\partial_k^\gamma a(t,\mathrm{x},k)| \leqslant C_{\alpha\beta\gamma}\langle t\rangle^{\delta-\alpha}\langle k\rangle^{m-|\gamma|},\ \ \alpha\in\mathbb{N},\ \beta,\gamma\in\mathbb{N}^d,$$

where $\langle t\rangle = (1+t^2)^{\frac12}$, $\langle k\rangle = (1+|k|^2)^{\frac12}$.

Assumption. $\exists\,\delta>0$ and $a_{\mathrm{out}}\in\Psi^2(\Sigma)$ elliptic, $a_{\mathrm{out}}(\mathrm{x},D_{\mathrm{x}})\geqslant m^2>0$, s.t. on $\mathbb{R}_+\times\Sigma$

$$\begin{aligned} &a(t,\mathrm{x},D_{\mathrm{x}}) - a_{\mathrm{out}}(\mathrm{x},D_{\mathrm{x}}) \in \Psi^{2,-\delta}_{\mathrm{td}}(\mathbb{R};\Sigma),\\ &r(t)\in\Psi^{0,-1-\delta}_{\mathrm{td}}(\mathbb{R};\Sigma).\end{aligned}$$

Asymptotic dynamics: $P_{\mathrm{out}} = \partial_t^2 + a_{\mathrm{out}}(\mathrm{x},,D_{\mathrm{x}})$.

Asymptotic ('out') vacuum:

$$c^{\pm,\mathrm{vac}}_{\mathrm{out}} = \frac12\begin{pmatrix}\mathbf{1} & \pm a_{\mathrm{out}}^{-\frac12}\\ \pm a_{\mathrm{out}}^{\frac12} & \mathbf{1}\end{pmatrix}.$$

Theorem. [Gérard, W.] Let

$$c^{\pm}_{\mathrm{out}}(t) := \lim_{t_+\to+\infty} U(t,t_+)c^{\pm,\mathrm{vac}}_{\mathrm{out}}U(t_+,t)$$

Then the corresponding $\Lambda^{\pm}_{\mathrm{out}}$ satisfies (Had).

4.2. Time-decaying ΨDO families.

Lemma. Let $\delta \in \mathbb{R}$ and (m_j) a real sequence decreasing to $-\infty$. Then if $a_j \in \Psi_{\rm td}^{m_j,-\delta}(\mathbb{R};\Sigma)$ there exists $a \in \Psi_{\rm td}^{m_0,-\delta}(\mathbb{R};\Sigma)$, unique mod $\Psi_{\rm td}^{-\infty,-\delta}(\mathbb{R};\Sigma)$, s.t.

$$a \sim \sum_{j=0}^{\infty} a_j, \text{ i.e. } a - \sum_{j=0}^{N} a_j \in \Psi_{\rm td}^{m_{N+1},-\delta}(\mathbb{R};\Sigma), \ \forall N \in \mathbb{N}.$$

Ellipticity is uniform in t.

Theorem. [Seeley] works also for $\Psi_{\rm td}^{m,0}(\mathbb{R};\Sigma)$ provided $a(t) \geqslant c_0\mathbf{1}$ for $c_0 > 0$.

Proposition. Let $a_i \in \Psi_{\rm td}^{2,0}(\mathbb{R};\Sigma)$, $i = 1,2$ elliptic, $a_i = a_i^*$ and $a_i(t) \geqslant c_0\mathbf{1}$, $c_0 > 0$. Assume $a_1 - a_2 \in \Psi_{\rm td}^{2,-\delta}(\mathbb{R};\Sigma)$, $\delta > 0$. Then $\forall \alpha \in \mathbb{R}$:

$$a_1^\alpha - a_2^\alpha \in \Psi_{\rm td}^{2\alpha,-\delta}(\mathbb{R};\Sigma).$$

Proposition. $\exists\, b(t) = a^{\frac{1}{2}}(t) + \Psi_{\rm td}^{0,-1-\delta}(\mathbb{R};\Sigma) = a_{\rm out}^{\frac{1}{2}} + \Psi_{\rm td}^{1,-\delta}(\mathbb{R}^\pm;\Sigma)$ s.t.

$$\mathrm{i}\partial_t b - b^2 + a + \mathrm{i}rb \in \Psi_{\rm td}^{-\infty,-1-\delta}(\mathbb{R};\Sigma).$$

Proof: The key is:

$$c_1, c_2 \in \Psi_{\rm td}^{0,-\mu}, \ c_1 - c_2 \in \Psi_{\rm td}^{-j,-\mu} \ \Rightarrow \ F(c_1) - F(c_2) \in \Psi_{\rm td}^{-j-1,-\mu}.$$

4.3. Proof.

The proof boils down to:

Proposition. $\exists$ Cauchy data $c_{\rm ref}^{\pm}(0)$ of Hadamard two-point function s.t.

$$c_{\rm out}^{\pm}(0) = c_{\rm ref}^{\pm}(0) + \Psi^{-\infty}(\Sigma).$$

The crucial lemma is:

Lemma. Let $W_{\rm out}(t) = U^{\rm ad}(0,t)U_{\rm out}^{\rm ad}(t,0)$. Then

$$\lim_{t\to+\infty} W_{\rm out}(t)\pi^+ W_{\rm out}(t)^{-1} = \pi^+ + \Psi^{-\infty}(\Sigma) \otimes L(\mathbb{C}^2), \text{ in } B(L^2(\Sigma) \otimes \mathbb{C}^2).$$

Proof: Cook method:

$$\lim_{t\to+\infty} W_{\rm out}(t)\pi^+ W_{\rm out}(t)^{-1} = \pi^+ + \int_0^{+\infty} \partial_t(W_{\rm out}(t)\pi^+ W_{\rm out}(t)^{-1})dt \text{ in } B(L^2(\Sigma) \otimes \mathbb{C}^2).$$

The integral term is:

$$\partial_t(W_{\rm out}(t)\pi^+ W_{\rm out}(t)^{-1}) = -\mathrm{i}U(0,t)[H^{\rm ad}(t),\pi^+]U(t,0) = U(0,t)[R_{-\infty}(t),\pi^+]U(t,0), \ \ R_{-\infty} \in \Psi_{\rm td}^{-\infty,-1-\delta}(\mathbb{R};\Sigma) \otimes B(\mathbb{C}^2).$$

4.4. **More general consequences.** Suppose now Σ is compact and we have an asymptotic dynamics $H_{\rm out}$ at $t=+\infty$ and also at $t=-\infty$. Modulo time-decaying, smoothing (hence compact) terms, we can now solve a global problem: $Pu=v$ with u and v with asymptotic data at $+\infty$ in $\operatorname{Ker}\boldsymbol{1}_{\mathbb{R}^+}(H_{\rm out})$ and asymptotic data at $-\infty$ in $\operatorname{Ker}\boldsymbol{1}_{\mathbb{R}^-}(H_{\rm out})$. This gives Fredholm property of P on suitable Hilbert spaces (somewhat analogous to anisotropic Sobolev spaces)! One can prove that P is actually invertible and P^{-1} is a Feynman parametrix in the sense of Duistermaat & Hörmander (this is a statement about the wave front set). This is closely related to essential self-adjointness of P!

For $\Sigma=\mathbb{R}^d$ one needs to impose and control the decay in spatial directions to get compact remainder terms. The techniques are similar but require a different pseudo-differential calculus.

Global Fredholm problems and inverses for P using different (but always microlocal) techniques: [Gell-Redman, Haber, Vasy '13], [Bär, Strohmaier '18], [Gérard, W. '17], [Vasy '17], etc.

Black hole spacetimes are more complicated...

5. Thermal and local-to-global effects

5.1. **Thermal states.**

Recall that if $a(t)=a\geqslant 0$ and $r(t)=0$,

$$c^{\pm}_{\rm vac}=\boldsymbol{1}_{\mathbb{R}^\pm}(H)=\frac{1}{2}\begin{pmatrix}\boldsymbol{1} & \pm a^{-\frac12}\\ \pm a^{\frac12} & \boldsymbol{1}\end{pmatrix}.$$

Thermal state at temperature $T=2\pi/\beta$ corresponds to:

$$c^{\pm}_{\beta}=(\boldsymbol{1}-{\rm e}^{\mp\beta H})^{-1},$$

Note $(c^{\pm}_{\rm vac})^2=c^{\pm}_{\rm vac}$ but not true for $c^{\pm}_{\beta}$. Note also $\lim_{\beta\to+\infty}c^{\pm}_{\beta}=c^{\pm}_{\rm vac}$.

Those choices are canonically associated to time-like Killing vector field ∂_t.

Consider two 'Wick-rotated' situations, with $t={\rm i}s$. Let $k=ds^2+h$, and $A=\partial_s+H$.

On $\mathbb{R}\times\Sigma$:

$$A^{-1}v(s)=\int_{\mathbb{R}}K(s-s')v(s')ds$$
$$K(s):={\rm e}^{-sH}\big(\boldsymbol{1}_{\mathbb{R}^+}(s)\boldsymbol{1}_{\mathbb{R}^+}(H)-\boldsymbol{1}_{\mathbb{R}^-}(s)\boldsymbol{1}_{\mathbb{R}^-}(H)\big).$$

On $\mathbb{S}_\beta\times\Sigma$,

$$A^{-1}v(s)=\int_{\mathbb{S}_\beta}K(s-s')v(s')ds'$$
$$K(s):={\rm e}^{-sH}\big(\boldsymbol{1}_{\mathbb{R}^+}(s)(1-{\rm e}^{-\beta H})^{-1}-\boldsymbol{1}_{\mathbb{R}^-}(s)(1-{\rm e}^{\beta H})^{-1}\big).$$

Let $\gamma f := f(0^+)$.

Proposition. $\gamma A^{-1}\gamma^*$ equals $c^+_{\rm vac}$, resp. c^+_β.

5.2. Unruh effect.

Let $M = \mathbb{R}^2$, $g = -dt^2 + d{\rm x}^2$, and $M^+ = \{x \in M : \ {\rm x} > t\}$. New coordinates on M^+:

$$t = a^{-1}{\rm e}^{ar}\sinh(a\eta)$$
$${\rm x} = a^{-1}{\rm e}^{ar}\cosh(a\eta)$$

Then $g = {\rm e}^{2ar}(-d\eta^2 + dr^2)$.

Theorem. The vacuum for ∂_t restricts to thermal state (with $\beta = 2\pi/\beta$) on M^+ for ∂_η.

On black-hole space-times with stationary exterior regions (or more precisely, space-times with *bifurcate Killing horizons*), similar result, but Hadamard extendability across the horizon enforces that $2\pi/\beta$ is exactly the *Hawking temperature* [Sanders '15; Gérard '18].

5.3. Reeh-Schlieder property.

Definition. $(x^0, \xi^0) \notin {\rm WF_a}(u)$ (the analytic wave front set of $u \in \mathcal{D}'(\mathbb{R}^n)$) if $\exists$ nbh. U of x^0 and Γ of ξ^0, and a bounded sequence $u_N \in \mathcal{E}'(\mathbb{R}^n)$ s.t. $u_k = u$ in U and

$$\left|\xi^N \widehat{u_N}(\xi)\right| \leqslant C(C(N+1))^N, \ \ \xi \in \Gamma.$$

Generalizes to real-analytic M.

Definition. For $F \subset M$, the normal set $N(F) \subset T^*M\backslash o$ is the set of (x^0, ξ^0) s.t. $x^0 \in F$, $\xi^0 \neq 0$, and $\exists f \in C^2(M;\mathbb{R})$ s.t. $df(x^0) = \xi^0$ or $df(x^0) = -\xi^0$ and $F \subset \{x : \ f(x) \leqslant f(x^0)\}$.

Theorem. [Kashiwara-Kawai] $\forall u \in \mathcal{D}'(M)$, $N(\operatorname{supp} u) \subset {\rm WF_a}(u)$.

Definition. analytic Hadamard condition ${\rm WF_a}(\Lambda^\pm)' \subset \mathcal{N}^\pm \times \mathcal{N}^\pm$.

Lemma. Let M be real-analytic, connected. If ${\rm WF}_a(u) \cap -{\rm WF}_a(u)$, and $O \subset M$ open non-empty, then

$$u|_O = 0 \ \Rightarrow \ u = 0.$$

Proof: $N(\operatorname{supp} u) = -N(\operatorname{supp} u)$, so assumption implies $N(\operatorname{supp} u) = \varnothing$. Hence $\partial \operatorname{supp} u = \varnothing$, so $\operatorname{supp} u = \varnothing$ or $\operatorname{supp} u = M$ (impossible if $u|_O = 0$). □

Theorem. [Strohmaier, Verch, Wollenberg '02] If $\Lambda^\pm$ analytic Hadamard then for any open $O \subset M$,

$$\operatorname{Vect}\big\{ \textstyle\prod_{i=1}^p \phi(u_i)\Omega_{\rm vac} : \ p \in \mathbb{N}, \ u_i \in C^\infty_{\rm c}(O)\big\}$$

dense in $\mathcal{H}$.
Proof: Suppose Φ is orthogonal. Then all distributions

$$\left(\prod_{i=1}^{p-1} \phi(u_i)\phi(\cdot)\Omega_{\rm vac}|\Phi\right)$$

vanish on O. Assumptions of Lemma are satisfied, so these distributions vanish on M. We conclude

$$\left(\prod_{i=1}^{p-1} \phi(u_i)\phi(\cdot)\Omega_{\rm vac}|\Phi\right)$$

In view of density of

$$\operatorname{Vect}\left\{ \prod_{i=1}^{p} \phi(u_i)\Omega_{\rm vac} : \ p \in \mathbb{N}, \ u_i \in C_{\rm c}^{\infty}(M)\right\},$$

this implies $\Phi = 0$. □

Theorem. [Gérard, W. '17] Analytic Hadamard two-point functions $\Lambda^{\pm}$ exist in analytic case.

5.4. **Outlook.**

Other methods: propagation estimates near radial sets

Open questions concern:

(1) Scattering + Hadamard condition on rotating black hole spacetimes (asymptotically thermal effects, extendability theoremes),
(2) Reeh-Schlieder property of i.e. HHI state.
(3) Coupling $\phi(x)$ with dynamical g
(4) 'Spectral geometry' of the Klein-Gordon operator P
etc.

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