Utility–based pricing and hedging via functional differentiation

Jochen Zahn
Christ Church
University of Oxford

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Abstract

We present an approach to utility–based pricing and hedging in incomplete markets. For the limit of an infinitesimally small number of options, we derive a formula for the marginal optimal hedging strategy that is compatible with the marginal indifference price of Davis [3]. For this, we use the concept of functional differentiation. The proposed framework is conceptually related and probably equivalent to a proposal of Kramkov and Sirbu [5]. We apply it to the case of a jump diffusion process and compare it conceptually and numerically with Merton’s and the minimal variance approach.
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Chapter 1

Introduction

The Black–Scholes framework for the pricing and hedging of European claims was so enormously successful in finance not only due to its computational simplicity, but also because it provides a unique price irrespective of risk preferences. This is possible because in their setting, where assets follow a geometric Brownian motion, risk can be eliminated (hedged) completely. While this may be a reasonable assumption for some asset classes, it is definitely not a good approximation for many others. Possibly, the inappropriate application of the Black–Scholes framework, in particular to credit derivatives, played a role in the recent financial crisis. In situations where European claims can not be replicated by a self-financing hedging strategy, i.e., risk can not be eliminated, one speaks of an incomplete market. This is the case when the stochastic process of the asset price is driven by more than one source of risk, such as in stochastic volatility or jump diffusion processes, or when market imperfections like discrete time trading or transaction costs are not negligible.

The main example of market incompleteness that we are going to study here are jump processes. There is ample evidence from many different markets that daily returns are not normally distributed, but exhibit so-called fat tails. These could be described by stochastic volatility or jump diffusion processes. While the use of stochastic volatility processes might be mathematically appealing (in particular if volatility is assumed to be log-normal), the volatility smile thus produced is not completely satisfactory [9, Chapter 14]. Furthermore, it has recently been shown by statistical analysis of high frequency financial data [2] that about a quarter of the quadratic variation of stock returns (in this study, Intel and Microsoft shares were used) should be attributed to a jump component. If such a jump component is present, then there is more than one source of risk: In addition to the risk inherent in the diffusion, there is the uncertainty as to when the jumps occur, and (possibly) how large these are. In the absence of other tradable assets that depend on the underlying
asset price (we assume here that there is no liquid plain vanilla market) one can thus not eliminate risk completely. Instead, one has to find a balance between these sources of risk that is in some sense optimal.

There are several approaches to the pricing and hedging of European claims in the case of market incompleteness. For the particular case of jump diffusion processes, Merton [7] proposed to hedge only the diffusion risk and try to diversify the jump risk. This approach is probably the most popular one in practice. It is equivalent to the naive interpretation of the Black–Scholes framework which states that the risk–neutral measure is obtained by adjusting the drift term such that the expected drift vanishes, and that the appropriate hedge amount is given by the $\Delta$ of the price thus obtained. The applicability of this approach crucially depends on the validity of the assumption that the jump risk can be diversified, i.e., jumps in different assets are not correlated. The occasional occurrence of market crashes casts strong doubts on this assumption.

A further approach for the pricing and hedging in incomplete markets is minimal variance hedging (also known as mean variance hedging and closely related to risk–minimising hedging). In this approach, the investor tries to minimise risk as measured by the variance of the return. It has the advantage that no new concepts have to be introduced and that all sources of risk are taken into account on an equal footing. However, the use of a quadratic criterion is somewhat arbitrary and penalises profits as well as losses. Furthermore, its aim is to minimise risk, but not to price the remaining risk. Finally, for discontinuous price processes, it will in general lead to a signed risk–neutral measure [11], i.e., the pricing measure is in general not positive.

Here, we will study a different approach to pricing and hedging in incomplete markets, namely marginal utility based pricing and hedging. In this approach, the investor is equipped with a monotonically increasing utility function $U(X)$ that associates with each terminal wealth $X_T$ a certain utility. By using a concave utility function, one can encode risk aversion. For continuous asset price processes, the relevant quantity is then the risk aversion $-U''(X)/U'(X)$. A drawback of the utility–based approach is that one has to consider investment and hedging at the same time (because the optimisation problem will always yield an investment component). In general, this is a difficult optimisation problem, and the result will depend on the number of options one is trading. However, in the limit of an infinitesimally small number of traded options, the two problems disentangle in a certain sense. In that case, one first has to solve the optimal investment problem (without claim). The solution to that problem determines the appropriate risk–neutral measure $Q$ for the option pricing problem,
which yields the so-called marginal indifference price. This well-known procedure is
due to Davis [3].

Even though this framework for utility-based pricing is known for quite a while, a
definition and a characterisation of the corresponding hedging strategy were obtained
rather recently by Kramkov and Sirbu [5]. They define the marginal optimal hedging
strategy via the associated wealth process and prove three theorems that charac-
terise it. For the case of a continuous process, they show that the marginal optimal
hedging strategy coincides with the minimal variance hedge for the risk-neutral mea-
sure $Q$ that determines the marginal indifference price. They also prove theorems that
are valid for discontinuous processes, but these require switching between different
numéraires and measures, so their application is not completely straightforward.

Here, we will present a somewhat different approach. The marginal optimal hedge
is defined directly as the wealth per traded option invested in the hedge, and it is
shown that it can be computed by functional differentiation. The basic idea is the
following: We consider the trading (and hedging) strategy $\pi_t(X,S)$ as a function of
time, the current wealth of the investor, and the asset price. The expected utility can
then be interpreted as functional on the space of admissible trading strategies. This
makes it possible to characterise the optimal hedging strategy by the vanishing of the
functional derivative w.r.t. $\pi$. For the case of an infinitesimally small number of traded
options one then finds a formula with a straightforward financial interpretation. This
approach is directly applicable also to discontinuous processes. In particular, we show
that for the case of a jump diffusion, the marginal optimal hedge does in general not
coincide with the minimal variance hedge for the risk-neutral measure $Q$.

In its present formulation, the approach described here lacks the mathematical
rigour of [5]. In particular, we do not give a precise definition of the space of admissible
trading strategies (including a topology). Thus, the functional differentials we are
using are rather formal. Once a rigorous formulation is achieved, one may discuss
whether the two approaches are equivalent. We strongly suspect that this is indeed the
case, since the ideas underlying the two definitions of the marginal optimal hedging
strategy are the same. Furthermore, we explicitly show that in the case of jump
diffusion, the two approaches yield the same result.

We would like to point out that the functional derivative approach proposed here
is not related to the Malliavin calculus, which can be used to efficiently compute
Greeks (for an introduction, we refer to [4]). The Malliavin calculus is a means
to define derivatives of random variables (the asset price process, e.g.) w.r.t. the
underlying stochastic process. The Malliavin derivative of a random variable is again
a random variable. In the framework proposed here, one computes the functional derivative of expected values that depend on the terminal wealth with respect to a parameter of the wealth process, i.e., the trading strategy.

This thesis is organised as follows: In the next chapter, we define the marginal indifference price and the marginal optimal hedge and show how to compute them via functional differentiation. In Chapter 3, we implement this procedure for a binomial tree and show that one recovers the Black–Scholes results in this case. In Chapter 4 we show how to apply the framework to an asset that follows a jump diffusion process. We also briefly describe the approach of Kramkov and Sirbu and show that it leads to the same results as the one described here. Also Merton’s approach and minimal variance hedging are discussed. Finally, we provide a nontrivial example that is analytically tractable, and compare the outcomes of the different approaches in this concrete case. We conclude with a summary and an outlook.
Chapter 2

Marginal pricing and hedging

In this chapter, we introduce the framework of utility–based pricing and hedging that we are using in the remainder of this thesis. We focus on a market with one traded asset $S_t$ and a risk–less bank account and try to value and hedge a European claim on the asset. The generalisation to situations with several risky assets is straightforward. We assume that the optimal investment strategy $\pi^*$ for the market consisting only of $S$ (and the bank account) exists and is known. Then the expected utility at time $t$ for wealth $X$ and asset price $S$ is given by

$$U_t(X, S) = E_t \left[ U(X^{\pi^*_T}) | X_t = X, S_t = S \right],$$

where the superscript on $X$ indicates the dependence of the wealth process on the investment strategy. Typically, $\pi_t$ will denote the wealth invested in the asset at time $t$. The explicit form of $X^{\pi^*_T}_t$ then depends on the concrete model one is discussing. For the present chapter, this explicit form of $X^{\pi^*_T}_t$ is not important. We only exploit the fact that we can interpret an expected value depending on $X^{\pi^*_T}_T$ as a functional depending on $\pi$, where $\pi$ is a function of time, wealth, and the asset price. This functional is assumed to be well–behaved, so that functional differentiation w.r.t. $\pi$ makes sense. We also assume that $S_t$ and $X^{\pi^*_T}_t$ are Markov processes, so that an expected value conditioned on the knowledge up to time $t$ in fact only depends on the time $t$ value of $X$, $S$, and the function $\pi_t$ for times $t' \geq t$.

**Definition 2.1.** In order to simplify the formulæ below, we introduce the following notation for expressions of the kind appearing on the r.h.s. of the above equation:

$$E \left[ U(X^{\pi^*_T}_t) | X_t = X, S_t = S \right] = E_t \left[ U(X^\pi_T) | X, S, \pi \right] = E \left[ U(X^\pi_T) | Z, \pi \right]$$

In the second step we subsumed the variables $t, X, S$ under one variable $Z$. Similarly, the variable $Z'$ then stands for $t', X', S'$. Note that the trading strategy $\pi$ does not
specify a condition for the expectation, but is a parameter of the process and thus the expected value is a functional of $\pi$.

In the indifference price framework [3], one introduces the European claim by forcing the investor to short an (infinitesimal) quantity $\epsilon$ of it. The final wealth of the investor is then given by

$$X_T - \epsilon C(S_T),$$

where $C$ is the payoff.

**Definition 2.2.** The *indifference price* $V_t^\epsilon(X, S)$ is the price that an investor would have to charge at time $t$ such that her expected utility is the same as in the case without the claim, i.e., $V^\epsilon$ is implicitly defined by

$$U_t(X, S) = \sup_{\pi} E_t [U(X_T - \epsilon C(S_T))|X + \epsilon V_t^\epsilon(X, S), S, \pi].$$

(2.1)

The *marginal indifference price* is given by the limit

$$V_t(X, S) = \lim_{\epsilon \to 0} V_t^\epsilon(X, S).$$

**Remark 2.3.** If the market is complete, the claim can be hedged by a self–financing trading strategy. Then the optimal trading strategy that achieves the supremum on the r.h.s. of (2.1) is given by adding ($\epsilon$ times) the self–financing hedging strategy to the optimal investment strategy $\pi^*$. Thus, the r.h.s. is

$$U_t(X + \epsilon V_t^\epsilon(X, S) - \epsilon V_t(S), S),$$

where $V_t(S)$ is the capital required to set up the self–financing hedging strategy. Thus, $V^\epsilon(X, S) = V(S)$, so one recovers the usual risk–neutral pricing.

Some authors (e.g. [12, 8]) also define the optimal hedging strategy for the investor who shorts $\epsilon$ claims as

$$\hat{\pi}^*_\epsilon = \pi^*_\epsilon - \pi^*,$$

where $\pi^*_\epsilon$ is the optimal trading strategy for the optimisation problem on the r.h.s. of (2.1). This corresponds to the change of the optimal trading strategy that is induced by the claim. Motivated by the definition of the (marginal) indifference price, we modify this definition slightly and go one (infinitesimal) step further.

**Definition 2.4.** The *optimal hedging strategy* $\hat{\pi}^*_\epsilon$ for $\epsilon$ European claims is implicitly defined by

$$\pi^*_\epsilon = \pi^* + \epsilon \hat{\pi}^*_\epsilon.$$
where \( \pi^* \) is the maximiser for the r.h.s. of (2.1). The marginal optimal hedging strategy \( \hat{\pi} \) is defined as

\[
\hat{\pi} = \lim_{\epsilon \to 0} \hat{\pi}_\epsilon.
\]

A different definition of the marginal optimal hedging strategy introduced by Kramkov and Sirbu [5] will be discussed in Section 4.10.

Remark 2.5. In the case of a complete market, the optimal hedging strategy coincides with the risk–eliminating hedging strategy, as discussed in Remark 2.3.

Using Definition 2.4, we can write (2.1) as

\[
U_t(X, S) = E_t[U(X_T - \epsilon^* C(S_T))]X + \epsilon V^*, S, \pi^* + \epsilon \hat{\pi}_\epsilon].
\]

Expanding the r.h.s. in \( \epsilon \) and using \( V^\epsilon = V + O(\epsilon) \) and \( \hat{\pi}_\epsilon = \hat{\pi} + O(\epsilon) \), we obtain

\[
U(Z) = U(Z) + \epsilon V \partial_X U(Z) - \epsilon E[U'(X_T)C(S_T)]Z, \pi^*
\]

\[
+ \epsilon \left( \frac{\delta}{\partial \pi} E[U(X_T)|Z, \pi^*], \hat{\pi} \right) + O(\epsilon^2),
\]

where we used the notation introduced in Definition 2.1 and denoted the partial derivative w.r.t. \( X \) by \( \partial_X \). To obtain the fourth term on the r.h.s., we used functional differentiation. It is the linear change in \( E[U(X_T)|Z, \pi] \) induced by changing \( \pi \) from \( \pi^* \) to \( \pi^* + \epsilon \hat{\pi} \). We refer to the appendix for a formal definition. This term vanishes, since \( \pi^* \) is optimal. Equating the remaining terms of first order in \( \epsilon \), one obtains [3]

\[
V_t(X, S) = \frac{E_t[U'(X_T)C(S_T)]X, S, \pi^* \partial_X U_t(X, S)}{\partial X U_t(X, S)}.
\] (2.2)

It follows that in order to determine the marginal indifference price \( V \) it suffices to know \( \pi^* \), i.e., one does not have to solve the full optimisation problem. Using the tower property, the above can be expressed as an expected value for quantities at times \( \tau < T \):

\[
V_t(X, S) = E_t \left[ \frac{U'(X_T)}{U_t(X_t)} E_{\tau} \left[ \frac{U'(X_T)}{U_t(X_T)} C(S_T)|X_T, S_T, \pi^* \right] |X, S, \pi^* \right]
\]

\[
= E_t \left[ \frac{U'(X_T)}{U_t(X_t)} V_{\tau}(X_T, S_T)|X, S, \pi^* \right]
\] (2.3)

In this form, the pricing problem can be solved backwards in time. In continuous time, the limit \( \tau = t + dt \) will yield a partial (integro–) differential equation (P(I)DE).

It turns out that the marginal optimal hedging strategy can be determined in a similar fashion. We recall that \( \pi^*_\epsilon = \pi^* + \epsilon \hat{\pi}_\epsilon \) is the maximiser of the r.h.s. of
Thus, the functional derivative of the expected utility w.r.t. \( \pi \), evaluated at \( \pi^* \), vanishes:

\[
\left\langle \frac{\delta}{\delta \pi} E_t [U(X_T - \epsilon C(S_T)) | X + \epsilon V^*, S, \pi^*] , \pi' \right\rangle = 0 \quad \forall \pi'.
\]

The l.h.s. of this expression is a linear functional of \( \pi' \). Formally, such a functional can be written as a distribution (again, we refer to the appendix). Thus, we write the above as

\[
\frac{\delta}{\delta \pi} E_t [U(X_T - \epsilon C(S_T)) | X + \epsilon V^*, S, \pi^*] = 0 \quad \forall \pi'.
\]

Expanding this in \( \epsilon \), one obtains, again using the optimality of \( \pi^* \),

\[
\partial_X E_t [U(X_T)|Z, \pi^*] \frac{\delta}{\delta \pi(Z')} E_t \left[ \frac{U'(X_T)}{\partial_X E_t [U(X_T)|Z, \pi^*]} C(S_T)|Z, \pi^* \right] = \int dZ'' \hat{\pi} (Z'') \frac{\delta^2}{\delta \pi(Z') \delta \pi(Z'')} E_t [U(X_T)|Z, \pi^*]. \tag{2.4}
\]

Note that the functional derivatives appearing here and in the following are functional derivatives w.r.t. \( \pi \), evaluated at \( \pi^* \), i.e., \( \pi^* \) is inserted after computing the functional derivative. Furthermore, we recall that the second order functional derivative is a symmetric bilinear map, so that it can be expressed as a symmetric bidistribution as above (for details we again refer to the appendix).

Using (2.2), the expression on the l.h.s. of (2.4) can be expressed differently, so that one obtains

\[
\partial_X E_t [U(X_T)|Z, \pi^*] \frac{\delta}{\delta \pi(Z')} E_t \left[ \frac{U'(X_T)}{\partial_X E_t [U(X_T)|Z, \pi^*]} C(S_T)|Z, \pi^* \right] = \int dZ'' \hat{\pi} (Z'') \frac{\delta^2}{\delta \pi(Z') \delta \pi(Z'')} E_t [U(X_T)|Z, \pi^*]. \tag{2.5}
\]

Now we argue heuristically that the second order functional derivative on the r.h.s. vanishes unless \( Z' = Z'' \), i.e., \( t' = t'' \), \( X' = X'' \) and \( S' = S'' \). Here the Markov property is crucial. First of all, we state that the functional derivative vanishes if either \( t' < t \) or \( t'' < t \), since the expected value is conditioned on time \( t \), so that it is independent of the strategy chosen at times before \( t \). Suppose that \( t' \neq t'' \). We assume \( t' < t'' \). We thus have \( t \leq t' < t'' \). By the tower property, we can write

\[
E_t [U(X_T)|X, S, \pi] = E_t [E' [U(X_T)|X', S', \pi_{[\epsilon, t']}]|X, S, \pi_{[\epsilon, t']}].
\]

Here we took care to write out the dependencies in full detail. The subscript of \( \pi \) gives the time interval for which the value of \( \pi \) matters. For example, the inner expected
value is conditioned on time $t'$, so the value of $\pi$ for times prior to $t'$ is not important in order to compute it. Now the outer expected value depends on $\pi_{t'}$ and $\pi_{t''}$ only through the inner expected value, so one can pull both functional derivatives inside.

In the following, we shall omit the outer expected value for simplicity and study

$$\frac{\delta^2}{\delta \pi(Z') \delta \pi(Z'')} E_{t'} \left[ U(X_T) | X_{t'}, S_{t'}, \pi_{t', t''} \right].$$  \hfill (2.6)

Using the same arguments as before, we may write this as

$$\frac{\delta}{\delta \pi(Z')} E_{t'} \left[ \frac{\delta}{\delta \pi(Z'')} E_{t''} \left[ U(X_T) | X_{t''}, S_{t''}, \pi_{t''}, T \right] \right].$$

Our goal is to evaluate this for the optimal trading strategy $\pi^*$. However, the inner expected value vanishes upon inserting $\pi^*$ (simply because $\pi^*$ is optimal). In particular, it vanishes for all values of the conditional variables $X_{t''}, S_{t''}$. So the outer expected value is an expected value of a quantity that vanishes for all possible paths from $t'$ to $t''$ and in particular for all possible trading strategies taken in the interval $[t', t'']$. Thus, the whole expression vanishes for $t' < t''$. Because of the symmetry of the second order functional derivative, this is also true for $t'' < t'$. Thus, the second order functional derivative vanishes for $t' \neq t''$. Now if $t' = t''$ but $X' \neq X''$, the second order functional derivative (2.6) vanishes, since the expected value does only depend on $\pi(Z')$ if $X' = X_{t'}$, so it can not depend on $\pi(Z')$ and $\pi(Z'')$ at the same time if $X' \neq X''$. The same reasoning can be applied to $S' \neq S''$. In the formalism of this thesis, we have thus argued for the following

**Lemma 2.6.** The second order functional derivative of the expected utility with respect to the trading strategy $\pi$ at two different points $Z', Z''$, evaluated at the optimal trading strategy $\pi^*$, vanishes for $Z' \neq Z''$. I.e., we can write

$$\frac{\delta^2}{\delta \pi(Z') \delta \pi(Z'')} E \left[ U(X_T) | Z, \pi^* \right] = \delta_D(Z' - Z'') \frac{\delta^2}{\delta \pi(Z')^2} E \left[ U(X_T) | Z, \pi^* \right],$$

with $\delta_D$ being the Dirac $\delta$ distribution in the variables $t, X, S$, and take this as an implicit definition of the second order functional derivative on the right hand side.

Thus, from (2.5), we obtain

**Proposition 2.7.** The marginal optimal hedging strategy $\hat{\pi}$ fulfils

$$\partial_X E \left[ U(X_T) | Z, \pi^* \right] \frac{\delta}{\delta \pi(Z')} E \left[ \frac{U'(X_T)}{\partial_X E \left[ U(X_T) | Z, \pi^* \right]} C(S_T) | Z, \pi^* \right] = \hat{\pi}(Z') \frac{\delta^2}{\delta \pi(Z')^2} E \left[ U(X_T) | Z, \pi^* \right].$$  \hfill (2.7)
It follows that the marginal hedging strategy \( \hat{\pi} \) can be determined very efficiently just by knowing the optimal trading strategy \( \pi^* \) for the case without the claim and disturbing it slightly and locally. One does not have to solve the full optimisation problem.

Equation (2.7) also has a straightforward financial interpretation: The functional derivative on the l.h.s. gives the marginal gain in the price \( V \) one can generate by shifting \( \pi \) from the optimal trading strategy \( \pi^* \). The factor in front of the functional derivative converts this into a marginal loss (since the investor is short the option) in utility. This loss in utility stemming from \( V \) is to be balanced to the loss in utility that is incurred to the wealth process (without the claim) by deviating from \( \pi^* \), which one finds on the right hand side.

Remark 2.8. The use of functional differentiation implies that the space of admissible trading strategies should be an open subspace of a vector space. This, however, is not always the case. An example are transaction cost models [8], where one uses the control variables \( m_t \) and \( l_t \), as the number of stocks that are bought, respectively sold, at time \( t \). These are restricted to be greater or equal to zero, so they can not form a vector space. In such cases, some care has to be taken at the boundary.
Chapter 3

A binomial tree implementation

In order to become acquainted with the framework, we now apply it to a binomial tree. Of course, this is the prime example of a complete market, so the study of this model can hardly produce any new results. However, there are several advantages of looking at it. First of all, the expected result is known, so this is a nice way of testing the framework. Furthermore, the optimal trading strategy $\pi^*$ is easy to find for suitable utility functions, so one can focus on finding $\hat{\pi}$. Finally, this example illustrates how to implement the present formalism on a binomial tree, which may also be useful in the incomplete case, e.g. for the treatment of transaction costs, cf. [8].

For our binomial tree implementation, we divide the time interval $[0,T]$ into $N$ pieces of length $\Delta t$, i.e., $\Delta t = T/N$. In one time step, $S_n$ moves with probability $p$ (respectively $q = 1 - p$) to $\omega_u S_n$ (respectively $\omega_d S_n$) with

$$\omega_{u/d} = \exp\left(-\frac{\sigma^2}{2} \Delta t \pm \sigma \sqrt{\Delta t}\right),$$

where $p$ is chosen such that it accounts for the drift $\mu$:

$$p = \frac{e^{\mu \Delta t} - e^{-\sigma \sqrt{\Delta t}}}{e^{\sigma \sqrt{\Delta t}} - e^{-\sigma \sqrt{\Delta t}}}$$

The wealth then moves with probability $p$ (respectively $q$) to

$$X_{n+1} = \beta X_n + \pi_n (\omega_{u/d} - \beta),$$

where we defined

$$\beta = \exp(r \Delta t),$$

with $r$ being the risk-free rate. Thus, $\pi_n$ specifies the wealth invested in the asset during the time step from $n$ to $n+1$. 

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Once a utility function is specified, one can find the expected utility by going backwards from \( n + 1 = N \) with

\[
U_n(X) = \sup_{\pi_n} \{ pU_{n+1}(X^u) + qU_{n+1}(X^d) \}. \tag{3.1}
\]

where we used the notation

\[
X^{u/d} = \beta X + \pi_n(\omega_{u/d} - \beta).
\]

The induction starts with \( U_N(X) = U(X) \). The optimal investment strategy \( \pi^*_n \) is the maximiser of the r.h.s. of (3.1), i.e., it solves (here and in the following, the optimal investment strategy \( \pi^* \) is used for the definition of \( X^{u/d} \))

\[
p(\omega_u - \beta)U'_{n+1}(X^u) + q(\omega_d - \beta)U'_{n+1}(X^d) = 0. \tag{3.2}
\]

Using this, we now have the following important equality:

\[
U'_n(X) = p \{ \beta + (\omega_u - \beta)\partial_X \pi^*_n \} U'_{n+1}(X^u) + q \{ \beta + (\omega_d - \beta)\partial_X \pi^*_n \} U'_{n+1}(X^d)
= \beta \{ pU'_{n+1}(X^u) + qU'_{n+1}(X^d) \}. \tag{3.3}
\]

**Example 3.1.** We calculate the optimal investment strategy for the example of logarithmic utility, \( U_{n+1}(X) = \ln X + A_{n+1} \). Then, (3.2) becomes

\[
p(\omega_u - \beta)U'_{n+1}(X^u) + q(\omega_d - \beta)U'_{n+1}(X^d) = 0.
\]

Using this, we now have the following important equality:

Using this, one finds that \( U_n(X) \) is again of the form \( U_n(X) = \ln(X) + A_n \) with

\[
A_n = A_{n+1} + \ln \beta + p \ln \left( \frac{\omega_u - \omega_d}{\beta} \right) + q \ln \left( \frac{\omega_u - \omega_d}{\omega_u - \beta} \right).
\]

However, the explicit form of \( \pi^* \) and \( A_n \) is of no importance in the following.

With (2.3), the marginal indifference price \( V_n \) can also be computed backwards in time. The induction starts at \( V_N(S) = C(S) \). Using (3.3), we obtain

\[
V_n(X, S) = \beta^{-1} \frac{pU'_{n+1}(X^u)V_{n+1}(X^u, \omega_u S) + qU'_{n+1}(X^d)V_{n+1}(X^d, \omega_d S)}{pU'_{n+1}(X^u) + qU'_{n+1}(X^d)}. \tag{3.4}
\]

By means of (3.2), this can be written as

\[
V_n(X, S) = \beta^{-1} \left\{ pV_{n+1}(X^u, \omega_u S) + qV_{n+1}(X^d, \omega_d S) \right\}. \tag{3.5}
\]
where we used
\[ \tilde{p} = \frac{\beta - \omega_d}{\omega_u - \omega_d} = (1 - \tilde{q}) \]

This is exactly the risk–neutral probability known from the Black–Scholes framework. Note that since the payoff is independent of \( X \), \( V_n \) will also be independent of \( X \) by (3.5).

We want to use (2.7) for the determination of the marginal optimal hedge. Since we already used the optimal investment strategy \( \pi^* \) for the derivation of (3.5), we can not use (3.5) to compute the functional derivative on the l.h.s. of (2.7). We will thus use (3.4) as our starting point for the calculation of the l.h.s. of (2.7). For simplicity, we calculate the functional derivative of \( V_n(X, S) \) w.r.t. to \( \pi_n(X, S) \), which corresponds to the choice \( Z' = Z \) in (2.7). We obtain

\[
\frac{\delta}{\delta \pi_n(X, S)} |_{\pi^*} V_n(X, S) = \beta U'_n(X)^{-2} \times \left\{ (p(\omega_u - \beta)U''_{n+1}(X^u)V_{n+1}(\omega u S) + q(\omega_d - \beta)U''_{n+1}(X^d)V_{n+1}(\omega_d S)) \times (pU''_{n+1}(X^u) + qU''_{n+1}(X^d)) \\
- (pU''_{n+1}(X^u)V_{n+1}(\omega u S) + qU''_{n+1}(X^d)V_{n+1}(\omega_d S)) \times (p(\omega_u - \beta)U''_{n+1}(X^u) + q(\omega_d - \beta)U''_{n+1}(X^d)) \right\}.
\]

Here we used (3.3) and wrote \( V_{n+1} \) as a function of \( S \) only, in accordance to the above. Some of the terms in this expression cancel, so one obtains

\[
\frac{\delta}{\delta \pi_n(X, S)} |_{\pi^*} V_n(X, S) = pq\beta U'_n(X)^{-2} \left\{ V_{n+1}(\omega u S) - V_{n+1}(\omega_d S) \right\} \times \left\{ (\omega_u - \beta)U''_{n+1}(X^d)(\omega u + (\beta - \omega_d)U''_{n+1}(X^u)U''_{n+1}(X^d) \right\}.
\]

Similarly, one finds

\[
U'_n(X) \frac{\delta^2}{\delta \pi_n(X, S)^2} |_{\pi^*} U_n(X) = \beta \left\{ pU''_{n+1}(X^u) + qU''_{n+1}(X^d) \right\} \times \left\{ (\omega_u - \beta)^2U''_{n+1}(X^u) + q(\beta - \omega_d)^2U''_{n+1}(X^d) \right\}.
\]

Using again (3.2), this is

\[
U'_n(X) \frac{\delta^2}{\delta \pi_n(X, S)^2} |_{\pi^*} U_n(X) = pq\beta(\omega_u - \omega_d) \times \left\{ (\omega_u - \beta)U''_{n+1}(X^d)(\omega u + (\beta - \omega_d)U''_{n+1}(X^u)U''_{n+1}(X^d) \right\}.
\]
Comparing both sides in (2.7), one thus obtains

\[ \hat{\pi}_n = \frac{V_{n+1}(\omega_u S) - V_{n+1}(\omega_d S)}{\omega_u - \omega_d}, \]

which is exactly the Black–Scholes ∆ hedge (note that \( \hat{\pi} \) specifies the wealth invested in the asset for hedging, not the number of assets that are used to hedge). We emphasise once more that once \( \pi^* \) is found, the recursive calculation of \( U_n \) and \( V_n \) is straightforward. In fact \( U_n \) has to be computed anyway in order to find \( \pi^*_n \), and \( V_n \) is interesting in itself, since it is the marginal indifference price. All that remains to be done in order to compute the marginal optimal hedging strategy is to compute the derivative w.r.t. \( \pi_n \) of \( V_n \) and the corresponding second derivative of \( U_n \).
Chapter 4

Jump diffusion processes

We now apply the framework developed so far to jump diffusion processes. Some motivation for studying these processes was given in the introduction. Jump diffusions are a prime example of market incompleteness. In the following, we consider only a single asset, i.e., a one-dimensional jump diffusion. The generalisation to higher dimensional processes is straightforward.

It should be pointed out that we are considering a market consisting only of the asset (and a bank account). Thus, there is no (liquid) plain vanilla option market from which a volatility surface could be implied. Instead, we are discussing the question how an investor with a given utility function would price and hedge plain vanilla options in an optimal way. The consistent inclusion of plain vanilla option prices into the model is a difficult task that would require more conceptual work.

This chapter is organised as follows: In the next section, we introduce the jump diffusion asset price process. We then use it to discuss Merton’s approach for the pricing and hedging of assets with a jump component in Section 4.2. Basic ingredients for the utility based framework are the wealth process, the expected utility, and the optimal investment strategy. These are discussed in Sections 4.3–4.5. We are then ready to compute the marginal indifference price in Section 4.6 and the corresponding marginal optimal hedge in Section 4.7. Up to that point, the discussion is based on a finite activity jump diffusion process. In Section 4.8 we discuss whether the framework is also applicable in the infinite activity case. Minimal variance hedging (and its variants) and its application to jump diffusion is discussed in Section 4.9. In Section 4.10, we present the approach of Kramkov and Sirbu and also apply it to jump diffusion. We find that the two definitions of the marginal optimal hedge are equivalent in that case. We conclude this chapter with the application of the different approaches to a jump diffusion process with a fixed jump size.
4.1 The asset process

We consider an asset $S_t$, in discounted units, whose log returns are described by a jump diffusion process, i.e.,

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t + (e^{J_t} - 1) S_t \, dN_t.$$  \hspace{1cm} (4.1)

Here $W_t$ is a Wiener and $N_t$ a Poisson process with frequency $\lambda$. The random variable $J_t$ determines the relative size $e^{J_t} - 1$ of the jump and is described by a distribution function $f_{J_t}$.

The evolution of a quantity $F$ depending on time $t$ and the asset price $S_t$ at time $t$ is given by (we write $F_t(S)$ instead of $F(t, S)$ in order to be consistent with the notation $V_t$ usually used for the value process of a claim)

$$dF_t = \partial_t F_t(S_t) \, dt + \left( \mu S_t \partial_S F_t(S_t) + \frac{\sigma^2}{2} S_t^2 \partial^2_S F_t(S_t) \right) \, dt$$
$$+ \sigma S_t \partial_S F_t(S_t) \, dW_t + \left( F_t(e^{J_t} S_t) - F_t(S_t) \right) \, dN_t.$$  \hspace{1cm} (4.2)

We write this as

$$dF_t = \partial_t F_t(S_t) \, dt + d^P F_t$$

with

$$d^P F_t = \left( \mu S_t \partial_S F_t(S_t) + \frac{\sigma^2}{2} S_t^2 \partial^2_S F_t(S_t) \right) \, dt$$
$$+ \sigma S_t \partial_S F_t(S_t) \, dW_t + \left( F_t(e^{J_t} S_t) - F_t(S_t) \right) \, dN_t.$$  \hspace{1cm} (4.2)

The superscript $P$ on the differential indicates the evolution under the real–world measure (as opposed to the risk–neutral measure that we introduce below). Assume that $F_t$ fulfils

$$F_t(S) = E_t[F_\tau(S_\tau)|S_t = S] \hspace{0.5cm} \forall t \leq \tau.$$  \hspace{1cm} (4.3)

Then, using (4.2), and setting $\tau = t + dt$, one obtains the partial integro–differential equation (PIDE)

$$\partial_t F_t(S) + L^P F_t(S) = 0,$$  \hspace{1cm} (4.3)

where $L^P$ is the integro–differential operator (we recall that $f_J$ is the distribution function of $J$)

$$L^P F_t(S) = \mu S \partial_S F_t(S) + \frac{\sigma^2}{2} S^2 \partial^2_S F_t(S) + \lambda \int dz \left( F_t(e^z S) - F_t(S) \right) f_J(z).$$  \hspace{1cm} (4.4)

\footnotetext[1]{We will later discuss whether and how infinite activity processes may be described in the present framework.}
4.2 Merton’s approach

We briefly discuss Merton’s approach to the pricing and hedging of jump diffusion processes [7]. The investor sets up a portfolio \( \Pi \) consisting of the claim and a quantity \(-\Delta\) of assets, so that the evolution of the portfolio is given by

\[
d\Pi_t = dV_t - \Delta_t dS_t
\]

\[
= \left( \partial_t V_t(S_t) + \mu S_t \partial_S V_t(S_t) + \frac{\sigma^2}{2} S_t^2 \partial^2_S V_t(S_t) - \Delta_t \mu S_t \right) dt
\]

\[
+ \sigma S_t \left\{ \partial_S V_t(S_t) - \Delta_t \right\} dW_t + \left\{ V_t(e^{Jt} S_t) - V_t(S_t) - \Delta_t (e^{Jt} - 1) S_t \right\} dN_t
\]

(4.5)

It is in general not possible to eliminate jump and diffusion risk at the same time, so some “optimal” choice is necessary. Merton’s proposal is to hedge only the diffusion risk and to diversify the jump risk, i.e., to set \( \Delta_t = \partial_S V_t \). The above then yields

\[
d\Pi_t = \left( \partial_t V_t(S_t) + \frac{\sigma^2}{2} S_t^2 \partial^2_S V_t(S_t) \right) dt
\]

\[
+ \left\{ V_t(e^{Jt} S_t) - V_t(S_t) - (e^{Jt} - 1) \partial_S V_t(S_t) \right\} dN_t.
\]

If jump risk is diversified, the investor does not need any risk premium for taking this risk, i.e., the expected value of \( d\Pi_t \) should vanish (here and in the following, we work in discounted units, so that the risk–free rate is zero). Thus, we obtain the pricing PIDE

\[
0 = \partial_t V_t(S) + \frac{\sigma^2}{2} S^2 \partial^2_S V_t(S) - S \partial_S V_t(S) \lambda \int dz \ (e^z - 1) f_J(z)
\]

\[
+ \lambda \int dz \ \{ V_t(e^z S) - V_t(S) \} f_J(z).
\]

(4.6)

Some remarks are in order here concerning the “diversification” of jump risk. First of all, in our toy model, there is only one asset, so diversification is not possible there. But also in practice this strategy is problematic: In a typical market crash, jumps occur in the whole market, so diversification may well turn out to be accumulation of risk (at least for net sellers of jump risk protection, which must be present, since there are huge buyers of jump risk protection, in particular pension funds).

Incidentally, Merton’s approach is probably the one that many practitioners would follow, even without knowing of Merton’s work. The reason is that it is in accordance to a naive interpretation of the Black–Scholes framework, stating that the risk–neutral measure is obtained by adjusting the drift term such that the expected drift vanishes (in discounted units). Using the risk–neutral measure obtained in this way, one finds Merton’s pricing PIDE (4.6). Another piece of wisdom that is often drawn from the
Black–Scholes framework is that the appropriate hedging strategy is given by the derivative of the price, \( \Delta = \partial_S V \). This is also in accordance with Merton’s result.

Another reason for the popularity of Merton’s approach is that the difficult-to-estimate drift does not enter the risk–neutral measure. We will see below that this is not the case for the marginal indifference price and minimal variance pricing.

### 4.3 The wealth process

In order to prepare for the utility based treatment of option pricing and hedging, we now consider in more detail the properties of the wealth process that are implied by jump diffusion. In discounted units, the wealth process \( X_t \), given a trading strategy \( \pi_t \), is given by

\[
dX_t = \pi_t(X_{t-}, S_{t-})\frac{dS_t}{S_t}
\]

\[
= \pi_t(X_{t-}, S_{t-})\mu dt + \pi_t(X_{t-}, S_{t-})\sigma dW_t + \pi_t(X_{t-}, S_{t-})(e^{J_t} - 1)dN_t
\]

We recall that \( \pi_t(X, S) \) is the wealth invested in the asset at time \( t \), given that the total wealth is \( X \) and the asset price is \( S \). Given a quantity \( G_t \), depending on time \( t \) and the asset price \( S_t \) and wealth \( X_t \) at time \( t \), we have, similarly to (4.2),

\[
dG_t = \{ \partial_t G_t(X_t, S_t) + \pi_t(X_t, S_t)\mu \partial_X G_t(X_t, S_t) + \mu S_t \partial_S G_t(X_t, S_t) \} dt
\]

\[
+ \frac{\sigma^2}{2} \left\{ \pi_t(X_t, S_t)^2 \partial_X^2 G_t(X_t, S_t) + 2\pi_t(X_t, S_t)S_t \partial_X \partial_S G_t(X_t, S_t) \right\} dt
\]

\[
+ \sigma \left\{ \pi_t(X_t, S_t) \partial_X G_t(X_t, S_t) + S_t \partial_S G_t(X_t, S_t) \right\} dW_t
\]

\[
+ \left( G_t(X_t + \pi_t(X_t, S_t)(e^{J_t} - 1), e^{J_t}S_t) - G_t(X_t, S_t) \right) dN_t.
\]

For the sake of notational simplicity, we wrote \( X_t, S_t \) instead of \( X_{t-}, S_{t-} \) on the right hand side. We write this as

\[
dG_t = \partial_t G_t(X_t, S_t) dt + \sigma^* G_t
\]

with

\[
d^* G_t = \{ \pi_t(X_t, S_t)\mu \partial_X G_t(X_t, S_t) + \mu S_t \partial_S G_t(X_t, S_t) \} dt
\]

\[
+ \frac{\sigma^2}{2} \left\{ \pi_t(X_t, S_t)^2 \partial_X^2 G_t(X_t, S_t) + 2\pi_t(X_t, S_t)S_t \partial_X \partial_S G_t(X_t, S_t) \right\} dt
\]

\[
+ \sigma \left\{ \pi_t(X_t, S_t) \partial_X G_t(X_t, S_t) + S_t \partial_S G_t(X_t, S_t) \right\} dW_t
\]

\[
+ \left( G_t(X_t + \pi_t(X_t, S_t)(e^{J_t} - 1), e^{J_t}S_t) - G_t(X_t, S_t) \right) dN_t.
\]
If $G_t$ fulfills
\[ G_t(X, S) = E_t[G_r(X_r)|X_t = X, S_t = S, \pi] \quad \forall t \leq \tau, \tag{4.7} \]
then, similarly to (4.3), one obtains the PIDE
\[ \partial_t G_t(X, S) + L^\pi G_t(X, S) = 0, \]
where $L^\pi$ is the integro–differential operator defined by
\[
L^\pi G_t(X, S) = \pi_t(X, S) \mu_{\partial X} G_t(X, S) + \mu_{\partial S} G_t(X, S) \\
+ \frac{\sigma^2}{2} \left\{ \pi_t(X, S)^2 \partial_X^2 + 2\pi_t(X, S)S \partial_X \partial_S + S^2 \partial_S^2 \right\} G_t(X, S) \\
+ \lambda \int dz \left\{ G_t(X + \pi_t(X, S)(e^z - 1), e^z S) - G_t(X, S) \right\} f_J(z). \tag{4.8}
\]

### 4.4 Expected utility

In the absence of consumption, the expected utility $U_t$ fulfills (4.7), when the optimal investment strategy $\pi^*$ is employed. In particular, the HJB equation
\[
\sup_{\pi} \left[ \partial_t U_t(X) + L^\pi U_t(X) \right] = 0 \tag{4.9}
\]
holds, where the supremum is achieved by the optimal investment strategy $\pi^*$. Note that we suppressed the $S$–dependence of the expected utility (and thus also of $\pi$), which is possible since the asset process (4.1) is invariant under scaling. According to (4.9), the optimal investment strategy $\pi_t^*$ fulfills
\[
\frac{\partial}{\partial \pi_t(X)}|_{\pi^*} L^\pi U_t(X) = 0. \tag{4.10}
\]

Note that we take the partial derivative w.r.t. $\pi_t(X)$ (and no functional derivative), as $L^\pi U_t(X)$ depends on $\pi$ only through $\pi_t(X)$. Using the explicit form (4.8) of $L^\pi$, we obtain
\[
\mu U_t'(X) + \pi_t^*(X) \sigma^2 U_t''(X) + \lambda \int dz \ U_t'(X^z)(e^z - 1)f_J(z) = 0, \tag{4.11}
\]
where we used the notation
\[
X^z = X + \pi_t^*(X)(e^z - 1) \tag{4.12}
\]
for the wealth after a jump. Having solved for $\pi^*$, we know that $U_t$ fulfills the partial integro–differential equation
\[
\partial_t U_t(X) + L^\pi U_t(X) = 0. \tag{4.13}
\]
For investment with a time horizon $T$, the boundary condition is given by the utility $U$ at time $T$, i.e.,

$$U_T(X) = U(X). \quad (4.14)$$

We want to state an important Lemma 4.1. Let $U_t$ be a solution to (4.13) and (4.14), with $\pi^*$ fulfilling (4.10). Then $U_t'$ is a solution to the PIDE

$$\partial_t U_t'(X) + L^\pi U_t'(X) = 0 \quad (4.15)$$

with the boundary condition

$$U'_T(X) = U'(X). \quad (4.16)$$

In particular,

$$U'_t(X) = E[U'_\tau(X_\tau)|X_t = X, \pi^*] \quad \forall \tau \geq t, \quad (4.17)$$

so $U'_t$ is a martingale.

**Proof.** Differentiating (4.13) and (4.14) w.r.t. $X$, one obtains (4.15) and (4.16), since

$$\partial_X L^\pi U(X) = \pi_t^*(X) \mu_\pi \partial_X U'(X) + \pi_t^*(X)^2 \frac{\sigma^2}{2} \partial_X^2 U'(X)$$

$$+ \lambda \int dz \ (U'(X^z) - U'(X)) f_J(z) + \partial_X \pi_t^*(X) \frac{\partial}{\partial \pi_t(X)} L^\pi U_t.$$

The last term on the r.h.s. of this equation vanishes due to (4.10). Equation (4.17) then follows from the fact that, upon setting $t = \tau = T$, one recovers the boundary condition (4.16), and upon choosing $\tau = t + dt$ and expanding the r.h.s. of (4.17), one gets (4.15):

$$U'_t(X) = U'_t(X) + (\partial_t U'_t(X) + L^\pi U'_t(X)) dt.$$

Thus, (4.17) is a solution to (4.15), fulfilling the boundary condition (4.16).

This Lemma is the counterpart of (3.3) in the continuous time setting (there, also the risk-free rate enters, which is absent here, since we work in discounted units).

### 4.5 Optimal investment

Our aim is now to solve the optimal investment problem, i.e., find $\pi^*$ that solves (4.10) and the corresponding expected utility function $U_t$ solving (4.13). We do this for logarithmic, power and exponential utility.
4.5.1 Logarithmic utility

We consider logarithmic utility, i.e., \( U(X) = \ln X \). We assume that the expected utility \( U_t \) is of the form
\[
U_t(X) = \ln X + A_t. \tag{4.18}
\]
This choice of \( U_t \) will be justified a posteriori. Equation (4.11) then becomes\(^2\)
\[
\frac{\mu}{X} - \pi_t^*(X) \frac{\sigma^2}{X^2} + \lambda \int dz \frac{e^z - 1}{X + \pi_t^*(X)(e^z - 1)} f_J(z) = 0.
\]
Multiplying with \( X \) and setting \( \tilde{\pi}_t^*(X) = \pi_t^*(X)/X \), this is
\[
\mu - \tilde{\pi}_t^*(X) \sigma^2 + \lambda \int dz \frac{e^z - 1}{1 + \tilde{\pi}_t^*(X)(e^z - 1)} f_J(z) = 0. \tag{4.19}
\]
Obviously, \( \tilde{\pi}^*(X) \) is independent of \( X \), i.e., \( \pi^*(X) \) is proportional to \( X \). It is also independent of \( t \) (this is a consequence of our ansatz).

We may now compute \( U_t \) and thus justify the ansatz (4.18). Having established that, with our ansatz, \( \tilde{\pi}_t^* = \pi_t^*(X)/X \) is independent of \( X \) and \( t \), we obtain, upon inserting (4.18) in (4.13),
\[
\dot{A}_t + \tilde{\pi}^* \mu - (\tilde{\pi}^*)^2 \frac{\sigma^2}{2} + \lambda \int dz \ln(1 + \tilde{\pi}^*(e^z - 1)) f_J(z) = 0.
\]
Given the boundary condition \( U_T(X) = \ln X \), this is solved by
\[
A_t = \left\{ \tilde{\pi}^* \mu - (\tilde{\pi}^*)^2 \frac{\sigma^2}{2} + \lambda \int dz \ln(1 + \tilde{\pi}^*(e^z - 1)) f_J(z) \right\} (T-t).
\]
Since there is no \( X \)-dependence in \( A_t \), this justifies our ansatz (4.18).

We now discuss two special cases in which (4.19) can be solved analytically:

Example 4.2. If we deal with a pure jump process with the jumps being only of one size \( J \), i.e., \( \sigma = 0 \), \( f_J(z) = \delta(z-J) \), then
\[
\tilde{\pi}^* = -\frac{\mu + \lambda(e^J - 1)}{\mu(e^J - 1)}. \tag{4.20}
\]
However, this is not the correct solution for all choices of \( \mu \) and \( J \). When a jump happens, wealth should not become negative, i.e.,
\[
\tilde{\pi}^*(e^J - 1) > -1. \tag{4.21}
\]
\(^2\)Since the asset price process is scale invariant, the optimal investment strategy will not depend on \( S \), so we suppress the \( S \)-dependence of \( \pi \) in this section.
For $\mu < 0$, this means $e^J - 1 > 0$, i.e., if the drift is lower than the risk–free rate, then the asset should allow for upward jumps. Conversely, for $\mu > 0$, this means $e^J - 1 < 0$. Under these conditions, the denominator in (4.20) is always negative. Thus, if optimal investment is possible, then $\tilde{\pi}^*$ has the same sign as the average drift $\mu + \lambda(e^J - 1)$, as expected.

**Example 4.3.** If only jumps of a certain size $J$ can happen, i.e., $f_J(z) = \delta(z - J)$, then (4.19) becomes

$$\sigma^2(e^J - 1)\tilde{\pi}^*_t(X)^2 + \left\{\sigma^2 - \mu(e^J - 1)\right\} \tilde{\pi}^*_t(X) - \left\{\mu + \lambda(e^J - 1)\right\} = 0.$$ 

We introduce the notation $\tilde{J} = e^J - 1$ for the relative jump size. The above is then solved by

$$\tilde{\pi}^* = -\frac{1}{2\tilde{J}} \left\{ 1 - \frac{\mu}{\sigma^2} \tilde{J} \pm \sqrt{\left(1 - \frac{\mu}{\sigma^2} \tilde{J}\right)^2 + \frac{4\mu + \lambda\tilde{J}}{\sigma^2} \tilde{J}} \right\}$$

Of these, only one solution is correct, since we also have to fulfil the constraint (4.21). This leads to the condition

$$\pm \sqrt{\left(1 - \frac{\mu}{\sigma^2} \tilde{J}\right)^2 + \frac{4\mu + \lambda\tilde{J}}{\sigma^2} \tilde{J}} < 1 + \frac{\mu}{\sigma^2} \tilde{J}.$$ 

For the plus sign, this is fulfilled if

$$\left(1 - \frac{\mu}{\sigma^2} \tilde{J}\right)^2 + \frac{4\mu + \lambda\tilde{J}}{\sigma^2} \tilde{J} < \left(1 + \frac{\mu}{\sigma^2} \tilde{J}\right)^2,$$

i.e.

$$\frac{\mu + \lambda\tilde{J}}{\sigma^2} \tilde{J} < \frac{\mu}{\sigma^2} \tilde{J}.$$

This is obviously not fulfilled for positive $\lambda$, regardless the sign of $\tilde{J}$. For the minus sign, one finds the above inequality with reversed sign, which is always fulfilled for positive $\lambda$. Thus we obtain

$$\tilde{\pi}^* = -\frac{1}{2\tilde{J}} \left\{ 1 - \frac{\mu}{\sigma^2} \tilde{J} - \sqrt{\left(1 - \frac{\mu}{\sigma^2} \tilde{J}\right)^2 + \frac{4\mu + \lambda\tilde{J}}{\sigma^2} \tilde{J}} \right\}. \quad (4.22)$$

This has the expected behaviour: $\tilde{\pi}^*$ always has the same sign as the average drift $\mu + \lambda\tilde{J}$. Also note that the expression under the square root is strictly positive, so that an optimal investment strategy fulfilling (4.21) always exist for a fixed jump size.
4.5.2 Power utility

For power utility, \( U(X) = X^\beta / \beta \) for \( \beta < 0 \). We assume that the expected utility \( U_t \) is of the form
\[
U_t(X) = B_t X^\beta / \beta,
\]
(4.23)

This will be justified a posteriori. Equation (4.11) then becomes
\[
\mu X^{\beta - 1} + \pi_t^*(X)(\beta - 1)\sigma^2 X^{\beta - 2} + \lambda \int dz \ (e^z - 1)(X + \pi_t^*(X)(e^z - 1))^{\beta - 1} f_J(z) = 0.
\]

Again, we set \( \tilde{\pi}_t^*(X) = \pi_t^*(X)/X \), and obtain
\[
\mu + \tilde{\pi}_t^*(X)(\beta - 1)\sigma^2 + \lambda \int dz \ (e^z - 1)(1 + \tilde{\pi}_t^*(X)(e^z - 1))^{\beta - 1} f_J(z) = 0.
\]

As in the case of logarithmic utility, \( \tilde{\pi}_t^*(X) \) is independent of \( X \) and \( t \). Inserting (4.23) into (4.13), we obtain
\[
\dot{B}_t + B_t \left\{ \tilde{\pi}_t^* \mu + (\tilde{\pi}_t^*)^2 \beta(\beta - 1)\sigma^2 + \lambda \int dz \ \{(1 + \tilde{\pi}_t^*(e^z - 1))^{\beta - 1}\} f_J(z) \right\} = 0.
\]

Given the boundary condition \( U_T(X) = X^\beta / \beta \), this is solved by
\[
B_t = e^{\left\{ \tilde{\pi}_t^* \mu + (\tilde{\pi}_t^*)^2 \beta(\beta - 1)\sigma^2 + \lambda \int dz \ \{(1 + \tilde{\pi}_t^*(e^z - 1))^{\beta - 1}\} f_J(z) \right\}(T-t)},
\]
which is independent of \( X \). This justifies our ansatz (4.23).

4.5.3 Exponential utility

Finally, we consider exponential utility, i.e., \( U(X) = -e^{-\alpha X} \). We assume that the expected utility \( U_t \) is of the form
\[
U_t(X) = -C_t e^{-\alpha X}.
\]
(4.24)

This choice of \( U_t \) will be justified a posteriori. Equation (4.11) then becomes
\[
\mu - \pi_t^*(X)\sigma^2 \alpha + \lambda \int dz \ (e^z - 1)e^{-\alpha \pi_t^*(X)(e^z - 1)} f_J(z) = 0.
\]

Obviously, a solution \( \pi_t^*(X) \) of this equation will be independent of \( X \) and \( t \). Inserting (4.24) into (4.13), we obtain
\[
-C_t + C_t \left\{ \alpha \pi_t^* \mu - (\alpha \pi_t^*)^2 \sigma^2 + \lambda \int dz \ \{e^{-\alpha \pi_t^*(e^z - 1)} - 1\} f_J(z) \right\} = 0.
\]

Given the boundary condition \( U_T(X) = -e^{-\alpha X} \), i.e., \( C_T = 1 \), this leads to
\[
C_t = e^{-\left\{ \alpha \pi_t^* \mu - (\alpha \pi_t^*)^2 \sigma^2 + \lambda \int dz \ \{e^{-\alpha \pi_t^*(e^z - 1)} - 1\} f_J(z) \right\}(T-t)}.
\]

Again, \( C_t \) is independent of \( X \), which justifies our ansatz (4.24).
4.6 The marginal indifference price

According to (2.3), the marginal indifference price fulfils, for \( \tau \geq t \),

\[
V_t(X, S) = E_t \left[ \frac{U_{\tau}'(X_{\tau})}{U_t'(X_t)} V_{\tau}(X_{\tau}, S_{\tau}) | X, S, \pi^* \right] \quad (4.25)
\]

This shows that

\[
\frac{dQ}{dP} = U_T'(X_T^{\pi^*}) \quad (4.26)
\]

is the Radon–Nikodym derivative that turns the real–world measure \( P \) into the appropriate risk–neutral measure \( Q \). Here we wrote \( X^{\pi^*} \) to indicate that the optimal investment strategy is used for the evolution of \( X \). For the evolution of the product of \( U' \) and \( V \), we obtain (for convenience, we divide by \( U' \))

\[
\frac{d(U_t'(X_t)V_t(X_t, S_t))}{U_t'(X_t)} = \frac{V_t(X_t, S_t)}{U_t'(X_t)} \left( dU_t'(X_t) - \{ U_{\tau}'(X_{\tau}^{J_t}) - U_t'(X_t) \} dN_t \right) \quad (4.27)
\]

\[
+ \{ \partial_t V_t(X_t, S_t) + \mu_S \partial_S V_t(X_t, S_t) + \mu \pi_t(X_t) \partial_X V_t(X_t, S_t) \} dt \quad (4.28)
\]

\[
+ \frac{\sigma^2}{2} \left\{ S_t^2 \partial^2_S V_t(X_t, S_t) + 2S_t \pi_t(X_t) \partial_S \partial_X V_t(X_t, S_t) + \pi_t^2(X_t) \partial^2_X V_t(X_t, S_t) \right\} dt \quad (4.29)
\]

\[
+ \sigma \pi_t(X_t) \frac{U_t'(X_t)}{U_t'(X_t)} \left\{ S_t \partial_S V_t(X_t, S_t) + \pi_t(X_t) \partial_X V_t(X_t, S_t) \right\} dW_t \quad (4.30)
\]

\[
+ \left\{ U_t'(X_t^{J_t}) \frac{U_t'(X_t)}{U_t'(X_t)} V_t(X_t^{J_t}, e^{\mu_S} S_t) - V_t(X_t, S_t) \right\} dN_t. \quad (4.31)
\]

In (4.27) and (4.32), we used again the notation introduced in (4.12). The term (4.27) contains the time evolution and the diffusion of \( U_t' \). The terms (4.28, 4.29, 4.31) stem from the evolution of \( V_t \) only. The term (4.30) comes from the joint diffusion of \( V_t \) and \( U_t' \) and the term (4.32) from the joint evolution of \( V_t \) and \( U_t' \) under jumps. Setting the expected value to zero, and using the martingale property (4.17) of \( U' \), one finds the following PIDE for \( V_t \):

\[
\partial_t V_t(X, S) + L_t^Q V_t(X, S) = 0, \quad (4.33)
\]
where $L^Q_t$ is the integro–differential operator defined by

$$
L^Q_t f(X, S) = \left\{ \mu + \sigma^2 \pi^*_t(X) \frac{U''_t(X)}{U'_t(X)} \right\} S \partial_S f(X, S)
+ \left\{ \pi^*_t(X) \mu + \sigma^2 \pi^*_t(X)^2 \frac{U''_t(X)}{U'_t(X)} \right\} \partial_X f(X, S)
+ \frac{\sigma^2}{2} \left\{ S^2 \partial_S^2 f(X, S) + 2S \pi^*_t(X) \partial_S \partial_X f(X, S) + \pi^*_t(S)^2 \partial_X^2 f(X, S) \right\}
+ \lambda \int dz \left\{ f(X^z, e^z S) - f(X, S) \right\} \frac{U'_t(X^z)}{U'_t(X)} f_J(z). \tag{4.34}
$$

Here we combined the second term in (4.27) and (4.32) to a new jump term.

**Example 4.4.** If the process is a pure diffusion, i.e., $\lambda = 0$, then, according to (4.11),

$$
\pi^*_t(X) = -\frac{\mu}{\sigma^2} \frac{U'_t(X)}{U''_t(X)}.
$$

Thus, the first term in (4.34) vanishes (and is thus independent of $X$), for any utility function. All other terms in $L^Q_t$ that bear some $X$–dependence involve $\partial_X$, so if the terminal condition is independent of $X$ (as is the case for a payoff), then all these terms vanish. Hence, (4.33) becomes the Black–Scholes PDE, in discounted units. Incidentally, this shows that the knowledge of the correct optimal investment strategy $\pi^*$ is crucial for finding the correct price (and thus also the optimal hedge) in this framework.

**Remark 4.5.** From our discussion in Section 4.5, we know that in the case of exponential, power, or logarithmic utility,

$$
\pi^*_t(X) = \frac{U''_t(X)}{U'_t(X)}
$$

and

$$
\frac{U'_t(X^z)}{U'_t(X)} = \frac{U'_t(X + \pi^*_t(X)(e^z - 1))}{U'_t(X)}
$$

are independent of $X$. Thus, for these types of utility, the only terms in $L^Q_t$ that depend on $X$ are those that involve at least one $\partial_X$. It follows that if the terminal condition is independent of $X$, the solution to (4.33) will also be independent of $X$. Since by definition the payoff only depends on $S$, the marginal indifference price is independent of $X$ for exponential, power, or logarithmic utility. The pricing PIDE then reduces to

$$
0 = \partial_t V_t(S) + \left\{ \mu + \sigma^2 \pi^*_t(X) \frac{U''_t(X)}{U'_t(X)} \right\} S \partial_S V_t(S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V_t(S)
+ \lambda \int dz \left\{ V_t(e^z S) - V_t(S) \right\} \frac{U'_t(X^z)}{U'_t(X)} f_J(z). \tag{4.35}
$$
Comparison with (4.6) shows that the jump distribution and frequency have changed. The new jump frequency is given by
\[ \lambda^Q = \lambda \int dz \frac{U'_i(Xz)}{U'_i(X)} f_J(z) \] (4.36)
and the new distribution by
\[ f^Q_J(z) = f_J(z) \frac{\lambda U'_i(Xz)}{\lambda^Q U'_i(X)} \]

**Remark 4.6.** An expression very similar to (4.35) was found in [6]. There, it is assumed that all market participants have power utility, and so the market–clearing utility must also be of this form. Furthermore, the market is invested fully in the asset, the positions in cash and options cancel each other\(^3\). This corresponds to \( \pi^*(X) = X \) in the present setting. In [6], the pricing PIDE is
\[ 0 = \partial_t V_t(S) - S \partial_S V_t(S) \lambda \int dz (e^{\gamma} - 1)e^{(\gamma - 1)z} f_J(z) + \frac{\sigma^2}{2} S^2 \partial^2_S V_t(S) + \lambda \int dz \{ V_t(e^{\gamma}S) - V_t(S) \} e^{(\gamma - 1)z} f_J(z), \] (4.37)
where \( \gamma \) is the exponent of the market–clearing utility. This is consistent with our PIDE (4.35) when \( \pi^*(X) = X \) and \( U_t(x) = X^\gamma / \gamma \) are employed and \( \mu \) is adjusted such that the average drift implied by the PIDE vanishes. In this sense, (4.37) is a special case of (4.35). However, the option value as seen by investors with individual utility functions, and in particular their optimal hedging strategies, are not discussed in [6].

**Remark 4.7.** The modified jump term in (4.35) takes into account how the option trade matches to the optimal investment strategy \( \pi^* \). However, the investment strategy a bank chooses is typically not derived from the model that is used to price options. Furthermore, drifts are notoriously hard to determine from market data. But the optimal investment strategy \( \pi^* \) crucially depends on the expected return and thus on the drift. A possible way out would be to interpret (4.35) in a different way: One considers \( \pi^* \) as the actual investment strategy and modifies the drift such that it is also the optimal investment strategy under this modified drift. Then (4.35) takes into account how the option trade matches to the actual investment strategy. Models in which the option value depends on the portfolio of assets (and options) the investor already holds, are discussed, e.g., in [1]. The difficulty in the present setting is that the actual investment strategy has to be known also for future times, which is probably not the case in general.

\[^3\text{This implies that the model is applicable to an “index” that comprises the whole market, i.e., in principle equities, commodities, real estate, etc.}\]
4.7 Marginal optimal hedging

We are now ready to discuss the marginal optimal hedging strategy \( \hat{\pi} \) for our jump diffusion process. The discussion will be rather heuristic and certainly deserves a thorough mathematical investigation. We recall from Proposition 2.7, that it fulfils

\[
\partial_X E \left[ U'(X_T) \right] \left[ \frac{\delta}{\delta \pi(Z')} E \left[ U(X_T) | Z, \pi^* \right] C(S_T) | Z, \pi^* \right] = \hat{\pi}(Z') \frac{\delta^2}{\delta \pi(Z')^2} E \left[ U(X_T) | Z, \pi^* \right]. \tag{4.38}
\]

cf. (2.7). On both sides of this equation, we find a functional derivative of an expected value. Our strategy for solving for \( \hat{\pi} \) will be to evaluate both sides on a trading strategy \( \pi' \) that is constant on the time interval \([t, t + dt)\) and around \( X, S \).

The marginal optimal hedge \( \hat{\pi}_t(X, S) \) is then implicitly defined by equating the l.h.s. evaluated in \( \pi' \) and the second order functional derivative of \( U_t \), evaluated in \( \pi' \otimes \hat{\pi} \).

The expected value that is functionally differentiated on the l.h.s. of (4.38) is nothing but the marginal indifference price \( V \) which is a solution to a PIDE. The functional derivative can thus be computed by “differentiating” this PIDE. This can be done as follows: Let \( V \) fulfil the PIDE

\[
\partial_t V_t + L_t V_t = 0,
\]

with terminal boundary condition, where \( L_t \) is an integro-differential operator depending on \( \pi_t \). Since we solve backwards in time this may formally be written as

\[
V_t = V_{t+dt} + L_t V_t dt,
\]

where \( V_{t+dt} \) is given. Applying an infinitesimal change \( \pi \to \pi + \epsilon \pi' \), where \( \pi' \) is constant in \( X \) and \( S \) and during the time interval \([t, t + dt)\) and vanishing for later times, one obtains

\[
\langle \frac{\delta}{\delta \pi} V_t, \pi' \rangle = \frac{\partial}{\partial \pi_t} L_t V_t \pi' dt,
\]

where on the r.h.s. one differentiates w.r.t. the \( \pi_t \) appearing in \( L_t \). From (4.33) and (4.34), and noting that in deriving these from (4.25) we did not use the the optimality

\footnote{To be more precise, one has to differentiate the system of PIDEs for \( U \) and \( V \). However, \( U \) is optimal, so its first order functional derivative vanishes.}
of $\pi^*$, we find
\[
\frac{\partial}{\partial \pi_t} L_t^0 V_t(X, S) = \sigma^2 \frac{U''(X)}{U'_i(X)} S \partial_S V_t(X, S) \\
+ \left\{ \mu + 2\sigma^2 \pi_t(X) \frac{U''(X)}{U'_i(X)} \right\} \partial_X V_t(X, S) \\
+ \sigma^2 \left\{ S \partial_S \partial_X V_t(X, S) + \pi_t(X) \partial_X^2 V_t(X, S) \right\} \\
+ \lambda \int dz \ \{ V_t(X^z, e^z S) - V_t(X, S) \} \frac{U''(X^z)}{U'_i(X)} (e^z - 1) f_J(z) \\
+ \lambda \int dz \ \partial_X V_t(X^z, e^z S) \frac{U''(X)}{U'_i(X)} (e^z - 1) f_J(z).
\]

Here we used again the notation $X^z$ for the wealth after a jump, cf. (4.12).

Assuming exponential, power, or logarithmic utility again, we know from Remark 4.5 that $V$ is independent of $X$, so the above reduces to
\[
\frac{\partial}{\partial \pi_t} L_t^0 V_t(X, S) = \\
\sigma^2 \frac{U''(X)}{U'_i(X)} S \partial_S V(S) + \lambda \int dz \ \{ V(e^z S) - V(S) \} \frac{U''(X^z)}{U'_i(X)} (e^z - 1) f_J(z).
\]

For the functional derivative on the r.h.s. of (4.38) we similarly obtain, for the same choice of $\pi'$,
\[
\langle \frac{\delta^2}{\delta \pi^2} U_t, \pi' \otimes \hat{\pi} \rangle = \hat{\pi}_t \frac{\partial^2}{\partial \pi^2} L_t^0 U_t \pi'dt,
\]
cf. (4.13). Here we used Lemma 2.6 which states that the symmetric bidistibution corresponding to the second order functional derivative of $U$ has support on the diagonal. With (4.8) one finds
\[
\frac{\partial^2}{\partial \pi^2} L_t^0 U_t(X) = \sigma^2 U''(X) + \lambda \int dz \ (e^z - 1)^2 U''(X^z) f_J(z).
\]

Thus, for power, logarithmic and exponential utility, we obtain, by comparing the r.h.s. and the l.h.s. of (4.38), evaluated in $\pi'$, the marginal optimal hedging strategy
\[
\hat{\pi}_t(S) = S \frac{\sigma^2 \partial_S V_t(S) + \lambda \int dz \ \frac{V_t(e^z S) - V_t(S)}{(e^z - 1) S} (e^z - 1)^2 U''(X^z) f_J(z)}{\sigma^2 + \lambda \int dz \ (e^z - 1)^2 U''(X^z) f_J(z)} \tag{4.39}
\]

Remark 4.8. In the case of exponential, power, or logarithmic utility (for which this formula is valid), the expression
\[
\frac{U''(X^z)}{U''(X)}
\]
is independent of $X$, according to the discussion in Section 4.5. Thus, the marginal optimal hedging strategy is independent of $X$ in that case.
Example 4.9. In the pure diffusion case $\lambda = 0$ one recovers Black–Scholes $\Delta$ hedging, $\hat{\pi} = S \partial_S V$. But if a jump component is present, the marginal optimal hedge is not given by $S \partial_S V$. Instead, it optimally balances diffusion and jump risk, given the specified utility function.

Remark 4.10. Taking the marginal optimal hedge as starting point and following the derivation of Merton’s formula (4.6) from (4.5), one does in general not recover the PIDE (4.35) for the marginal indifference price. This is not surprising, since in the present framework the investor wants to be compensated for taking risk.

4.8 Infinite activity processes

For the preceding discussion, we assumed that the asset price process has finite activity, i.e., that the jump distribution $f_J(z)$ is integrable around $z = 0$. In this section, we discuss whether this condition may be relaxed.

If the jump distribution is not integrable around $z = 0$, one has to introduce a compensator term $\nu$ in order to define the stochastic differential. Instead of (4.1), we then have the following process for $s = \log S$:

$$s_t = \mu t + \sigma W(t) + \int_0^t \int_{|z|<1} z \{N(dt, dz) - \nu(dz)dt\} + \int_0^t \int_{|z|>1} z N(dt, dz).$$

Here $N$ is the joint jump distribution function and the compensator $\nu$ is chosen such as to make the integral over $z = 0$ well-defined. By the Itô formula for Lévy processes, one then has a different representation of the operator $L^P$, cf. (4.4):

$$L^P f(S) = \mu S \partial_S f(S) + \frac{\sigma^2}{2} S^2 \partial_S^2 f(S) + \int dz \left( f(e^z S) - f(S) - \chi(z) z S \partial_S f(S) \right) f_J(z).$$

Here $\chi$ denotes the characteristic function of the interval $[-1, 1]$. We used $f_J$ as the jump distribution function again, but dropped the constant $\lambda$, since it does not make sense to speak of a jump frequency in the infinite activity case. On the other hand, $f_J$ is no longer normalised. Similarly, for $L^n$, cf. (4.8), one obtains

$$L^n G_t(X, S) = \pi_t(X, S) \mu \partial_X G_t(X, S) + \mu S \partial_S G_t(X, S)$$

$$+ \frac{\sigma^2}{2} \left\{ \pi_t(X, S)^2 \partial_X^2 + 2 \pi_t(X, S) S \partial_X \partial_S + S^2 \partial_S^2 \right\} G_t(X, S)$$

$$+ \int dz \left\{ G_t(X + \pi_t(X, S)(e^z - 1), e^z S) - G_t(X, S)$$

$$- \chi(z) z (\pi_t(X) \partial_X G_t(X, S) + S \partial_S G_t(X, S)) \right\} f_J(z).$$
The condition for the optimal investment strategy $\pi^*$, cf. (4.11), now becomes

$$\mu U'_t(X) + \pi^*_t(X) \sigma^2 U''_t(X) + \int dz \left\{ U'_t(X^z)(e^z - 1) - \chi(z) z U'_t(X) \right\} f_J(z) = 0. \tag{4.40}$$

For a Lévy process, the integral

$$\int dz \min(z^2, 1) f_J(z)$$

has to be finite. In the integral on the r.h.s. of (4.40), the expression in curly brackets is of order $z^2$ around $z = 0$, so for a Lévy process the integral is well-defined.

For the jump term in $L^Q$, cf. (4.34), one now finds, by adjusting the terms (4.27) and (4.32) appropriately,

$$\int dz \left\{ f(X^z, e^z S) - f(X, S) - \chi(z) z \pi^*_t(X) \partial_X f(X, S) + S \partial_S f(X, S) \right\} \times \frac{U'_t(X^z)}{U'_t(X)} f_J(z).$$

Note that no compensator term involving the derivative of the expected utility occurs. Thus, also in the infinite activity case the risk-neutral jump distribution is obtained by multiplication with $U'_t(X)/U'_t(X)$. For a well-behaved $f$, the expression in curly brackets is of order $z^2$ around $z = 0$, so the integral is well-defined for a Lévy process.

The PIDE for the marginal indifference price in the case of logarithmic, power, or exponential utility, cf. (4.35), is then modified to

$$0 = \partial_t V_t(S) + \left\{ \mu + \sigma^2 \pi^*_t(X) \frac{U''_t(X)}{U'_t(X)} \right\} S \partial_S V_t(S) + \frac{\sigma^2}{2} S^2 \partial^2_S V_t(S)$$

$$+ \int dz \left\{ V_t(e^z S) - V_t(S) - \chi(z) z S \partial_S V_t(S) \right\} \frac{U'_t(X^z)}{U'_t(X)} f_J(z).$$

The marginal optimal hedge in the infinite activity case, cf. (4.39) is then given by

$$\hat{\pi}_t(S) = \frac{\sigma^2 \partial_S V_t(S)}{S} + \int dz \frac{V_t(e^z S) - V_t(S) - \chi(z) z S \partial_S V_t(S)}{(e^z - 1)S} \frac{(e^z - 1)^2 U''_t(X^z)}{U'_t(X)} f_J(z).$$

The integral in the denominator is well-defined since $(e^z - 1)^2$ is of order $z^2$ around $z = 0$. The integral in the numerator is also well-defined: If $V$ is well-behaved, then the fraction involving the price is of order $z$ around $z = 0$, so the integrand

\[ ^5 \text{Using (4.40), one can show, analogously to the finite activity case, that the marginal indifference price is independent of } X \text{ for these types of utility functions.} \]
is of order $z^3$. Problems may occur in the limit $t \to T$ for payoffs that are not continuously differentiable. These, however, are not essentially different from those that are already present in a purely diffusive setting. As there, any discontinuities in the payoff (or derivatives thereof) are smeared by the heat kernel in the diffusive part for times $t < T$.

We may thus conclude that the framework can also be applied to infinite activity processes.

### 4.9 Minimal variance hedging

In minimal variance pricing and hedging\footnote{Sometimes also called mean–variance pricing and hedging.}, one tries to minimise the variance of the value of a portfolio consisting of the claim and the hedge. Thus, it can be expressed as the following joint optimisation problem:

$$
(V_0, \vartheta) = \arg \inf_{v, \vartheta} E \left[ \left( C - v - \int_0^T \vartheta_t dS_t \right)^2 \right]. 
$$

(4.41)

As above, $C$ is the payoff of the claim. Since the aim is to minimise the variance of the real–world returns, one should use the physical measure $P$ here. As shown in [11, Proposition 2], the above optimisation problem for $V_0$ can be solved by evaluating the payoff under the so–called variance–optimal signed measure $\mathcal{V}$. As shown in [11, Theorem 13], $\mathcal{V}$ is a measure if the process $S$ is continuous, but only a signed measure in the general case. This will be illustrated below in our example of jump diffusion. Working with signed measures is obviously something one would like to avoid (otherwise there are positive claims to which one assigns negative values). But in general the only way out would be to start with a martingale measure equivalent to $P$ in (4.41). However, there is a whole continuum of such measures, so this would introduce a lot of arbitrariness. Furthermore, the choice of the appropriate risk–neutral measure is precisely what one usually expects from the pricing framework.

In minimal variance hedging, risk, as measured by the variance of the returns, is minimised. Thus, it penalises profits as well as losses. Furthermore, it is not clear in which sense the investor is compensated for the remaining risk (see also Remark 4.13 below).

As shown in [10], the optimisation problem (4.41) is closely related to finding the Föllmer–Schweizer decomposition of the payoff $C$. For this, one first decomposes the
process \( S_t \) as
\[
S_t = S_0 + M_t + \int_0^t \alpha_s d\langle M \rangle_s,
\]
where \( M \) is a martingale and \( \langle M \rangle \) its sharp bracket. In the present case, one obtains
\[
\begin{align*}
    dM_t &= dS_t - S_t \left\{ \mu + \lambda \int dz \left( e^{z} - 1 \right) f_J(z) \right\} dt \\
    d\langle M \rangle_t &= S_t^2 \left\{ \sigma^2 + \lambda \int dz \left( e^{z} - 1 \right)^2 f_J(z) \right\} dt
\end{align*}
\]
and thus
\[
\alpha_t = \frac{1}{S_t} \frac{\mu + \lambda \int dz \left( e^{z} - 1 \right) f_J(z)}{\sigma^2 + \lambda \int dz \left( e^{z} - 1 \right)^2 f_J(z)}. \tag{4.42}
\]
We remark that \( S_t \alpha_t \) can be interpreted as a generalisation of the market price of risk, with the excess return in the numerator and the quadratic variation in the denominator. We also note that the so-called mean–variance tradeoff process \([10]\)
\[
\tilde{K}_t = \int_0^t \alpha_s^2 d\langle M \rangle_s
\]
is purely deterministic, which guarantees the applicability of the results from \([10, 11]\) mentioned below.

Given the above data, the Föllmer–Schweizer decomposition of the payoff \( C \) is of the form
\[
C = C_0 + \int_0^T \theta^C_u dS_u + L^C_T,
\]
where \( C_0 \) is a number and \( L^C_T \) a martingale under \( P \) which is orthogonal to all \( \int \gamma dM \).
The strategy \( \theta^C \) is called the locally risk minimising strategy. Its economic significance stems from the fact that the hedging error \( L^C \) is a martingale, so that on average the hedging error vanishes. As shown in \([10, \text{Theorem 3}]\), \( \theta^C \) is related to \( \vartheta \) by
\[
\vartheta_t = \theta^C_t + \alpha_t \left( V^C_t - V_0 - \int_0^t \vartheta_u dS_u \right) \tag{4.43}
\]
where
\[
V^C_t = C_0 + \int_0^t \theta^C_u dS_u + L^C_t \tag{4.44}
\]
is called the value process of \( C \). Furthermore, \( V_0 = C_0 \), according to \([10, \text{Corollary 10}]\).

Remark 4.11. Equation (4.43) can be interpreted as follows: Assume the trader is short in the claim. Then at \( t = 0 \) the claim is hedged with \( \theta_0^C \) quantities of the asset. For \( t > 0 \) the expression in brackets in the second term on the r.h.s. of (4.43)
represents the losses incurred so far from the trading strategy. As shown above, $\alpha$ is a generalisation of the market price of risk. It follows that if the trading strategy earned money (so that the expression in brackets is negative), and the asset has positive mean returns (which means $\alpha > 0$), then the second term in (4.43) tells the trader to short the asset, i.e., to try to get rid of the profit. The reason for this strange behaviour is the statement (4.41) of the optimisation problem, which penalises profits as well as losses. In fact it can be shown that the optimisation problem $E[(a - \theta dS)^2]$ for a fixed constant $a$ leads to $\theta_0 = \alpha_0 a$, which can directly be compared to the second term on the r.h.s. of (4.43). Obviously, this behaviour does not make sense economically. This is probably the reason why many authors consider the risk–minimising strategy $\theta^C$ instead.

Now we study the optimal–variance measure $\mathcal{V}$ for the case of jump diffusion. According to [11, Example 1], its Radon–Nikodym derivative is given by

$$
\frac{d\mathcal{V}}{dP} = \frac{\mathcal{E}(-\int \alpha dS)}{\mathcal{E}[\mathcal{E}(-\int \alpha dS)]},
$$

Note that if $S$ is a martingale under $P$, then $\alpha$ vanishes, so that $\mathcal{V} = P$ in this case.

Using $\mathcal{V}$, one finds, similarly to the calculation in Section 4.6, the pricing PIDE

$$
\partial_t V + (\mu - \sigma^2 S \alpha) S \partial_S V + \frac{\sigma^2}{2} S^2 \partial^2_S V + \lambda \int dz \left\{ V(e^z S) - V(S) \right\} \{1 - \alpha (e^z - 1)\} f_J(z) = 0. \quad (4.45)
$$

Note that the new jump measure gives negative contributions if the support of the jump distribution is such that $\alpha (e^z - 1) S > 1$ for some $z \in \text{supp} f_J$. For $\alpha > 0$, this condition will always be fulfilled for unbounded upward jump distributions, which includes Merton’s log–normal jump distribution [7]. This is an illustration of the above mentioned fact that the optimal–variance measure is in general signed.

We now compute the risk–minimising strategy $\theta^C$, which is defined by the Föllmer–Schweizer decomposition (4.44) of $C$. In order to construct this decomposition, we write $dL^C$ as

$$
dL^C_t = a_t dW_t + b_t(J_t) dN_t + \chi_t dt,
$$

where $\chi$ is chosen such that it compensates the average drift from the jump term, so that $L^C$ is a martingale under $P$. Using the fact that $L^C$ is a martingale, the required orthogonality to $dM_t$ leads to the condition

$$
\sigma a_t + \lambda \int dz \, b_t(z)(e^z - 1)f_J(z) = 0.
$$
Writing down the stochastic differential of (4.44) and comparing the coefficients of the diffusion and the jump term, one finds the conditions

\[ a_t = \sigma (S \partial_S V_t(S) - \theta_t^C S), \]
\[ b_t(J) = V_t(e^J S) - V_t(S) - \theta_t^C (e^J - 1) S. \]

This leads to

\[ \theta_t^C = \frac{\sigma^2 \partial_S V_t(S) + \lambda \int dz \frac{V_t(e^{xz} S) - V_t(S)}{(e^{xz} - 1) S} (e^z - 1)^2 f_J(z)}{\sigma^2 + \lambda \int dz \frac{(e^z - 1)^2 f_J(z)}{(e^{xz} - 1) S}}. \] (4.46)

**Remark 4.12.** When the risk–neutral measure $Q$ from marginal indifference pricing, cf. (4.26), is used as a starting point instead of the real–world measure, then one obtains

\[ \theta_t^C = \frac{\sigma^2 \partial_S V_t(S) + \lambda \int dz \frac{V_t(e^{xz} S) - V_t(S)}{(e^{xz} - 1) S} (e^z - 1)^2 U'(x + \pi^*_t(z)(e^{xz} - 1)) f_J(z)}{\sigma^2 + \lambda \int dz \frac{(e^z - 1)^2 f_J(z)}{(e^{xz} - 1) S}}. \]

This does coincide with the optimal marginal hedging strategy in two special cases: Either $\pi^*_t = 0$, i.e., the original real–world measure $P$ is already a martingale. Or one employs exponential utility. However, for general utility functions, $\theta_t^C$ obtained from $Q$ does not coincide with the marginal optimal hedge, in contrast to the case of continuous price processes, cf. [5] and the following section.

**Remark 4.13.** Using (4.42) it is straightforward to show that taking $\theta_t^C$ as hedging strategy and following the derivation of Merton’s formula (4.6) from (4.5), one recovers the pricing PIDE (4.45). This shows that with minimal variance hedging one tries to minimise risk (as measured by the variance), but one is not compensated for it. This is in contrast to the setting of utility maximisation, cf. Remark 4.10. Possible modifications of the framework to include also a risk premium are discussed, e.g., in [1].

### 4.10 The approach of Kramkov and Sirbu

In [5], Kramkov and Sirbu proposed a different definition of optimal hedging strategies. They define them via the wealth process they imply. We present this approach concisely, restricting to one dimension and lacking the mathematical rigour of the original presentation.

Kramkov and Sirbu call $X(x, q)$ the optimal wealth process for an initial capital $x$ and a quantity $q$ of claims. The utility–based wealth process $G(x, q)$ is defined as

\[ G(x, q) = X(c(x, q)) - X(x, q) \]
where \( c(x,q) \) is the indifference price and \( X(x) = X(x,0) \). The marginal hedging strategy \( H(x) \) is a wealth process that is defined as the derivative of \( G(x,q) \) w.r.t. \( q \) at \( q = 0 \). An interest rate \( r = 0 \) is assumed, i.e., they work in discounted units. They prove three theorems that characterise \( H \) in different fashions. Their main theorem [5, Theorem 3] states that

\[
H_t = V_0 + \int_0^t K_u dS_u + \int_0^t \{ H_u - V_u \} \frac{dR_u}{R_u}.
\]  

(4.47)

Here \( V \) is the marginal indifference price and \( K \) is given by its unique Föllmer–Schweizer decomposition (cf. Section 4.9)

\[
V_t = V_0 + \int_0^t K_u dS_u + L_t
\]

with respect to the risk–neutral measure \( Q \) that yields the marginal indifference price. Finally, \( R \) is the risk–tolerance wealth process, i.e., the maximal positive wealth process with final condition

\[
R_T = - \frac{U'(X_T)}{U''(X_T)}.
\]  

(4.48)

Equation (4.47) is interpreted as follows: One starts with initial wealth \( V_0 \) and hedges with \( K \) quantities of the asset. The hedging strategy is not self–financing, thus gains (or losses) are invested in (or financed from) the optimal trading strategy.

Unfortunately, (4.47) is only valid for continuous price processes. Under much milder assumptions\(^9\), the following holds [5, Theorem 2]:

\[
H_t = \frac{R_t}{R_0} \left\{ V_0 + \int_0^t \tilde{K}_u d\frac{R_0 S_u}{R_u} \right\}
\]  

(4.49)

where \( V_0 \) is the marginal indifference price at \( t = 0 \) and \( \tilde{K} \) (not related to the mean–variance tradeoff process from the previous section) is given by the Föllmer–Schweizer decomposition

\[
\frac{R_0}{R_t} V_t = V_0 + \int_0^t \tilde{K}_u d\frac{R_0 S_u}{R_u} + \tilde{N}_t
\]  

(4.50)

with respect to the measure \( \mathbb{R} \) given by

\[
\frac{d\mathbb{R}}{dQ} = \frac{R}{R_0}.
\]  

(4.51)

\(^7\)Since we will only be interested in cases where the price and the hedging strategy do not depend on \( x \), we drop the \( x \)-dependence in order to simplify the notation.

\(^8\)This is in contrast to the situation in minimal variance hedging, where the minimal variance hedging strategy tries to get rid of gains by investing badly, cf. Remark 4.11.

\(^9\)In particular, we assume that Assumption 5 of [5] is fulfilled, i.e., the risk–tolerance wealth process \( R \) exists.
We now assume logarithmic or power utility with power\(^{10}\) \(\beta\) (a similar discussion can be done for exponential utility). Then, as is evident from (4.48),

\[
\frac{R}{R_0} = \frac{X}{X_0}.
\]

In order to deduce \(\tilde{K}\) from the Föllmer–Schweizer decomposition (4.50), we write the infinitesimal change of \(\tilde{N}\) as

\[
d\tilde{N}_t = a_t dW_t + b_t(J_t)dN_t + \chi_t dt,
\]

where \(\chi\) is chosen such that \(\tilde{N}\) is a martingale under \(R\). The process

\[
\tilde{M}_t = \int_0^t \tilde{K}_u \frac{X_0 S_u}{X_u} du
\]

is also a martingale under \(R\), with

\[
d\tilde{M}_t = \tilde{K}_t \frac{X_0}{X_t} (1 - \tilde{\pi}^*) S_{t-} \left\{ \sigma dW_t + \frac{e^{J} - 1}{1 + \tilde{\pi}_t(e^{J} - 1)} dN_t \right\} + \eta_t dt. \tag{4.52}
\]

Here we used again \(\tilde{\pi}^* = \pi^*/X\) and introduced a compensator term \(\eta\) that is irrelevant for the present discussion. Since both \(\tilde{M}\) and \(\tilde{N}\) are martingales under \(R\), and using

\[
\frac{U'(X + \pi^*(e^z - 1))}{U'(X)} = (1 + \tilde{\pi}^*(e^z - 1))^{\beta - 1},
\]

cf. the modified jump term of the risk–neutral measure \(Q\) implied by (4.35), the orthogonality condition is now

\[
\sigma a_t + \lambda \int d z \ (1 + \tilde{\pi}^*(e^z - 1))^{\beta - 1} b_t(z)(e^z - 1)f_J(z) = 0. \tag{4.53}
\]

Note that in the jump term, the denominator of the second term in (4.52) is cancelled by the presence of \(X\) in the measure \(R\), cf. (4.51).

Now in order to fulfil (4.50), an infinitesimal change on the l.h.s. must be balanced on the r.h.s. by a similar change. The stochastic differential of the l.h.s. is given by

\[
d\frac{X_0 V_t}{X_t} = \frac{X_0}{X_t} \left\{ \sigma S_{t-} \partial_S V_t(S_{t-}) dW_t - \sigma \tilde{\pi}^*_t V_t(S_{t-}) dW_t 
\right.
\]

\[
\left. + \left( \frac{V_t(e^{J} S_{t-})}{1 + \tilde{\pi}_t(e^{J} - 1)} - V_t(S_{t-}) \right) dN_t \right\}.
\]

\(^{10}\)Logarithmic utility corresponds to \(\beta = 0\).

\(^{11}\)Here we use Itô calculus and the fact that when a jump occurs, \(\frac{X_0 S}{X}\) changes to \(\frac{X_0 e^{J} S}{X(1 + \tilde{\pi}^*(e^z - 1))}\).
Thus, we conclude

\[ a_t = \frac{X_0}{X_{t-}} \sigma \left\{ (S_t \partial S_t - \tilde{\pi}_t^* V_t) - \tilde{K}_t (1 - \tilde{\pi}_t^*) S_t \right\}, \]

\[ b_t(J) = \frac{X_0}{X_{t-}} \frac{V_t (e^J S) - V_t (S) - \tilde{K}_t (1 - \tilde{\pi}_t^*) (e^J - 1) S_t - \tilde{\pi}_t^* (e^J - 1) V_t (S)}{1 + \tilde{\pi}_t^* (e^J - 1)}. \]

Inserting this into (4.53), one obtains

\[ \tilde{K}_t = \frac{\sigma^2 \partial S V_t (S) + \lambda \int dz \ (1 + \tilde{\pi}_t^* (e^z - 1))^\beta \frac{V_t (e^z S) - V_t (S) (e^z - 1)^2 f_J (z)}{(e^z - 1)^2}}{(1 - \tilde{\pi}_t^*) \left\{ \sigma^2 + \lambda \int dz \ (1 + \tilde{\pi}_t^* (e^z - 1))^\beta \frac{(e^z - 1)^2 f_J (z)}{(e^z - 1)^2} \right\} - \tilde{\pi}_t^* V_t (S)}. \]

We now want to express (4.49) in a form similar to (4.47). We consider the change \( \Delta H_t \) of \( H_t \) when a jump occurs:

\[ \Delta H_t = \frac{X_t}{X_0} (1 + \tilde{\pi}_t^* (e^J - 1)) \left\{ V_0 + \tilde{M}_t + \tilde{K}_t \frac{X_0}{X_{t-}} (1 - \tilde{\pi}_t^*) S_t \frac{e^J - 1}{1 + \tilde{\pi}_t^* (e^J - 1)} \right\} - \frac{X_{t-}}{X_0} \left\{ V_0 + \tilde{M}_{t-} \right\}

= \tilde{\pi}_t^* (e^J - 1) H_{t-} + \tilde{K}_t (1 - \tilde{\pi}_t^*) S_{t-} (e^J - 1)

= \tilde{\pi}_t^* (e^J - 1) \left\{ H_{t-} - V_{t-} (S_{t-}) \right\} + \tilde{K}_t (e^J - 1) S_{t-}.

Here we used

\[ \tilde{K}_t = (1 - \tilde{\pi}_t^*) \tilde{K}_t + \tilde{\pi}_t^* V_t \]

\[ = \frac{\sigma^2 \partial S V_t (S) + \lambda \int dz \ (1 + \tilde{\pi}_t^* (e^z - 1))^\beta \frac{V_t (e^z S) - V_t (S) (e^z - 1)^2 f_J (z)}{(e^z - 1)^2}}{\sigma^2 + \lambda \int dz \ (1 + \tilde{\pi}_t^* (e^z - 1))^\beta \frac{(e^z - 1)^2 f_J (z)}{(e^z - 1)^2}}. \]

Doing the same analysis for the diffusion part and noting that, when a jump occurs,

\[ \tilde{\pi}_t^* (e^J - 1) = \frac{\Delta X_t}{X_t}, \]

we conclude that \( dH_t \) can be written as

\[ dH_t = \tilde{K}_t dS_t + (H_{t-} - V_{t-}) \frac{dX_t}{X_t}. \]

Thus, \( H_t \) is given by

\[ H_t = V_0 + \int_0^t \tilde{K}_u dS_u + \int_0^t \{ V_{u-} - P_{u-} \} \frac{dX_u}{X_u}, \]

which is basically the same as (4.47), the main difference being that \( \tilde{K} \) quantities of the asset are used for hedging. But for the case of power or logarithmic utility considered
here, $\tilde{K}$ coincides with the marginal optimal hedging strategy (4.39) calculated in Section 4.7 via functional differentiation. Since the idea behind the definition of the marginal optimal hedge is the same in both approaches, we suspect that this is generally the case. However, for a proof one would have to define the functional derivatives rigorously, i.e., specify the space of admissible trading strategies, including a topology.

4.11 An example

We conclude this chapter by considering a concrete, analytically tractable example and comparing the prices and hedging strategies obtained using the different approaches considered so far.

We study a jump diffusion process with a fixed jump size $J$ and logarithmic utility. In the three different approaches, one finds a pricing PIDE of the form

$$\partial_t V_t(S) + \frac{\sigma^2}{2} S^2 \partial^2 S V_t(S) + \left\{ r - \tilde{\lambda} \tilde{J} \right\} S \partial_S V_t(S) - r V_t(S) + \tilde{\lambda} \left\{ V_t(e^{\tilde{J}} S) - V_t(S) \right\} = 0,$$

where we used $\tilde{J} = e^J - 1$. The only difference lies in the value of $\tilde{\lambda}$ that is employed.

The above is solved by

$$V_t(S) = \sum_{k=0}^{\infty} \frac{(\tilde{\lambda}(T-t))^k e^{-\tilde{\lambda}(T-t)}}{k!} V_t(e^{k\tilde{J}} S, r, \tilde{\lambda} \tilde{J})$$

(4.55)

where $V_t(S, r, q)$ is the Black–Scholes price for the claim, given a risk–free rate $r$ and a dividend yield $q$. This can be seen as follows: Using

$$\sum_{k=0}^{\infty} \partial_t \frac{(\tilde{\lambda}(T-t))^k e^{-\tilde{\lambda}(T-t)}}{k!} f(e^{k\tilde{J}} S)$$

$$= - \sum_{k=0}^{\infty} \frac{(\tilde{\lambda}(T-t))^k e^{-\tilde{\lambda}(T-t)}}{k!} \tilde{\lambda} \left\{ f(e^{(k+1)\tilde{J}} S) - f(e^{k\tilde{J}} S) \right\}$$

and inserting (4.55) into (4.54), one obtains

$$\sum_{k=0}^{\infty} \frac{(\tilde{\lambda}(T-t))^k e^{-\tilde{\lambda}(T-t)}}{k!} \left\{ V_t(e^{k\tilde{J}} S, r, \tilde{\lambda} \tilde{J}) + \frac{\sigma^2}{2} (e^{k\tilde{J}} S)^2 V''_t(e^{k\tilde{J}} S, r, \tilde{\lambda} \tilde{J}) \right.$$  

$$+ \left\{ r - \tilde{\lambda} \tilde{J} \right\} e^{k\tilde{J}} S V'_t(e^{k\tilde{J}} S, r, \tilde{\lambda} \tilde{J}) - r V_t(e^{k\tilde{J}} S, r, \tilde{\lambda} \tilde{J}) \right\} = 0,$$

which is fulfilled by the definition of $V_t(S, r, q)$. Also the terminal condition is fulfilled, since the Poisson factor in (4.55) vanishes for $k \neq 0$ for $t \rightarrow T$.

\[12\] Here we re–introduced the risk–free rate $r$
Figure 4.1: Implied volatilities for the marginal indifference price (with logarithmic utility) for a jump diffusion with fixed jump size $\tilde{J} = -0.25$ using different jump frequencies $\lambda$ and parameters $\sigma = 0.2, r = 0, \bar{\mu} = \mu + \lambda \tilde{J} = 0.05$ and $T = 1$.

4.11.1 Comparison of the price

In Merton’s approach, the risk–neutral measure is obtained by adjusting the drift such that the process is a martingale, i.e.,

$$\mu = r - \lambda \tilde{J},$$

This amounts to the choice

$$\bar{\lambda}^M = \lambda$$

in the above terminology. The resulting price is called $V^M$ in the following.

For the marginal indifference price the modified jump frequency is given by (4.36).

Using (4.22) one thus obtains

$$\bar{\lambda}^m = \frac{\lambda}{1 + \tilde{J} \tilde{\pi}^*} = \frac{2 \lambda}{1 + \frac{\mu - r}{\sigma^2} \tilde{J} + \sqrt{(1 - \frac{\mu - r}{\sigma^2} \tilde{J})^2 + 4 \frac{\mu - r + \lambda \tilde{J}}{\sigma^2} \tilde{J}}}.$$

The resulting price is called $V^m$ in the following.
Figure 4.2: Implied volatilities for a jump diffusion with fixed jump size $\tilde{J} = -0.25$ using the different pricing methods for the parameters $\sigma = 0.2$, $r = 0$, $\lambda = 0.25$, $\tilde{\mu} = \mu + \lambda \tilde{J} = 0.05$ and $T = 1$. For the marginal indifference price logarithmic utility was used.

For minimal–variance pricing and hedging, the pricing PIDE (4.45) with (4.42) leads to (4.54) with

$$\tilde{\lambda}^{mv} = \lambda (1 - \alpha \tilde{J} S) = \lambda \left(1 - \frac{\tilde{J} \mu + \lambda \tilde{J} - r}{\sigma^2 + \lambda \tilde{J}^2}\right).$$

The resulting price is called $V^{mv}$ in the following.

Figure 4.1 shows shows the marginal indifference price (converted to implied volatilities) as a function of moneyness for different jump frequencies $\lambda$ for the parameters $\sigma = 0.2$, $r = 0$, $\tilde{J} = -0.25$, $\tilde{\mu} = \mu + \lambda \tilde{J} = 0.05$ and $T = 1$. We see that for $\lambda \to 0$, the price converges to the Black–Scholes price, as expected.

Figure 4.2 compares the prices (converted to implied volatilities) for the different methods as a function of moneyness for the parameters $\sigma = 0.2$, $r = 0$, $\lambda = 0.25$, $\tilde{J} = -0.25$, $\tilde{\mu} = \mu + \lambda \tilde{J} = 0.05$ and $T = 1$. As a reference, also the square root of the variance of the process is indicated. We see that the marginal indifference price and
the minimal variance price are quite close together, but the difference to Merton’s price is notable.

In order to illustrate the dependence on the drift (and thus the optimal investment strategy and the market price of risk, respectively), Figure 4.3 shows the same plot as before, but with an expected drift $\tilde{\mu} = \mu + \lambda \tilde{J} = -0.05$. Now the marginal indifference price and the minimal variance price are below Merton’s price. This can be understood as follows: If the expected drift is positive, the investor will be invested in the asset. Since jumps are always downwards in our model, she is exposed to jump risk. Writing a put on the asset in this situation enlarges this exposure. She will thus ask for a risk premium. On the other hand, if the expected drift is negative, the investor is short the asset and is then exposed to the risk of no jumps happening. Writing a put in this situation diminishes the exposure to this risk. Thus, she can sell the put with a discount.
Figure 4.4: Marginal optimal hedging strategies (with logarithmic utility) for a jump diffusion with fixed jump size $\tilde{J} = -0.25$ using different jump frequencies $\lambda$ and parameters $\sigma = 0.2$, $r = 0$, $\hat{\mu} = \mu + \lambda \tilde{J} = 0.05$ and $T = 1$.

### 4.11.2 Comparison of the hedge

In Merton’s approach, the hedge is given by

$$\Delta_t^M(S) = \partial_S V_t^M(S) = \sum_{k=0}^{\infty} \frac{(\lambda(T-t))^k e^{-\lambda(T-t)}}{k!} e^{kJ_t} e^{kJ_S(\sigma^2 + \lambda \tilde{J}^2)}.$$

where $\Delta_t(S, r, q)$ is the Black–Scholes $\Delta$ for the claim, given a risk–free rate $r$ and a dividend yield $q$.

For the marginal optimal hedging strategy, one finds, using (4.39)

$$\Delta_t^m(S) = S \hat{\pi}_t(S) = \frac{\sigma^2 \partial_S V_t^m(S) + \lambda \frac{V_t^m(e^{J_t} - e^{J_S})}{JS} - \frac{j^2}{(1+J^* \sigma^2)}}{\sigma^2 + \lambda \frac{j^2}{(1+J^* \sigma^2)}}.$$

For the risk–minimising hedging strategy (4.46), one obtains

$$\Delta_t^{mv}(S) = \frac{\sigma^2 \partial_S V_t^{mv}(S) + \lambda \frac{V_t^{mv}(e^{J_t} - e^{J_S}) - \frac{j^2}{(1+J^* \sigma^2)}}{JS} \tilde{J}^2}{\sigma^2 + \lambda \tilde{J}^2}.$$
Figure 4.5: The different hedging strategies (expressed in units of the asset) for a put with strike $K = 100$ using the same parameters as in Figure 4.2.

Figure 4.4 shows the marginal optimal hedge for a put with strike $K = 100$, expressed in units of the asset, for different jump frequencies $\lambda$ and the same parameters as in Figure 4.1. For $\lambda \to 0$, the hedge converges to the Black–Scholes $\Delta$, as expected. For larger $\lambda$, the steepness of $\Delta$ is reduced, as one would expect for higher quadratic variation. Furthermore, the area above the different curves differs, contrary to the case of varying $\sigma$ in the Black–Scholes case. The reason is that the optimal marginal hedge is not given by the derivative w.r.t. $S$ of $V$, but by (4.39).

Figure 4.5 compares the different hedging strategies for a put with strike $K = 100$, expressed in units of the asset. Close to the money, the difference of Merton’s hedge and the optimal marginal hedge is nearly 0.1, while Merton’s $\Delta$ is about 0.3. This is a deviation of about 30%. Note that this strong deviation stems mainly from the new hedging formula (4.39) and not so much from using a different price. This can be seen from Figure 4.6 where, for the same parameters as above, Merton’s hedge and the optimal marginal hedge are compared to the derivative w.r.t. $S$ of the marginal indifference price $V^m$. This derivative is quite close to Merton’s hedge, so for hedging...
Figure 4.6: Comparison of Merton’s hedge, the marginal optimal hedge and the derivative of the marginal indifference price for the same parameters as in Figure 4.5.

purposes it seems to be more important to use the appropriate hedging formula than to use the correct price.
Chapter 5

Summary & Outlook

In this thesis, we introduced the notion of a marginal optimal hedging strategy and showed that it can be characterised by means of functional differentiation of the marginal indifference price and the expected utility w.r.t. the trading strategy. We showed that on a binomial tree, one rediscovers Black–Scholes Δ–hedging. We also applied the concept to a one-dimensional jump diffusion process and derived formulas for the marginal indifference price and the marginal optimal hedge in that case. We discussed Merton’s approach and minimal variance hedging in the same context. The approach of Kramkov and Sirbu, which is derived from the same idea as the framework considered here, was also presented. It was shown that the two approaches indeed lead to the same hedging strategy, at least for jump diffusion. Finally, we compared the prices and hedges obtained with Merton’s, minimal variance, and the marginal utility approach in the case of a jump diffusion with fixed jump size. We found that while the results obtained with the minimal variance and the marginal utility approach are relatively close, the difference to Merton’s approach is notable, especially for the hedge.

Many issues are left open, on the conceptual as well as one the practical side. Our use of functional derivatives was rather formal. Once a rigorous definition is achieved, it should be possible to decide whether the approach discussed here is really equivalent to that of Kramkov and Sirbu.

From the practical point of view, a major drawback of the present approach is that one needs to know the average drift, which is notoriously hard to estimate. In Remark 4.7, we already pointed to a possible way out: The average drift enters through the optimal investment strategy, and the marginal indifference price (and also the marginal optimal hedge) depends on how well the option trade matches to this optimal investment strategy. The underlying assumption is of course that the rational investor is invested in his optimal investment strategy. Thus, in order to
make sure that for the valuation of the claim the matching to the actual investment strategy is taken into account, one could calibrate the drift such that the actual and the optimal investment strategy coincide. It remains to be investigated whether this can be carried out consistently in practice.

Finally, it would be interesting to apply the present framework to more realistic models such as a lognormal jump distribution, variance–gamma, or even stochastic volatility processes. In particular, it remains to be investigated how observed plain vanilla option prices can be consistently used for calibration in such models.
Appendix A

Functional derivatives

We give a formal definition of functional derivatives. A functional $F$ is a function on a space of functions. In the case that matters here, $F$ is an expected value that depends, among others, on a trading strategy $\pi$, i.e., a function of $t, X, S$.

Let $\mathcal{F}$ be a vector space of functions that map from a set $\mathcal{M}$ to the real numbers $\mathbb{R}$. We assume that some topology on $\mathcal{F}$ is given. A continuous functional $F$ on $\mathcal{F}$ is a continuous map $F : \mathcal{F} \to \mathbb{R}$ that assigns a real number $F(f)$ to each function $f \in \mathcal{F}$. If $F$ is also linear, then it is an element of the dual space $\mathcal{F}^*$. In this case, we denote the map by

$$\langle \cdot, \cdot \rangle : \mathcal{F}^* \times \mathcal{F} \to \mathbb{R},$$

i.e., we could write $F(f) = \langle F, f \rangle$. The functional derivative of $F$, evaluated at $g \in \mathcal{F}$, is now a linear functional, defined by

$$\langle \frac{\delta}{\delta f} F(g), h \rangle = \left. \frac{d}{d\epsilon} F(g + \epsilon h) \right|_{\epsilon=0},$$

provided the r.h.s. exists and is continuous and linear in $h$. This definition also makes sense if $F$ is only defined on an open subset $\mathcal{F}'$ of $\mathcal{F}$, since for $g \in \mathcal{F}'$ and $\epsilon$ small enough, $g + \epsilon h \in \mathcal{F}'$. Similarly, second (and analogously also higher) order functional derivatives are defined as symmetric bilinear maps

$$\langle \frac{\delta^2}{\delta f^2} F(g), h \otimes h' \rangle = \left. \frac{d}{d\epsilon} \frac{d}{d\epsilon'} F(g + \epsilon h + \epsilon' h') \right|_{\epsilon=0,\epsilon'=0}.$$

If a functional is linear, it is usually written as an integration, i.e., for $\mathcal{M}$ being a subset of $\mathbb{R}^n$,

$$\langle F, f \rangle = \int_{\mathcal{M}} d^n x \ F(x) f(x),$$

1The expected utility does of course also depend on $t, X, S$, but this dependence does not matter for purpose of defining the functional derivative.
where $F(x)$ is in general a distribution. Since the functional derivative is linear, it can be written as

$$\left\langle \frac{\delta}{\delta f} F(g), h \right\rangle = \int d^n x \ \frac{\delta}{\delta f(x)} F(g) h(x).$$

This can be seen as an implicit definition of $\frac{\delta}{\delta f(x)} F(g)$, which in general will be a distribution in $x$. Similarly, one defines

$$\frac{\delta^2}{\delta f(x) \delta f(x')} F(g)$$

as the distribution corresponding to the second order functional derivative (we recall that it is a bilinear map).

If $F$ is linear, then

$$\left\langle \frac{\delta}{\delta f} F(g), h \right\rangle = \langle F, h \rangle,$$

or

$$\frac{\delta}{\delta f(x)} F(g) = F(x),$$

i.e., the functional derivative is independent of $g$ (analogously to the case of the derivative of a linear function).

When $\mathcal{M}$ is discrete and finite, the integral can be replaced by a sum. In this case, topological subtleties do no matter any more and the functional derivative coincides with the usual partial derivative if one writes $F(f)$ as $F(f_1, \ldots, f_n)$, where $f_i$ is the value of the function $f$ on the $i$th point.

It is straightforward to prove that if $g$ maximises the functional $F$, then the functional derivative evaluated at $g$ vanishes. Otherwise there would be a function $h \in \mathcal{F}$ s.t.

$$\left\langle \frac{\delta}{\delta f} F(g), h \right\rangle \neq 0.$$

Since $\mathcal{F}$ is a vector space, $h$ can be chosen such that the l.h.s. is positive. Then for sufficiently small $\epsilon$ we would have $F(g + \epsilon h) > F(g)$, i.e., $g$ would not maximise $F$, in contradiction to the assumption.
References


