The spectral action and the renormalization group in curved spacetime

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The spectral action principle is an elegant way for obtaining the standard model action from geometrical considerations, including pre- and post-dictions on the Higgs mass, the number of fermions per generation, etc. However, it is only well-defined for compact Riemannian spaces and involves classical fields.

\[ S[A] = \text{Tr}(f(D_A/\mu)) + \langle J\psi, D_A\psi \rangle \]

We argue that the most prominent part of the bosonic spectral action is obtained on arbitrary globally hyperbolic spacetimes from the anomalous scaling of the fermions. A similar statement is known for compact Riemannian spaces when the fermionic part is regularized by a spectral cut-off [Andrianov & Lizzi 10].
1. The spectral action

2. Local covariant fields

3. The scaling of the stress-energy tensor

4. Conclusion & Outlook
1. The spectral action

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Spectral triple

An even real spectral triple \((\mathcal{A}, \mathcal{H}, D, \gamma, J)\) consists of

- a separable Hilbert space \(\mathcal{H}\),
- an unbounded self-adjoint operator \(D\) on \(\mathcal{H}\) with compact resolvent,
- a unital \(*\)-algebra \(\mathcal{A}\) and a faithful representation \(\pi\) on \(\mathcal{H}\) such that \([D, \pi(a)]\) is bounded,
- a \(\mathbb{Z}/2\) grading \(\gamma : \mathcal{H} \to \mathcal{H}\) such that
  \[ [\gamma, a] = 0, \quad \{\gamma, D\} = 0, \]
- an antilinear isometry \(J : \mathcal{H} \to \mathcal{H}\) such that
  \[ J^2 = \varepsilon, \quad JD = \varepsilon' DJ, \quad J\gamma = \varepsilon'' \gamma J, \]
  \[ b^0 := Jb^* J^{-1}, \quad [a, b^0] = 0, \quad [[D, a], b^0] = 0. \]

Here \(\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}\) determine the \(KO\)-dimension.
Examples

Let $M$ be a compact, Riemannian spin manifold, $\Sigma$ its spinor bundle. Then

$$\mathcal{A} = C^\infty(M), \quad \mathcal{H} = L^2(M, \Sigma), \quad D = -i\gamma^\mu \nabla^s_\mu, \quad \gamma = \gamma^5, \quad J = C,$$

defines a commutative spectral triple. Under further conditions, $M$ can be reconstructed as a differential manifold from these data. [Connes 08]

Also finite spectral triples exist. A commutative example is the two-point space

$$\mathcal{A} = \mathbb{C}^2, \quad \mathcal{H} = \mathbb{C}^2, \quad \pi(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad D = \begin{pmatrix} 0 & \lambda \\ \bar{\lambda} & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

An almost commutative spectral triple is the product of a commutative and a finite one:

$$(C^\infty(M, \mathcal{A}_F), L^2(M, \Sigma) \otimes \mathcal{H}_F, -i\gamma^\mu \nabla^s_\mu \otimes 1 + \gamma^5 \otimes D_F, \gamma^5 \otimes \gamma_F, C \otimes J_F).$$
Two spectral triples are unitarily equivalent, if there is an isomorphism $\alpha : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ and a unitary $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that

$$UD_1 = D_2 U, \quad U\pi_1(a) = \pi_2(\alpha(a))U, \quad U\gamma_1 = \gamma_2 U, \quad UJ_1 = J_2 U.$$

Given an inner automorphism $\alpha(a) = uau^*, \, u \in \mathcal{A}, \, u^*u = uu^* = 1$ and defining $U = (u^*)^0 u$, the spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ is unitarily equivalent to $(\mathcal{A}, \mathcal{H}, D_{\mathcal{A}}, \gamma, J)$ with representation $\pi \circ \alpha$ and

$$D_{\mathcal{A}} = D + A + \varepsilon' JAJ^{-1}, \quad A = u[D, u^*].$$

One identifies these inner automorphisms $Inn(\mathcal{A})$ with the gauge group.

Considering Morita equivalent $\mathcal{A}$’s, one allows for a larger class of inner fluctuations, where $A = A^*$ can be written as $A = \sum a_n[D, b_n]$.

The requirement of irreducibility of $\mathcal{A}$ and $J$ on $\mathcal{H}$ naturally leads to $\mathcal{A} = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$. Breaking to $\mathcal{A} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ leads to the gauge group $U(1) \times SU(2) \times SU(3)$ [Chamseddine & Connes 10].
The spectral action I

The spectral action is given as [Chamseddine & Connes 96]

\[ S[A] = \text{Tr}(f(D_A/\mu)) + \langle J\psi, D_A\psi \rangle \]

The fermionic part consists of all SM terms that contain fermions.

In the bosonic part, one has to introduce a cut-off function \( f \) and a mass scale \( \mu \). It can be expanded as

\[ \text{Tr}(f(D_A/\mu)) = \int d\mu g \left( 2f_4 \mu^4 a_0 + 2f_2 \mu^2 a_2 + f(0)a_4 + O(\mu^{-2}) \right) \]

Here \( f_j \) are moments of \( f \) and \( a_j \) are the Seeley-deWitt coefficients of \( D_A \).

This is supposed to be the classical action at unification scale. It has the form of the SM, including an Einstein-Hilbert term, a cosmological constant, and a conformal coupling of the Higgs. Applying renormalization group flow through the big desert to present day collider energies, one can derive predictions, e.g., for the Higgs mass.
Despite its elegance, the spectral action has some shortcomings:

1. It only works on compact, Riemannian spaces,
2. requires a somewhat arbitrary cut-off, and
3. involves classical fields.

We try to overcome these difficulties in the following way:

- Consider $M$ globally hyperbolic and a bundle $E = DM \otimes G$ with $DM$ the Dirac bundle and $G$ a vector bundle associated to a principal $G$ bundle, e.g. $G = M \times \mathcal{H}_F$ and $G = \text{Inn}(\mathcal{A}_F)$.
- Consider a Dirac operator $D$, e.g. of the form of an inner fluctuation.
- Take the Lagrangean $(\psi^\dagger, D\psi)$, and quantize $\psi$ as a local covariant field in a background of gravitational, gauge, and Yukawa fields.

**Observation**

The renormalization group flow of the corresponding stress-energy tensor generates terms corresponding to $a_4$, which contains all standard model terms except for the Higgs mass term.
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Local, covariant fields

We introduce the setting of local, covariant fields [Hollands & Wald 01] for the example of the scalar field.

Let $M$ be a globally hyperbolic spacetime (including metric and background fields, e.g. $m^2$). Define $\ast$-algebra $A[M]$ generated by $\varphi(f)$, $f \in C_\infty^c(M)$, subject to

$$
\begin{align*}
\varphi(\alpha f_1 + \beta f_2) &= \alpha \varphi(f_1) + \beta \varphi(f_2), \\
\varphi(Pf) &= 0,
\end{align*}
$$

Choose a quasi-free Hadamard state $\omega_M$ on $A[M]$ and define

$$
W_n(x_1, \ldots, x_n) = \frac{\delta^n}{i^n \delta f(x_1) \cdots \delta f(x_n)} \exp \left( \frac{1}{2} \omega_M(f, f) + i \varphi(f) \right)
$$

These can also be smeared with $t \in \mathcal{E}'(M^n)$ such that

$$
WF(t \cap (V_+^n \cup V_-^n)) = \emptyset.
$$

This gives the $\ast$-algebra $W[M]$. 
Wick powers are elements of $W[M]$ that are characterized axiomatically. In particular they should be defined simultaneously and in a consistent way on all backgrounds $M$.

For an isometric embedding $\chi : M' \to M$, there is a $*$-homomorphism $\alpha_\chi : W[M'] \to W[M]$.

If $M'$ and $M$ are related by rescaling, i.e., $g' = \lambda^2 g$, $m'^2 = \lambda^{-2} m^2$, . . . , then there is a $*$-isomorphism $\sigma_\lambda : W[M'] \to W[M]$. The scaling dimension of $\Phi$ is

$$d_\Phi = \inf \{ \delta \in \mathbb{R} | \lim_{\lambda \to 0} \lambda^\delta \sigma_\lambda \Phi = 0 \}.$$ 

The order of a field $\Phi$ is the highest number $n$ of elements $W_n$ occurring in $\Phi$. 
Wick powers, axioms

Loc. & Cov.: For an isometric embedding \( \chi : M' \to M \) one has
\[
\alpha_{\chi}(\Phi(f)) = \Phi(\chi_* f).
\]

Scaling: The Wick products \( \Phi \) scale almost homogeneously, i.e.,
\[
\lambda^{d\Phi} \sigma_{\lambda}(\Phi(f)) = \Phi(f) + \sum \log^n \lambda \Psi_n(f),
\]
where \( \Psi_n \) are local covariant fields of lower order than \( \Phi \), with the same scaling dimension \( d_{\Psi_n} = d_{\Phi} \) and almost homogeneous scaling. Fields of order 0 and 1 scale homogeneously. Furthermore, the scaling dimension coincides with the engineering dimension.

\( \mu \text{SC} \): If \( \omega \) is a quasi-free Hadamard state, then \( \omega(\Phi(x)) \) is smooth.

Smoothness: Wick products depend smoothly on the background fields.

Analyticity: In the case of an analytic spacetime, the Wick products depend analytically on the background fields.

Commutator, Hermiticity, Leibniz rule: \( \ldots \)
Wick powers, uniqueness and existence

Two different definitions $\varphi^k$ and $\tilde{\varphi}^k$ differ by a finite number of terms:

$$\tilde{\varphi}^k(x) = \varphi^k(x) + \sum_{j=0}^{k-2} \binom{k}{j} C_{k-j}(x) \varphi^j(x),$$

where $C_k$ is a polynomial in the background fields, and derivatives of these, which scales as $C_k \to \lambda^k C_k$ under $g \to \lambda^{-2} g$, $m^2 \to \lambda^2 m^2$, depending analytically on the curvature coupling.

Existence relies on the parametrix:

$$h_{\Lambda}^{\pm}(x, x') = \frac{1}{16\pi^2} \lim_{\varepsilon \to \pm 0} \left( \frac{V_0(x, x')}{\Gamma_{\varepsilon}(x, x')} + \log \frac{-\Gamma_{\varepsilon}(x, x')}{\Lambda^2} V(x, x') \right).$$

The Wick square may now be defined as

$$\varphi^2(x) = \lim_{y \to x} (\varphi(x)\varphi(y) - h_{\Lambda}^{+}(x, y)) + a(\xi)m^2(x) + b(\xi)R(x).$$
Consider \( M, M' \), where \( g' = \lambda^2 g, \ m'^2 = \lambda^{-2} m^2 \). The Wick square \( \varphi^2 \) is defined on both these spacetimes with the same choice of \( \Lambda, a, b \). One computes, for \( n = 4 \),

\[
\lambda^2 \sigma_\lambda(\varphi^2(x)) = \varphi^2(x) + \frac{1}{16\pi^2} \log \lambda^2 V(x, x) \\
= \varphi^2(x) + \frac{1}{16\pi^2} \log \lambda^2 (m^2(x) + (\xi - \frac{1}{6}) R(x))
\]

This inhomogeneous scaling (in particular that of time-ordered products) leads to the renormalization group flow [Hollands & Wald 02]: The interacting field for interaction \( L_1 \) depends on the Wick powers in \( L_1 \) and the time-ordered products. A change of these due to a scale transformation can be absorbed in a redefinition of the parameters in \( L_1 \).
Wick powers are observables, i.e., there should be a description of how to measure them. The design of a corresponding apparatus involves a scale.

The scale transformation $\sigma_\lambda$ relates observables of observers $O$ and $O'$ with different apparatus sizes.
We consider sections in a vector bundle $E = DM \otimes G$, where $DM$ is the Dirac spinor bundle and $G$ a vector bundle associated to a principal bundle. Then the Dirac operator and the auxiliary operator are

$$D = -\gamma^\mu (\nabla^s_\mu - iA_\mu) + m, \quad \tilde{D} = \gamma^\mu (\nabla^s_\mu - iA_\mu) + m.$$ 

Their product is the normally hyperbolic operator $P$:

$$P = D\tilde{D} = -g^{\mu\nu} \nabla_\mu \nabla_\nu + \frac{R}{4} + \frac{i}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu} + m^2 - \gamma^\mu \partial_\mu m.$$ 

If $\Delta^\pm$ and $h^\pm_\Lambda$ are retarded and advanced propagators and parametices for $P$, then

$$S^\pm = \tilde{D} \Delta^\pm, \quad H^\pm_\Lambda = \tilde{D} h^\pm_\Lambda$$

are the corresponding objects for $D$. 
For test sections $f \in \Gamma_c^\infty(E)$ and $g \in \Gamma_c^\infty(E^*)$, one defines $\psi(g)$ and $\psi^\dagger(f)$ subject to

$$[\psi(g_1), \psi(g_2)]_+ = [\psi^\dagger(f_1), \psi^\dagger(f_2)]_+ = 0,$$

$$[\psi(g), \psi^\dagger(f)]_+ = i\langle g, Sf \rangle,$$

$$\psi(D^*g) = \psi^\dagger(Df) = 0.$$

A Wick square is then defined as

$$\psi^\dagger \psi(x) = \lim_{y \to x} \left( \psi^\dagger(y) \psi(x) + H^-_\Lambda(x, y) \right) + r(x).$$

This makes sense if, for any two-point function $\omega_2$ of a Hadamard state, $H^-_\Lambda$ and $\omega_2$ differ (at least locally) by smooth terms (this is the definition of a state of Hadamard form in [Sahlmann & Verch 00]). As $\omega_2$ is a bi-solution, $H^-_\Lambda$ must be a bi-solution modulo smooth terms.
The bi-solution property

The bi-solution property is known to hold in the flat case and, for \( m = \text{const}, A = 0 \), on all geodesically convex subsets of some neighborhood of an arbitrary Cauchy surface of an arbitrary causal domain with the topology of \( \mathbb{R}^4 \) [Hack 10].

**Theorem**

The bi-solution property, i.e.,

\[
D_x H^\pm_\Lambda(y, x) \in \Gamma^\infty(E^* \boxtimes E), \quad D_y H^\pm_\Lambda(y, x) \in \Gamma^\infty(E^* \boxtimes E),
\]

holds on all causal domains.
The fundamental solutions $\Delta^\pm$ of $D\tilde{D}$ exist and are unique [Bär, Ginoux & Pfäffle 07]. The retarded and advanced propagator $S^\pm = \tilde{D}\Delta^\pm$ are fundamental bi-solutions [Dimock 82]. On a causal domain, $\Delta^\pm$ can be approximated such that $\Delta^\pm - \Delta_k^\pm \in C^k$ by

$$\Delta_k^\pm = \sum_{j=0}^{[\frac{n}{2}] - 1 + k} V_j R_\pm (2 + 2j),$$

where $V_j$ are the Hadamard coefficients determined by the transport equation and $R_\pm(j)$ are Riesz distributions fulfilling

$$\Gamma R_\pm(j) = j(j - n + 2)R_\pm(j + 2),$$

$$2j\nabla R_\pm(j + 2) = R_\pm(j)\nabla \Gamma,$$

$$R_\pm(0) = \delta.$$

The fundamental bi-solution property implies relations between the $V_j$ on $\Sigma = \{(x, x') | \Gamma(x, x') = 0\}$.
On a causal domain, the parametrices $h^\pm$ can be approximated such that $h^\pm - h^\pm_k \in C^k$ by

$$h^\pm_k = \left[\frac{n}{2}\right]^{1+k} - 1 + k \sum_{j=0}^{[\frac{n}{2}]-1+k} V_j T_\pm(2 + 2j),$$

Here $T_\pm(j)$ are distributions fulfilling

$$\Gamma T_\pm(j) = j(j - n + 2) T_\pm(j + 2) + \text{smooth terms},$$

$$2j \nabla T_\pm(j + 2) = T_\pm(j) \nabla \Gamma + \text{smooth terms},$$

$$T_\pm(0) = 0.$$ 

The relations between the $V_j$ on $\Sigma = \{(x, x')|\Gamma(x, x') = 0\}$ are then sufficient to show the bi-solution modulo smooth terms property of $H^\pm = \tilde{D}h^\pm$. 

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The stress-energy tensor

Classically, the stress-energy tensor for the Dirac field is given by

$$T_{\mu\nu} = \frac{1}{2} \left( (\nabla_{(\mu} \psi^{\dagger}_{\nu)} - \psi^{\dagger}_{\nu}(\gamma_{\nu} \nabla_{\mu}) \psi) - (\psi^{\dagger}_{\nu}, \gamma_{\nu} \nabla_{\mu}) \right) - g_{\mu\nu} L.$$ 

The anomalous scaling of the quantized expression is given by

$$\delta T_{\mu\nu} = \lambda^4 \sigma_\lambda (T_{\mu\nu}) - T_{\mu\nu} = -\frac{1}{16\pi^2} \log \lambda^2 \text{tr}[D_{\mu\nu} V],$$

with

$$D_{\mu\nu} = \frac{1}{2} \gamma_{(\nu}(\nabla^c_{\mu)} - \nabla_{\mu}) \tilde{D}$$

and

$$V = \sum_k \Gamma^k V_{k+1}.$$ 

Thus, it suffices to determine $V_1$ and $V_2$.

For the case $m = \text{const}$, $A = 0$ and a generic curved background, one has [Hack 10]

$$\delta T_{\mu\nu} = -\frac{1}{8\pi^2} \log \lambda^2 \left\{ \frac{1}{2} m^4 g_{\mu\nu} - \frac{1}{6} m^2 G_{\mu\nu} + \frac{1}{80} K_{\mu\nu} \right\},$$

with

$$K_{\mu\nu} = \frac{2}{\sqrt{|\det g|}} \frac{\delta}{\delta g^{\mu\nu}} \int \; d\mu(x) \; C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}.$$
Interpretation

Naively,

\[ \omega(T_{\mu\nu}) = \frac{2}{\sqrt{|\det g|}} \frac{\delta S_{\text{eff}}}{\delta g^{\mu\nu}}. \]

In this sense, a change in the stress-energy tensor corresponds to a change in the effective action that generates it. Such an interpretation is necessary for a consistent interpretation of the semi-classical Einstein equation

\[ G_{\mu\nu} = 8\pi G \omega(T_{\mu\nu}). \]

It should be valid for all observers, regardless the scale of their measurement devices. Thus, we should attribute different actions to the observers \(O\) and \(O'\):

\[ S' = S - \frac{1}{8\pi^2} \log \lambda^2 \int d\mu \, \delta L. \]
Yukawa background field

By dimensional arguments, no couplings between derivatives of $m$ and the curvature can appear in $\delta T_{\mu\nu}$. Thus, it suffices to work on Minkowski space. One has to compute $V_1$ and $V_2$ that solve the transport equation

$$2(x - x')^\mu \partial_\mu V_j - 2jV_j = 2j \left(-\partial_\mu \partial^\mu + m^2(x) - \partial m(x)\right) V_{j-1}$$

with $V_0 = 1$. One needs $[V_1]$, $[V_1;\mu]$, $[V_1;\mu\nu]$, and $[V_2]$.

Including the already known coupling to the curvature, one obtains

$$\delta T_{\mu\nu} = -\frac{1}{8\pi^2} \log \lambda^2 \left\{ -\nabla_\mu m \nabla_\nu m + \frac{1}{6} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^\lambda \nabla_\lambda - R_{\mu\nu}) m^2 - \frac{1}{2} g_{\mu\nu} (-\nabla_\lambda m \nabla^\lambda m - m^4 - \frac{1}{6} R m^2) \right\}.$$ 

The expression in curly brackets is the stress-energy tensor for

$$\delta L_m = -\frac{1}{2} \left( \nabla_\mu m \nabla^\mu m + \frac{1}{6} R m^2 + m^4 \right).$$
For the case of an electromagnetic background field, one analogously obtains
\[
\delta T_{\mu\nu} = -\frac{1}{8\pi^2} \log \lambda^2 \left\{ -\frac{2}{3} F_{\mu}{}^\lambda F_{\nu\lambda} + \frac{1}{6} \eta_{\mu\nu} F^{\lambda\rho} F_{\lambda\rho} \right\},
\]
where the expression in curly brackets is the stress-energy tensor of the Maxwell Lagrangean
\[
\delta L_M = -\frac{1}{6} F_{\mu\nu} F^{\mu\nu}
\]
Combining this with the Yukawa and the purely gravitational term, we obtain
\[
\delta L = -\frac{1}{2} \left( \nabla_\mu m \nabla^\mu m + \frac{1}{6} Rm^2 + m^4 \right) - \frac{1}{6} F_{\mu\nu} F^{\mu\nu} + \frac{1}{80} C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}.
\]
Comparison with the spectral action

In a Euclidean setting, one obtains

\[ a_4 \propto \left\{ \frac{1}{2} \nabla_{\mu} m \nabla^{\mu} m + \frac{1}{12} m^2 R + \frac{1}{2} m^4 + \frac{1}{6} F_{\mu \nu} F^{\mu \nu} - \frac{1}{80} C_{\mu \nu \lambda \rho} C^{\mu \nu \lambda \rho} + \frac{1}{120} \Delta R + \frac{11}{1440} R^* R^* + \frac{1}{6} \Delta m^2 \right\}. \]

Up to the \textcolor{red}{red} terms, which are either total derivatives or topological, this coincides with \( \delta L \).

The terms

\[ \frac{1}{2} f_4 \mu^4 + \frac{1}{2} f_2 \mu^2 \left( \frac{1}{6} R - m^2 \right) \]

of the spectral action cannot be obtained from scale transformations, as they involve the scale \( \mu \).

One might argue that there is a scale in the theory, namely \( m_P \). Naively taking this into account and relaxing the condition of almost homogeneous scaling, one can also obtain terms of the above form.
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Conclusion & Outlook
We argued that the anomalous scaling behavior of quantum fields leads to the major part of the bosonic part of the spectral action.

⇒ The nontrivial input of the spectral action seems to be a means to generate the non-Weyl invariant terms in a “controlled” way.

The background fields were taken as non-dynamical, so only gravity feels the presence of these terms. But the same terms appear if they are taken to be dynamical, equipped with some arbitrary kinetic term.

More appropriate would be a treatment of Weyl spinors transforming under different representations. This is also possible and leads to the same result (modulo possible anomalies).
Dynamical fields

Up to now, we considered the background fields as nondynamical. Thus, the terms induce via scale transformations only contribute gravitationally. In order for the above to work for dynamical fields, one has to introduce some (arbitrary) kinetic term for the $m$ and $A$. The scaling behavior is efficiently computed by the background field method:

$$m = m_0 + m_1, \quad A^\mu = A_0^\mu + A_1^\mu.$$  

Here $m_1$ and $A_1$ are quantized, while $m_0$ and $A_0$ are part of the background. Then

$$L_1 = m_1 \psi^\dagger \psi + i A_1^\mu \gamma^\mu \psi^\dagger \psi$$

is the interaction Lagrangean. We obtain (in the flat case)

$$\delta L_1 = -\frac{1}{8\pi^2} \log \lambda^2 \left\{ m_1 \left( \partial^\mu \partial_\mu m_0 - 2m_0^3 \right) - \frac{2}{3} A_1^\mu \partial_\nu F_0^{\mu\nu} \right\}$$

$$= -\frac{1}{8\pi^2} \log \lambda^2 \left\{ -\frac{1}{2} \left( \partial^\mu m \partial_\mu m + m^4 \right) - \frac{1}{6} F_\mu^\nu F^{\mu\nu} \right\} \mathcal{O}(m_1, A_1) + \text{tot. der.}$$
Subhomogeneous scaling

Suppose there would be a preferred energy scale $\mu$, say the Planck scale, which is the same on all backgrounds (as is the speed of light for all inertial observers). A natural generalization would then be subhomogeneous scaling, tentatively defined as

$$\lambda^{d\Phi} \sigma(\Phi(f)) = \Phi(f) + \sum \log^n \lambda \Psi_n(f) + \sum_{m=1}^{d\Phi} (\lambda^m - 1) \mu^m \tilde{\Psi}_m(f),$$

where $\Psi_n$ and $\tilde{\Psi}_m$ are fields of lower order than $\Phi$, which scale subhomogeneously with scaling dimension $d\Phi$ and $d\Phi - m$, respectively. In the case of the stress-energy tensor, the new terms in the allowed ambiguity are then

$$r_{\mu\nu} = g_{\mu\nu} \left( a\mu^3 m^3 + \mu^2 (bR + cm^2) + d\mu^3 m + e\mu^4 \right).$$

But here the ratio of $b$ and $c$ and their sign is unrestricted.
Chiral fermions I

For left-handed $\chi$ and right-handed $\varphi$, transforming under different representations, take the Lagrangean

$$(\bar{\chi}_a \quad \bar{\varphi}_\alpha) \begin{pmatrix} -i \bar{\sigma}^\mu \nabla^a_b \quad m^{a\beta} \\ m^{\alpha b} \quad -i \sigma^\mu \nabla^\alpha_\beta \end{pmatrix} (\chi_b \quad \varphi_\beta).$$

There is a corresponding auxiliary operator

$$\tilde{D} = \begin{pmatrix} i \sigma^\mu \nabla^a_b & m^{a\beta} \\ m^{\alpha b} & i \bar{\sigma}^\mu \nabla^\alpha_\beta \end{pmatrix}$$

so that

$$D \tilde{D} = -\eta^{\mu\nu} \begin{pmatrix} \nabla^a_c \nabla^b_c \quad 0 \\ 0 \quad \nabla^\alpha_\gamma \nabla^\gamma_\beta \end{pmatrix}$$

$$+ \begin{pmatrix} -\frac{i}{2} \bar{\sigma}^{[\mu} \sigma^\nu] F^{a\beta}_{\mu\nu} + m^{a\gamma} m^{\gamma b} \\ -i \sigma^\mu (\nabla_\mu m)^{a\beta} \\ -i \sigma^\mu (\nabla_\mu m)^{\alpha b} \\ -\frac{i}{2} \sigma^{[\mu} \bar{\sigma}^{\nu]} F^{\alpha\beta}_{\mu\nu} + m^{\alpha c} m^{c\beta} \end{pmatrix}$$
Now we use again the background field method. We write

\[ A_{\text{tot}} = A + \tilde{A}, \quad m_{\text{tot}} = m + \tilde{m}, \]

where \( A \) and \( m \) are background fields. The interaction term is then

\[
(\bar{\chi}_a \bar{\phi}_\alpha) \left( \begin{array}{cc}
-\bar{\sigma}^\mu \tilde{A}^{ab}_\mu & \tilde{m}^{a\beta} \\
\tilde{m}^{a\alpha}_b & -\sigma^\mu \tilde{A}^{\alpha\beta}_\mu
\end{array} \right) (\chi_b \phi_\beta).
\]

For the anomalous scaling of the interaction Lagrangean, one thus finds

\[
\delta L \propto -\frac{2}{3} \text{tr} (\tilde{A}^\mu \nabla^\nu F_{\mu\nu}) + i \text{tr} (\tilde{A}^\mu [m, \nabla_\mu m]) + \text{tr} (\tilde{m} \nabla^\mu \nabla_\mu m) - 2 \text{tr} (\tilde{m} m m m) + \frac{i}{2} \varepsilon^{\mu\nu\lambda\rho} \left( \text{tr}_a (\tilde{A} \nabla_\nu F_{\lambda\rho}) - \text{tr}_\alpha (\tilde{A} \nabla_\nu F_{\lambda\rho}) \right).
\]

Here \( \text{tr}_a \) and \( \text{tr}_\alpha \) denote the trace over the internal indices of left- and right-handed spinors, respectively. Furthermore, \( \text{tr} = \text{tr}_a + \text{tr}_\alpha \).

Neglecting the last term and modulo total derivatives, this is

\[
\delta L \propto \text{tr} \left( -\frac{1}{6} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \nabla_\mu m \nabla^\mu m - \frac{1}{2} m m m m \right) \mathcal{O}(\tilde{m}, \tilde{\Lambda}).
\]