UV-IR Mixing on the Noncommutative Minkowski Space in the Yang-Feldman Formalism

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LQP 29, Leipzig, November 2011
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2. UV-IR mixing in the Euclidean setting

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We study scalar quantum field theory on Moyal space \((\mathcal{X}(\mathbb{R}^d), \star)\), which implements commutation relations of the coordinates

\[ x^\mu \star x^\nu - x^\nu \star x^\mu = i\sigma^{\mu\nu}. \]

This is motivated by

- A semiclassical analysis leading to uncertainty relations between the coordinates of an event. These may be implemented by assuming noncommutativity of the coordinates [Doplicher, Fredenhagen & Roberts 94].
- The appearance of such commutation relations in a particular limit of open string theory \((\sigma^0i = 0)\) [Schomerus 99; Seiberg & Witten 99].
The noncommutativity can either be implemented in a formal or a strict sense,

\[(f \ast h)(x) = e^{i \partial_x^y \partial_x^\mu \partial_x^\nu \partial_x^x} f(y) h(z) \big|_{x=y=z} \quad \text{vs} \quad \hat{f} \ast \hat{h}(p) = (2\pi)^{-d/2} \int \! d^d k \, \hat{f}(k) \hat{h}(p - k) e^{-i p \sigma k} \]

The Seiberg-Witten map uses the formal expansion to relate gauge theories on commutative and noncommutative spaces.

The formal expansion is also the basis for the twist approach of Wess et al.

In the formal approach the fact that noncommutative spaces are intrinsically nonlocal is hidden in the appearance of derivatives of arbitrary order.

It is in general not clear whether the expansion converges.

Here we consider strict noncommutativity.
Euclidean vs Lorentzian

- In QFT on ordinary flat spacetime, it is often convenient to work in Euclidean signature, using the Osterwalder-Schrader theorem.
- It is straightforward to derive modified Feynman rules in the noncommutative case from a Euclidean path integral [Filk 96].
- Due to the absence of Osterwalder-Schrader reflection positivity (for $\sigma^0i \neq 0$), it is not clear what this tells us about the Lorentzian case.
- A naive application of these rules in the Lorentzian setting leads to a violation of unitarity for $\sigma^0i \neq 0$ [Gomis & Mehen 00].
- The reason for this is an inappropriate definition of time-ordering [Bahns, Doplicher, Fredenhagen & Piacitelli 02].
- In the Lorentzian case, one can use the Hamiltonian or the Yang-Feldman approach.
- Most of the work on NCQFT was done in the Euclidean setting.
- In particular, the UV-IR mixing was found there [Minwalla, Raamsdonk & Seiberg 00].
In the Hamiltonian approach, one postulates a Hamiltonian $H(t)$ and uses the Dyson series.

In the Yang-Feldman approach, one directly uses the equation of motion.

In the commutative case, the Hamiltonian approach yields the Feynman rules. The Yang-Feldman rules are more complicated than the Feynman rules, but are believed to be equivalent.

In the NC case, the two approaches differ. The combinatorics of the Hamiltonian approach is in general more complicated.

In the Hamiltonian approach, the interacting field does, at tree level, not fulfill the equation of motion [Bahns 04]. At higher loop orders, it exhibits divergences that have no analog in the Euclidean setting [Bahns 10]. A systematic treatment seems cumbersome.
Setup

- We assume $d$ to be even and $\sigma = \text{diag}(\epsilon, \ldots, \epsilon)$ with
  \[ \epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

We often decompose $p = (p_0, p_1, p_s)$, hence $(\sigma p)^2 = p_1^2 - p_0^2 - p_s^2$.

- We consider Lagrangeans of the form
  \[ L = \frac{1}{2} \partial_\mu \phi \ast \partial^\mu \phi - \frac{m^2}{2} \phi \ast \phi - \frac{\lambda}{n} \phi \ast n, \]
  with $n \in \{3, 4\}$, leading to equations of motion
  \[ (\Box + m^2)\phi = -\lambda \phi \ast (n-1). \]

- There is a suitable algebra for the $\ast$ product, including coordinates $x^\mu$ and plane waves $e^{ikx}$ [Gracia-Bondia & Varilly 88].
In the Euclidean, the UV-IR mixing stems from the fact that some graphs that would be UV divergent in the commutative case are regularized by the external momentum, e.g. \( \Sigma(p) = \frac{1}{(\sigma p)^2} \).

This is finite for nonvanishing \( p \), but diverges as \( p \to 0 \). The integration over \( p = 0 \) can cause problems if such graphs are embedded into bigger ones.

In the Yang-Feldman formalism, one also finds self-energies of the form \( \Sigma(p) = \frac{1}{(\sigma p)^2} \). In some graphs, the integration is confined to the mass shell, so one might hope that the problem is absent or weakened.

In the Lorentzian setting, the sets \((\sigma p)^2 = \text{const}\) are noncompact hypersurfaces. There are then two potential problems:

1. The integration over the singularity at \((\sigma p)^2 = 0\).
2. The integration over the hypersurface \((\sigma p)^2 = \text{const}\).

Both problems do indeed lead to divergences, also in cases that are not problematic in the Euclidean setting.
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The modified Feynman rules

The modified Feynman rules amount to attach a phase factor to each vertex,

\[ k_3 \]
\[ \begin{array}{c}
  k_2 \\
  k_4
\end{array} \]
\[ k_1 \]

\[ = \exp \left( \frac{i}{2} \sum_{i<j} k_i \sigma k_j \right) \]

but to retain the Feynman propagator \( \Delta_F \) and the usual combinatorics. The planar tadpole \( \therefore \) is as in the commutative case and is subtracted. The nonplanar tadpole has a finite, but IR divergent self-energy:

\[ k \]
\[ = \int \text{d}p \, \hat{\Delta}_F(p) e^{-ik\sigma p} = \Delta_F(\sigma k) \sim (\sigma k)^{2-d} \]

As it is finite “on-shell” and non-local, a naive treatment would suggest not to subtract it.
Following [Minwalla, Raamsdonk & Seiberg 00] we consider a graph of the following form:

For $n$ insertions, it has an IR divergence if $n(d-2) \geq d$. Note that there are no UV problems, as $\Delta_F(x)$ falls off exponentially for large $x$ (in the massive case).

It was thought for a while that UV-IR mixing spoils renormalizability. However, . . .
Renormalization in the Euclidean

There are two known ways to renormalize the Euclidean $\phi^*_4$ model:

1. Add a Grosse-Wulkenhaar term [Grosse & Wulkenhaar 05]:

$$L(x) = \frac{1}{2} (\partial_\mu \phi)^2(x) + \frac{m^2}{2} \phi^2(x) + \frac{\Omega^2}{2} (\sigma x)^2 \phi^2(x) + \frac{\lambda}{4!} \phi^*_4(x).$$

This model is perturbatively renormalisable and asymptotically safe. However, the added term breaks translation invariance. A naive transfer to the Lorentzian setting leads to infrared divergences for $\Omega \geq 1$ [Z. 11] ($\Omega = 1$ being the fixed point in the Euclidean).

2. Following [Gurau, Magnen, Rivasseau & Tanasa 08], one may also use a translation-invariant, but non-local term:

$$\hat{L}(p) = \frac{1}{2} (\hat{\partial}_\mu \hat{\phi})^2(p) + \frac{m^2}{2} \hat{\phi}^2(p) + \frac{a}{2} \frac{1}{(\sigma p)^2} \hat{\phi}^2(p) + \frac{\lambda}{4!} \hat{\phi}^*_4(p).$$

This model is perturbatively renormalisable.
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The Yang-Feldman formalism

**Ingredient:** Eom with a well-posed Cauchy problem.

**Example:** \( \phi^4 \) model, i.e., \((\Box + m^2)\phi = -\lambda \phi \ast \phi \ast \phi \).

**Ansatz:** \( \phi = \sum_{n=0}^{\infty} (-\lambda)^n \phi_n. \)

\[ \Rightarrow (\Box + m^2)\phi_{n+1} = \sum \sum_{n_i=n} \phi_{n_1} \ast \phi_{n_2} \ast \phi_{n_3}. \]

- \( \phi_0 \) is the free field. We identify it with the incoming field.
- \( \phi_1 = \Delta_{\text{ret}} \times (\phi_0 \ast \phi_0 \ast \phi_0). \)
- \( \phi_2 = \Delta_{\text{ret}} \times (\phi_1 \ast \phi_0 \ast \phi_0 + \phi_0 \ast \phi_1 \ast \phi_0 + \phi_0 \ast \phi_0 \ast \phi_1). \)

With an appropriate infrared cutoff, the interacting field is finite at each order [Z. 03]. UV problems reappear in the adiabatic limit. A clear disentanglement of UV and IR problems does not seem to be possible. Hence, we work in a formal adiabatic limit.
Ordering

We define $f \overline{\ast} g = g \ast f$. This corresponds to $\sigma \mapsto -\sigma$. Obviously, $\phi \overline{\ast} \phi = \phi \ast \phi$. Hence, the r.h.s. of the equation of motion is invariant under the replacement of any of the $\ast$ products by a $\overline{\ast}$ product. The naively quantized expression does not possess this symmetry. We propose to restore it by defining

$$\phi \overline{\ast} \overline{\ast} \phi(k) = c_d \int \prod_i d^d k_i \, \delta(k - \sum k_i) \{\hat{\phi}(k_1), \hat{\phi}(k_2), \hat{\phi}(k_3)\}$$

$$\times e^{-\frac{i}{2}(k_1 \sigma k_2 + k_1 \sigma k_3 + k_2 \sigma k_3)}.$$ 

Hence, we have

$$\hat{\phi}_n(k) = c_d \hat{\Delta}_R(k) \int \prod_i d^d k_i \, \delta(k - \sum k_i) \sum_{\sum n_i = n-1} \hat{\phi}_{n_1}(k_1)\hat{\phi}_{n_2}(k_2)\hat{\phi}_{n_3}(k_3)$$

$$\times \left\{ \cos\left(\frac{1}{2} k_1 \sigma k_2 \right) e^{-\frac{i}{2}(k_1 + k_2) \sigma k_3} + \cos\left(\frac{1}{2} k_1 \sigma k_3 \right) e^{-\frac{i}{2}(k_1 + k_3) \sigma k_2} \right. \right.$$ 

$$\left. + \cos\left(\frac{1}{2} k_2 \sigma k_3 \right) e^{-\frac{i}{2}(k_2 + k_3) \sigma k_1} \right\}.$$
We introduce a graphical notation, in which a double line stands for a retarded propagator and an open single line for an uncontracted free field. In this notation, $\phi_1$ is given by

$$\phi_1 = \cos\left(\frac{1}{2} k_1 \sigma k_2\right) e^{-\frac{i}{2}(k_1+k_2)\sigma k_3} + \cos\left(\frac{1}{2} k_1 \sigma k_3\right) e^{-\frac{i}{2}(k_1+k_3)\sigma k_2} + \cos\left(\frac{1}{2} k_2 \sigma k_3\right) e^{-\frac{i}{2}(k_2+k_3)\sigma k_1}$$

Similarly, $\phi_2$ is represented by

$$
\begin{align*}
\begin{array}{c}
\phi_2 = \cos\left(\frac{1}{2} k_1 \sigma k_2\right) e^{-\frac{i}{2}(k_1+k_2)\sigma k_3} + \cos\left(\frac{1}{2} k_1 \sigma k_3\right) e^{-\frac{i}{2}(k_1+k_3)\sigma k_2} + \cos\left(\frac{1}{2} k_2 \sigma k_3\right) e^{-\frac{i}{2}(k_2+k_3)\sigma k_1}
\end{array}
\end{align*}
$$
Loops

Loops are obtained by contracting two single lines, yielding a two-point function $\Delta_+$. As it is not symmetric, the ordering is important. For the contracted part of $\phi_1$,

\[
\hat{\phi}_1^c(k) = c_d \hat{\Delta}_R(k) \hat{\phi}_0(k) \int d^d p \, \hat{\Delta}_+(p) \{2 + \cos p\sigma k\}
\]

\[
= c_d \hat{\Delta}_R(k) \hat{\phi}_0(k) \{2\Delta_+(0) + \Delta_1(\sigma k)\}.
\]

Here we used $\Delta_1(x) = \frac{1}{2}(\Delta_+(x) + \Delta_+(-x))$. The first term diverges and corresponds to the usual tadpole. Hence, we subtract it. The second term is finite and nonlocal, so we do not subtract it (following [Bahns, Doplicher, Fredenhagen & Piacitelli 05]). We call it the nonplanar tadpole.
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The snowman graphs

The analoga of the nonplanar snowman graphs are

Here $\Sigma_{np}(k) = \Delta_1(\sigma k)$. We obtain

$$\hat{\phi}_2^c(k) = c_d \hat{\Delta}_R(k) \hat{\phi}_0(k) \int d^d p \, \hat{\Delta}_R(p) \Sigma_{np}(p) \hat{\Delta}_1(p) \{2 + \cos p \sigma k\}.$$ 

We use the formal equality $\hat{\Delta}_+(p) \left(\hat{\Delta}_R(p) + \hat{\Delta}_A(p)\right) = c_d \partial m^2 \hat{\Delta}_+(p)$ with

$$\int d^d p \, f(p) \partial m^2 \hat{\Delta}_+(p) = c_d \int d^{d-1} p \left(\frac{1}{4\omega_p^3} f(p^+) - \frac{1}{4\omega_p^2} \partial_0 f(p^+)\right),$$

where $p^+ = (\omega_p, p)$. 
The calculation of $\Pi$

Writing $\Delta_1(x) = h(x^2)$, with $h \in S'(\mathbb{R})$ rapidly decreasing, we obtain

$$\Pi(x) = c_d \int d^{d-1}p \left[ \left( \frac{1}{4\omega_p^3} \cos(p^+ k) - \frac{1}{4\omega_p^2} k_0 \sin(p^+ k) \right) h((\sigma p^+)^2) + \frac{1}{2\omega_p} h'((\sigma p^+)^2) \cos(p^+ x) \right].$$

Using $(\sigma p^+)^2 = -m^2 - p_s^2$, one finds that the green terms are integrable, whereas for the violet one, we obtain, by integration over $p_1$,

$$\Pi(x) = c_d \int d^{d-2}p_s \ h'(-m^2 - p_s^2) \Delta_1^{(2)}(x_0, x_1; \sqrt{m^2 + p_s^2}) \cos(p_s \cdot x_s).$$

Here $\Delta_1^{(2)}(x; m)$ is $\Delta_1$ in $2d$ and mass $m$. It is logarithmically divergent at $x_1 = \pm x_0$. Hence, the red term in

$$\hat{\phi}_2^c(k) = c_d \hat{\Delta}_R(k) \hat{\phi}_0(k) \left( 2\Pi(0) + \Pi(\sigma k) \right)$$

diverges, whereas the blue one is finite (as $k$ is on-shell).
We have found a logarithmic divergence in the $\phi^4$ model with one nonplanar insertion, and independent of the dimension. In the Euclidean case, one needed $d \geq 4$ and at least 2 nonplanar insertions.

The nonplanar tadpole with arbitrary many nonplanar insertions is finite, contrary to the Euclidean case, where the divergences of

In the adiabatic limit, a suitable counterterm (subtracting the whole graph, not just $\Sigma_{np}$) would be momentum independent, i.e., local.

We could also consider what happens when non-planar tadpoles are inserted into a fish graph, i.e., in the Euclidean

Similar graphs will be considered for the $\phi^3$ model.
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In the Euclidean setting, the nonplanar $\phi^3$ fish graph is diverges as $(\sigma k)^{4-d}$. Hence, the fish graph with nonplanar insertions,

\[
\begin{array}{c}
\text{...}
\end{array}
\]

is divergent for $d \geq 6$ and $n$ insertions if $(d - 4)n \geq d$, i.e., for $n \geq 3$ in $d = 6$. 
In the Yang-Feldman formalism, the fish graph

\[ \Sigma(p) = c_d \int d^d k \ \hat{\Delta}_1(k) \hat{\Delta}_R(p - k) \{1 + \cos(k \sigma p)\} . \]

The blue term corresponds to the ordinary fish graph and is renormalized. The red term corresponds to the nonplanar fish graph from the Euclidean.
The nonplanar fish graph

The nonplanar self-energy

\[ \Sigma_{np}(p) = c_d \int d^d k \, \hat{\Delta}_1(k) \hat{\Delta}_R(p-k) \cos(k \sigma p). \]

can be defined as an oscillatory integral for \(0 < p^2 < 4m^2\) [Döscher & Z. 09]:

\[ \Sigma_{np}(p) = c_d \int_0^\infty dk \frac{k^{d-2}}{\omega_k(p^2-4\omega_k^2)} \frac{\sin(k \sqrt{|(\sigma p)^2|})}{k \sqrt{|(\sigma p)^2|}}. \]

This diverges as \(\log |(\sigma p)^2|\) for \(d = 4\) and as \((|\sigma p|^2)^{4-d}\) for \(d \geq 6\).

Regularizing \(\hat{\Delta}_R\) (choosing \(\varepsilon\) finite), one can define \(\Sigma_{np}\) for all \(p\). One finds the same scaling behavior at \((\sigma p)^2 = 0\) and

\[ \text{WF}(\Sigma_{np}) = \{(k, y) | (\sigma k)^2 = 0, y = \lambda k, \lambda \neq 0\}. \]

Assumption: This is still true for \(\varepsilon \to 0\), and \(\Sigma_{np}(p)\) falls off rapidly for \((\sigma p)^2 \to -\infty\).
There are two possibilities to close a fish graph loop:

$$\Sigma_{np} \Sigma_{np} \ldots \Sigma_{np} \Sigma_{np}$$

While the graphs of the first type are finite, those of the second type are in general divergent for $n \geq 2$. 
The infrared divergence

In the case $n = 2$ one obtains the self-energy

$$
\Sigma(k) = c_d \int d^d p \, \hat{\Delta}_1(p) \hat{\Delta}_R^3(k - p) \Sigma_{np}^2(k - p) \{1 + \cos p \sigma k\}.
$$

We are interested in $k \in H_m^+$. Furthermore, $p \in H_m^\pm$.

- For $p \in H_m^+$, there is a $d - 2$ dimensional submanifold $S$ of $H_m^+$ where $(\sigma(k - p^+))^2 = 0$.
- $\Sigma_{np}^2$ is not well-defined at $S$ in the sense of Hörmander.
- $\Sigma_{np}^2$ has scaling degree 2 at a submanifold of codimension 1. Hence, its extension to $S$ introduces a momentum-dependent renormalization freedom [Brunetti & Fredenhagen 00].

For $d = 6$, already 2 nonplanar insertions lead to nonlocal divergences, in contrast to the 3 insertions needed in the Euclidean setting.
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We found two different mechanisms of UV-IR mixing in the Yang-Feldman formalism:

1. The integration over the singularity \((\sigma p)^2 = 0\).
2. The integration over noncompact hypersurfaces \((\sigma p)^2 = \text{const}\).

The second possibility has no counterpart in the Euclidean theory and leads to divergences in situations that are finite there. Also the first possibility has a different power counting behavior than in the Euclidean.
The introduction of nonlocal counterterms seems to be unavoidable. They should be restricted to be functions of $(\sigma p)^2$ [Liao & Sibold 02].

In the Euclidean $\phi^4_4$ model, a $(\sigma p)^{-2}$ “mass” counterterm does the job. In the situations discussed here, this would also suffice. But it seems very difficult to make statement that are valid to all orders:

- The Yang-Feldman formalism is combinatorically more complicated than the Feynman rules. Even in the commutative case, there is no good notion of power counting, as cancellations of different term have to be taken into account.

- Because of UV-IR mixing, one has to control both the UV and the IR. In the Euclidean, this is done by multiscale analysis [Gurau, Magnen, Rivasseau & Tanasa 08]. Can one generalize this to the Lorentzian setting?

- Is there a mapping of the Lorentzian theory to some Euclidean one? If yes, it must be quite involved, since the divergences in the two settings are quite different.