GK “Groups and Geometry”, Georg-August Universität Göttingen Winter school Geometric Langlands Program

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Abstract

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These notes were taken by Thomas Schick (typing during the talks). They are rather imperfect and have not undergone a thorough revision. All remaining mistakes might be blamed on the note-taker.

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Talk 1: Class field theory (Heinloth)

Class field theory was first formulated for number fields. These behave similarly to function fields over finite fields:

$$\text{number-fields} \longrightarrow \text{curves over} F_q \xrightarrow{\text{recently}} \text{Riemann surfaces=curves}/\mathbb{C}$$

We start with the middle objects.

0.1 Notation. Let C/F_q be a smooth projective curve over a finite field F_q, geometrically connected.

Let $k(C)$ be the field of meromorphic functions on C.

Fix $x \in C$ a point. Then O_x is the complete local ring at x, $O_x \cong F_q[[t]]$, and $K_x = Quot(O_x) \cong F_q((t))$.

0.2 Theorem. ***Artin's reciprocity law (unramified version)*** *$\{$Finite quotients of the Idele class group$\}$ are in one to one correspondence to $\{$Galois groups of abelian unramified extensions of $k(C)\}$.*

Here, the left hand side is the biquotient

$$k(C)^* \backslash \prod_x{}' K_x^* / \prod_x O_x^* \cong Div(k(C)^*) \backslash \bigoplus_x \mathbb{Z}x \cong prinDiv \backslash Div(C) \cong Pic_C(F_q).$$

$Pic_C(F_q)$ is the set of line bundles on C, i.e. $Pic_C(F_q) \cong \prod_{\mathbb{Z}} Pic_C^d(F_q)$, decomposed according to degree.

For the right hand side, we get

$\{$Automorphism groups of abelian Galois coverings $\tilde{C} \to C\} \cong \{f$finite quotients of $\pi_1^{ab}(C)\}$

How is this correspondence constructed? Each $x \in C$ is mapped to the group generated by the corresponding Frobenius element.

One has to show that this factors over the principal Divisors, which is non-obvious.

To see this, geometry helps: Replace groups by their representations.

Then the RHS becomes: Representations of $\pi_1(C)$, i.e. local systems on C.

And the LHS becomes: Characters of $Pic(F_q) \subset$ Functions on Pic.

Grothendieck observed: interesting functions (in particular those arising above) should arise from sheaves (the "functions $\leftrightarrow$ sheaves"-dictionary).

0.3 Theorem. *(Deligne): There is a natural corresondence*

$$\{\text{1-dim representations of } \pi_1(C)\} \xrightarrow{(1:1)?} \{\text{finite characters} Pic(F_q) \to \bar{\mathbb{Q}}_l^* \subset\}$$

$$\downarrow \text{faisceaux} \leftrightarrow \text{fonctions}$$

$$\{\text{local systems } A_L \text{ on } Pic \text{ with} A_L|_0 \cong \bar{\mathbb{Q}}_l^* \text{trivial}\}$$

We have the action $+ : C \times Pic \to Pic; (x, L) \mapsto L(x) = L \otimes O_C(x)$. We require in addition

$$(+)^* A_L \cong L \boxtimes A_L \text{on } C \times Pic, \text{ where } L \text{ is some local system on } C.$$

Proof. From characters to locals systems is the "Faisceaux-fuction" dictionary, the correspondence between the LHS and local systems was proved by Deligne. This finally proves Artin reciprocity. □

0.1 Functions-Sheaves dictionary

0.4 Definition. $f\colon \tilde{C} \to C$ is called *etale/local diffeomorphism* $\iff$ f is smooth with 0-dimensional fibers. In particular, for differential forms: $f^*\Omega_C = \Omega_{\tilde{C}}$.

0.5 Example. $Spec(F_q) = x$, $\tilde{x} \to x$ with $\tilde{x} = Spec(k[x]/p(x))$. Jacobi criterion for smoothness says:

etale $\iff$ the polynomial $p(x)$ is seperable.

We may assume: $\tilde{x} = Spec(F_{q^n}) \to Spec(F_q) = x$. Then $Aut(\tilde{x}/x) = Gal(F_{q^n}/F_q) = \mathbb{Z}/n =< Frob = (\cdot)^q >$.

0.6 Example. $\pi\colon \tilde{C} \to C$ etale, Galois. For $x \in C$, $\pi^{-1}(x) = \amalg_{x' \in \pi^{-1}(x)} x'$. Then $Aut(\tilde{C}/C)$ contains $Stab(x') \xrightarrow[\text{counting}]{\cong} Aut(\tilde{x}/x) = \mathbb{Z}/n =< Frob_x >$. This defines the element $Frob_x \in Aut(\tilde{C}/C)$, well defined upto conjugation. Nonetheless, its value on characters of representations is well defined.

0.7 Definition. Essentially (with a little cheat)

$$\pi_1(C) := \lim_{\tilde{C}/C \text{ Galois etale}} Aut(\tilde{C}/C) \supset \{Frob_x \mid x \in C\},$$

the subset is defined upto conjugation.

Fact: Can define local system as representation $\pi_1(C) \to Gl_n(\bar{\mathbb{Q}}_l)$. This leads to a sheaf theory as in topology. This also leads to a derived category of sheaves $D^b_{et}(C)$ (sheaves on the etale site where in addition we make the convention that the cohomology is constructible).

We also get a trace map

$$D^b_{et}(C) \to Fun(\amalg_N C(F_{q^N}))$$

given by

$$K^\bullet \mapsto \mathrm{tr}_K(Frob, K|_x) = \sum_i (-1)^i \,\mathrm{tr}(Frob_x, H^i(K)|_x).$$

easy fact: $\mathrm{tr}_{K\oplus L} = \mathrm{tr}_K + \mathrm{tr}_L$, $\mathrm{tr}_{K\otimes L} = \mathrm{tr}_K \mathrm{tr}_L$. for $f\colon X \to Y$, $\mathrm{tr}_{f^*K} = f^* \mathrm{tr}_K$.

hard fact: $\mathrm{tr}_{Rf_!K} = f_! \,\mathrm{tr}_K(y) := \sum_{x \in \pi^{-1}(y)} \mathrm{tr}_K(x)$ (this is the Grothendieck-Lefschetz trace formula).

0.8 Remark. $\{Frob_x\}_{x\in X} \subset \pi_1(X)$ are dense.

in particular, $\rho\colon \pi_1(X) \to Gl_n(\bar{\mathbb{Q}}Q_l)$ is determined by $\mathrm{tr}(\rho(Frob_x))$. This is due to Cebotuavi(?).

Proof of Deligne's theorem: Let L be a 1-dimensional local system on C. We need to construct A_L on $Pic(C)$.

We have a map $C^d \to Pic^d; (x_k) \mapsto O(\sum x_i)$. This factors through the symmetric product $C^{(d)} = C^d/S_d$; the latter parameterizes effective divisors of degree d.

Set

$$L^{\boxtimes d} := \otimes \,\mathrm{pr}_i^* L, \qquad L^{(d)} := (\sum_x (L^{\boxtimes d}))^{S_d}.$$

Its stalk at $\sum_{n_i x} \in C^{(d)}$ is $\otimes_i (L_{x_i}^{\otimes n})$, so $L^{(d)}$ is locally constant. Then $Sym^* L^{(d)} = L^{\boxtimes d}$.

The fibers of the map $C^{(d)} \to Pic^d$ over $\mathcal{L}$ is $P(H^0(C, \mathcal{L}))$, i.e. is a projective space. In particular it is simply connected. (because $\pi_q(P^n_{\bar{F}_q}) = 0$)

The Riemann-Roch theorem asserts: The map $AJ\colon C^{(d)} \to Pic^d$ is a fibration if $d > 2g-2$, i.e. $\pi_1(C^{(d)}) = \pi_1(Pic^d)$ for $d > 2g-2$, in particular $L^{(d)}$ descends to A_L on Pic^d.

$Sym^* L^{(d)} = L^{\boxtimes d}$ tell us:

$$\begin{array}{ccc}
(x, D) & \longrightarrow & (x + D) \\
C \times C^{(d)} & \xrightarrow{\tilde{+}} & C^{(d+1)} \\
\downarrow & & \downarrow \\
C \times Pic^d & \xrightarrow{+} & Pic^{d+1} \\
(x, L) & \longrightarrow & L(x).
\end{array}$$

Note $+^* A_L = L \boxtimes A_L$.

Now extend definition of A_L to arbitrary d by additivity, e.g. fix $x \in C(F_q)$. Then on Pic^i we have: $+Nx\colon Pic^i \to Pic^{i+N}; L \mapsto L(Nx)$. Define $A_L := (+Nx)^* A_L \otimes L_x^{\otimes -N}$.

Talk 2: Geometric function theory (Ben-Zvi)

0.2 Goals of this talk

Survey about the geometric Langlands conjecture:

in short: "it gives a Geometric Fourier transform for a reductive group G over an algebraic curve C". This shall be made a little more precise during the talk.

0.3 Classical Fourier transform

Fourier transfrom for a locally compact abelian group G, e.g. $G = \mathbb{R}$, or $G = S^1$. We should think that these groups live over a point.

Realize representations of G by the (left convolution) action of G on $L^2(G)$.

Let $\hat{G} = \mathrm{Hom}(G, U(1))$ be the group of characters/dual group, or the unitary irreducible representations of G. Recall that $\hat{\mathbb{R}} = \mathbb{R}$, and $\hat{S^1} = \mathbb{Z}$.

We then have Fourier-Pontrjagin duality:

$$F\colon L^2(G) \xrightarrow{\cong} L^2(\hat{G}).$$

For $G = \mathbb{R}$, we know that $f(x) = \int_{\hat{\mathbb{R}}} \hat{f}(t) e^{ixt}\, dt$, where $\hat{f} = F(f)$ is the Fourier transform.

Idea: We decompose (as continuous integral) f as "linear combination" of characters e^{ixt}, parameterized by $t \in \hat{\mathbb{R}}$.

Note the F diagonalizes the action of G on $L^2(G)$:

convolution/translation/differentiation d/dx corresponds to multiplication/multiplicatoin by t, i.e. we get a *spectral decompostion of differentiation.*

We can extend this to other function spaces:
if $e^{\lambda x}$ is allowed, and distributions as well, then we have
$F(e^{\lambda x})$ corresponds (under F) to the delta "function" δ_λ on $\hat{\mathbb{R}}$.
Again: characters are mapped to points.

Consider $Lie(\mathbb{R}) = \mathbb{R}$. It has indecomposable representations which are not irreducible.

0.9 Example. $\{e^{\lambda x}, xe^{\lambda x}\}$ is a space on which d/dx acts as $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Under Fourier transform, this corresponds to the action of multiplication by t on $\mathbb{R}[t]/(t-\lambda)^2$

0.4 Geomeric Fourier transform

Replace $\mathbb{R}$ by the affine line A^1.

What replaces $L^2(\mathbb{R})$? Polynomials on A^1 are way too small. The space should be rich enough to contain e^{xt}.

Consider the ring of algebraic differential operators on A^1,

$$D := \mathbb{C} < x, d/dx = \partial > /(\partial x - x\partial - 1).$$

Now $e^{\lambda x}$ is replaced by $De^{\lambda x} \cong D/D(\partial - \lambda) \subset C^\infty(\mathbb{R}, \mathbb{C})$. Note that $C^\infty(\mathbb{R})$ is a D-module, and $De^{\lambda x}$ a D-submodule.

Similarly, we replace δ_λ by $D_t\delta_\lambda = D_t/D_t(t-\lambda)$ (on this side, the variable is t).

In short: replace functions by D-modules. Then Fourier transform becomes:

$$F\colon D_x - mod \xrightarrow{\cong} D_t - mod; M \mapsto F(M) := \int M(x)e^{ixt}\,dt := (\pi_2)_*(\pi_1)^* M \otimes P.$$

Where: $A^1 \times A^1$ has two projections $\pi_{1,2}$ to A^1, and on $A^1 \times A^1$ lives $P = \{e^{xt}\}$.

In this picture

- $M_\lambda = De^{\lambda x}$ are "characters" of A^1.

  $M_\lambda|_{x_1+x_2} \cong M_\lambda|_{x_1} \otimes M_\lambda|_{x_2}$.

- $\mu\colon A^1 \times A^1 \to A$ multiplication, then $\mu^* M_\lambda \cong M_\lambda \boxtimes M_\lambda$.

0.5

$e^{\lambda x}$, δ_λ satisfy nice algebraic differentail equations. Unlike $Df \subset C^\infty$, because for generical f: $Df \cong D$ is a *big* D-module (called non-holonomic), $e^{\lambda x}$, δ_λ are *holonomic*.

0.6 More general FT

In general, we replace Function spaces by categories (of sheaves), which we understand as "vector space valued functions" on a space.

Moreover, we have to understand the geometry of D-modules.

Fourier transform gives an isomorphism

$$F\colon \text{holonomic } D - mod \xrightarrow{\cong} \text{holonomic } D - mod.$$

Fix a variety X. To this we assign "function spaces" as categories of modules.

0.10 Example. R commutative ring (and $X = Spec(R)$) is equivalent to $R - mod = QCoh(X)$ (quasicoherent sheaves), containing R and O, respectively.

If R is a k-algebra, then R-mod is a k-linear category, which we understand s a partially defined k-algebra.

For an arbitrary variety X, we consider $QCoh(X)$, the category of quasicoherent sheaves on X.

0.7 Pass to derived categories

Algebraically, we go from associative k-algebras to dg k-algebras (differential graded).

Similarly, k-linear categories are replaces by dg categories as follows:

to A a k-linear category we assign $Com^\bullet(A)$ (complexes in A). We go further by killing all acyclic complexs to get $D_{dg}(A)$.

To a variety X we assign derived categories of (quasi)-coherent sheaves $D(X)$ (several choices of extra conditions: perfect, bounded, ...).

There is some kind of "geometric functional analysis" governing this world.

Now, X gives rise to $D(X)$, but in general $D(X)$ will not determine X completely; it still is a very fine invariant of X.

0.8 Operators on functions

A typical operator on the space of functions is given by a kernel "function"

$$Kf(t) = \int K(x,t)f(x)\,dx.$$

Similarly, operators between $D(X)$ and $D(Y)$ are given by "integral kernels" $K \in D(X \times Y)$,

On $X \times Y$ we have the two projections p_X, p_Y to X and Y. Then we get

$$F \mapsto (\pi_Y)_*((\pi_q)^*F \otimes K)$$

By a theorem of Toen, every operator has this form.

0.9 Geometric Langland conjecture

Let C be a curve, G a reductive group (everything over $\mathbb{C}$).

Consider $Bun_G(C)$, the moduli space of G-bundles on C. $D(Bun_G(C))$, the derived category of D-modules on $Bun_G(C)$ (considered as space of "functions" on $Bun_G(C)$).

On $D(Bun_G(C))$ we have a bunch of operators: Hecke operators.

Geometric Langlands says that this is equivalent to $O(Loc_{\hat{G}}C)$, the derived category of O-modules (coherent sheaves on the space of local systems), where $Loc_{\hat{G}}C$ is the space of $\hat{G}$-local systems.

The Hecke operators are mapped under this equivalence to multiplication operators, i.e. of the form "$\otimes V$". I.e. we get the Fourier-transform property that complicated operators become simple.

0.11 Example. In Heinloth's first talk: $G = Gl_1$, and $Bun_G C$ is then $Pic(C)$.

0.10 D-modules

Fix X a smooth variety. We get the sheaf of differential operatos on X, the associative algebra (a quasicoherent sheaf)

$$D_X := O_X < \tau_x >_{\tau_X \text{ vector field}} /(\partial f - f\partial = f', \partial_1\partial_2 - \partial_2\partial_2 = [\partial_1, \partial_2]).$$

A D-module M is a quasicoherent sheaf M with action $D \otimes M \to M$. Equivalently, M is a quasicoherent sheaf with a flat connection $\nabla \colon M \to M \otimes \Omega^1$.

In slogans: if an O-module is an "analytic vector space valued function", a D-module is an "infinitesimally constant vector space valued function" (coming from the parallel transport)

On the other hand, D-modules are closely relatd to sheaves on T^*X as follows:

D has a natural filtration by order of differential operators. The associated graded algebra $gr(D) = Sym\tau_X = O_{T^*X}$ (symmetric power of space of vector fields). Modules over this ring are sheaves on T^*X.

Observe that in the relation for D_X, if we add the parameter $\partial f - f\partial = hf'$, $\partial_1\partial_2 - \partial_2\partial_2 = h[\partial_1, \partial_2]$, then $h = 0$ gives $gr(D)$, $h \neq 0$ gives D, i.e. D is a deformation of O_{T^*X}.

Therefore we see a shadow: $\{D\text{-modules}\}$ (are similar to) $\{\text{sheaves on } T^*X\}$.

under this shadow: holonomic D-modules go to sheaves with Lagrangian suport in T^*X.

1 Talk 3: The classical limit of the Geometric Langlands conjecture (Pantev)

1.1 Goal of this talk

Goal is to explain and *proof* a classical limit of the Geometric Langlands Corresondence. Nasty details will be omitted.

1.2 Classical Langlands duality for groups

This will be done in an unusual non-canonical way.

For the whole talk, we work over $\mathbb{C}$.

Let G be a complex reductive group. $T \subset G$ a maximal torus.

There are two lattices: $char_G = \mathrm{Hom}(T, \mathbb{C}^*)$ and $cochar_G = \mathrm{Hom}(\mathbb{C}^*, T)$. They are dual to each other, we consider them as Coxeter systems.

1.1 Definition. Two groups G, G' are Langlands dual if

$$char_G \cong cochar_{G'}.$$

with the whole structure, coming from the groups G and G', respectively.

1.2 Notation. If G is a group, ${}^LG$ is a Langlands dual group.

Note: a priori the definition depends on choices (e.g. choice of a maximal torus).

1.3 Fact. ${}^L(-)$ is a duality on the category of complex reductive groups.

1.4 Example. If $\mathfrak{g}$ is a simple Lie algebra (Lie algebra of simple reductive group), then ${}^L\mathfrak{g} \cong \mathfrak{g}$ if $\mathfrak{g}$ is of type A,D,E,F,G, whereas ${}^L\mathfrak{g}^B = \mathfrak{g}^C$ and vice versa (for type B,C).

1.5 Example. The following groups are self-Langlands-dual: GL_n, G_2, E_8, $(\mathbb{C}^\times)^k$.

For other groups we get (more complicated than for Lie algebras)

G	${}^LG$
Sl_n	Sl_n/μ_1
$Spin_{2n}$	So_{2n}/μ_2
$Sp(n)$	$SO(2n+1)$
$Spin(2n+1)$	$Sp(n)/\mu_2$

1.3 Geometric objects

C compact smooth curve of genus $g \geq 2$.

G, ${}^LG$ a pair of Langlands dual groups.

Basic spaces: Bun_G, the moduli space of (semistable) principal G-bundles on C.

$Loc_{{}^LG}$, the moduli space of ${}^LG$ local systems on C.

1.6 Definition. A ${}^LG$ local system on C is a pair $\mathbb{V} := (V, \nabla)$, where V is a principal algebra ${}^LG$-bundle on C, and ∇ is an integrable connection on V. (We think of such a connection as a lifting of the infintesimal symmetries of C to V, which are compatible with the group action).

Explicitly, V gives a sequences of sheaves of Lie algebras on C:

$$0 \to ad(V) \to A(V) \to T_C \to 0,$$

where T_C is the holomorphic tangent bundle to C, $ad(V) := V \times_{ad} {}^L\mathfrak{g}$, $A(V) := \pi_* T_{tot}(V)^{{}^LG}$. This sequence is the push forward of the tangent sequence of $Tot(V) \xrightarrow{\pi} C$, where $Tot(V)$ is the total space of the bundle V, followed by taking invariants. Tangent sequence: $0 \to T_\pi \to T_{Tot(V)} \to \pi^* T_C \to 0$.

A connection is just a splitting of the Atiyah sequence, as a sequence of O_C-modules. The connection is integrable, if the splitting preserves the Lie algebra structure; this is automatic for a curve C.

1.7 Remark. (1) For all $\mathbb{V} = (V, \nabla)$ and any $x \in C$, we get a homomorphism $mon\colon \pi_1(C, x) \to {}^LG = \mathrm{Aut}(V_x)^{{}^LG}$ (with topological fundamental group). This is the monodronomy representation, given by parallel transport.

The construction of monodronomy involves exponentiation, therefore is transcendental. It will not carry any information about the algebraic structure.

We have $V = \Delta \times {}^L G/mon$, and $\nabla = d$ in this representation.

This assignement gives an analytic isomorphism

$$Loc_{^L G} \cong \mathrm{Hom}(\pi_1(C), {}^L G)//{}^L G.$$

The right hand side is affine, the left hand not, os we can not lift this isomorphism to an algebraic isomorphism (the left hand side carries enough information to recover C.

(2) If $\rho\colon {}^L G \to Gl_n(\mathbb{C})$ representation, look at $\rho(V), \rho(\nabla)$, a vector bundle with connection. Then $\rho(V)^{\rho(\nabla)}$ is a locally constant sheaf of $\mathbb{C}$-vector spaces.

There is a Tannakian formalism which allows to reconstruct from all those locally constant sheaves the local system.

1.4 Hecke correspondences

The space $Bun_{^L G}$ has a natural family of self-correspondences. $p_x\colon Hecke_x \to Bun_{^L G}$, $q_x\colon Hecke_x \to Bun_{^L G}$, $x \in C$.

Here $Hecke_x$=moduli space of triples $(V, V', \beta\colon V|_{C-x} \xrightarrow{\cong} V'|_{C-x})$, V, V' principal ${}^L G$-bundles. The maps to $Bun_{^L G}$ project to V and V', respectively (every structure algebraic).

These can be combined (union over all $x \in C$) to $Hecke \to Bun_{^L G}$, $Hecke \to Bun_{LG} \times C$.

1.8 Proposition. *The maps $p_x, q_x\colon Hecke \to Bun_{^L G}$ are smooth, locally trivial fibrations. The fibers are affine Grassmanians for ${}^L G$ of the form ${}^L G((t))/{}^L G[[t]]$. These are infinite dimensional ind-schemes.*

Because the fibers are so huge (integration along the fibers not really possible), the spaces have to be cut down.

Fix therefore a dominant $\mu \in cochar^+_{^L G} = char^+_G$. Define $Hecke^\mu_x \subset Hecke_x$ as $\{(V, V', \beta)\}$ such that for every irreducible representation $\rho\colon {}^L G \to Gl_n(\mathbb{C})$ the map $\rho(\beta)$ gives an inclusion of locally free sheaves $\rho(V) \subset \rho(V')(< \lambda, \mu > x)$. Here μ is the highest weight of rho; and we bound the order of the pole at x.

Then $Hecke_x = colim Hecke^\mu_x$. We get $p^\mu_x, q^\mu_x\colon Hecke^\mu_x \to Bun_{^L G}$ by restriction. p^μ_x and q^μ_x are locally trivial in the smooth topology. The fibers are smooth if and only if mu is minuscle as a character of G, but they are compact.

The composition map

$$Hecke_x \times_{q_x, p_x} Hecke_x \to Hecke_x; (V, V', \beta), (V', V'', \gamma) \mapsto (V, V'', \gamma\beta)$$

is compatible with weights, and we get an isomorphism

$$Hecke^\mu_x \times_{Bun_{^L G}} Hecke^\nu_x \to Hecke^{\mu+\nu}_x.$$

1.5 Hecke operators

$H^\mu\colon D^b(Bun_{^LG}, D) \to D^b(Bun_{^LG} \times C, D)$,

here we take the (ordinary) bounded derived category of D-modules on $Bun_{^LG}$ (this does not generalize to families, one should work with dg-modules).

$$H^\mu\colon M \mapsto {}^Lq^\mu_!({}^Lp^{\mu*}M \otimes IC_{Hecke^\mu})$$

Here IC is the middle perversity extension of the trivial local system from the smooth points.

1.9 Definition. A D-module $M \in D^b(Bun_{^LG}, DJ)$ is a Hecke eigensheaf of eigenvalue $\mathbb{V} \in Loc_G$ if for all μ

$$H^\mu(M) = M \boxtimes \rho^\mu(\mathbb{V})[-d^\mu]$$

where d^μ is the dimension of the fiber of p^μ_x (or q^μ_x, which are equal).

(automorphic D-modules)

Talk 4: Topological field theory (Ben-Zvi)

1.6 Geometric lLanglands conjecture: formulation

1.10 Conjecture. *Fix a curve C, $G, {}^LG$ as in Pantev's first talk.*

Then there is an equivalence

$$D^b(loc_G, O) \xrightarrow{\sim} D^b(Bun_{^LG}, D)$$

where $Bun_{^LG}$ is the moduli space of all bundles, and the functors sends a skyscraper at $\mathbb{V}$ to the Hecke eigensheaf for $\mathbb{V}$.

If μ is an irreducible representation of G, then under this correspondence the tensor product with $\rho^\mu(E)$, where E is the universal local system over $loc_G \times C$; this operator is sent to H^μ.

1.7 Point of view of Kapustin-Witten

Slogan: Geometric Langlands is a manifestation of an underlying isomorphism of 4-dimensional (conformal) topological gauge theories:

$$Z_{G,\Psi} = Z_{\hat{G},-1/\Psi}.$$

This "symmetry" is called electric-magnetic duality (a non-commutative generalization of the duality between electric and magnetic fields in Maxwell's theory of electromagnetism).

1.8 Topological Field theories

Axiomizes roughest aspects of QFT (only feeling the topology).

0 tier TFT: An n-dimensional TFT $Z/\mathbb{C}$ is an assignement $Z\colon M^n \mapsto Z(M) \in \mathbb{C}$ (M^n n-dimensional smooth compact oriented manifold) with the following properties:

(1) oriented diffeomorphism invariant

(2) multiplicativity: $Z(\emptyset) = 1$, $Z(M \amalg N) = Z(M)Z(N)$.

This is related to physics as follows: $Z(M)$ is the partition function: it should be the integral over all fields (a path integra) $\int e^{-S(\phi)} D\phi$, ϕ a field, S the action functional.

1 tier TFT: An n-dimensional topological field theory is as above, and to any $N^{n-1} \mapsto Z(N) \in Vect_{\mathbb{C}}$ with the following properties:

(1) functoriality

(2) multiplicativity: $Z(\emptyset) = \mathbb{C}$, $Z(M \amalg N') = Z(M) \otimes Z(N')$, $Z(M^{op}) = Z(M)^*$.

(3) If M^n is a manifold with boundary, $Z9M) \in Z(\partial M)$

(4) $Z(N \times [0,1]) = \dim \mathrm{Hom}(Z(N), Z(N)) = Z(\partial(N \times [0,1]))$.

(5) Functoriality under composition of bordisms:

Then $Z(T_f) = \mathrm{tr}(Z(f))$, where $f\colon M \to M$ is a diffeomorphism, and T_f is the mapping torus. In particular, $Z(N \times S^1) = \dim(Z(N))$.

We can think of this as a map $Z \mapsto Z_{S^1}$ from n-dimensional TFT to $(n-1)$-dimensional TFT. More generally, if Σ^d is a d-manifold, we get "dimensional reduction" along Σ by precomposing with the product with Σ.

1.11 Notation. We adopt the following labelling convention: if $v \in Z(\partial M)$, M with boundary labelled by v we assign to this (via Z) the number $Z(M)(v) \in \mathbb{C}$.

1.12 Theorem. *{1-tier 2D TFT} are in 1-1 correspondence with {commutative Frobenius algebras} as follow:*

$Z(S^1) = H$ is a vector space. The disc with outgoing boundary gives a map $\mathbb{C}1 \to H$, the unit.

The disc with incoming boundary gives a map $H \xrightarrow{tr} \mathbb{C}$, a trace.

$Z(closingpairofpants)$ is $\mu\colon H \otimes H \to H$ the multiplication.

One has to check: Z of cylinder with in 2 incoming components $=\mathrm{tr}(\mu)$ is non-degenerate.

Commutativity follows from the diffeomorphism which "interchanges the two holes" in the disc with two holes.

More generally, if Z is an n-dimensional TFT, then $Z(S^{n-1})$ carries a product (commutative if $n > 1$). $Z(S^{n-1})$ acts on $Z(N^{n-1})$, using a point $x \in N$ (by cutting out a little disc near $(x, 1/2)$ in $N \times [0,1]$).

Generalizations:

(1) families version: if $N \to B$ is a family of $(n-1)$-manifolds, $Z(N \to B)$ should be a local system on B.

(2) graded version: $Z(N^{n-1})$ a graded vector space, etc.

(3) dg TFT/TCFT: $Z(N)$ should be a dg algebra. Then, given a bordism M from N_1 to N_2, $Z(M)$ should be a chain map of degree 0 from $Z(N_1)$ to $Z(N_2)$.

In the families version: given a k-chain of manifolds bounding N_1, N_2, we should get a $(1-k)$-chain in $Hom(Z(N_1), Z(N_2))$ of the right degree.

In this version, $Z(S^{n-1})$ is an E_n-algebra.

1.13 Example. Physics: a Calabi-Yau variety X gives rise to two 2D TFT, the A-model and the B-model; called "$N = 2$ SUSY σ-manifold".

The A-model only depends on the symplectic structure of X (its Frobenius algebra is the quantum cohomology $QH^*(X)$), the B-model depends on the complex structure (its Frobenius ist $\oplus HH(\Lambda^i T_X)$, the Hochschild homology).

2-tier TFT: goal is to express locality of $Z(N^{n-1})$. Now, $Z\colon Y^{n-2} \mapsto Z(Y)$, a $\mathbb{C}$-linear (dg) category.

Properties: functorial, monoidal.

Refinement: $Z(N^{n-1}) \in Ob(Z(\partial N))$.

To a $(n-1)$-bordism we should assign a functor between the categories of the boundaries, compatible with composition of bordisms.

To an n-manifolds with corners, such that the boundary is an $(n-1)$-manifold with boundary, we (roughly) get a natural transformation between the functors associated to the boundary pieces, to write down details becomes slightly more involved.

1.14 Theorem. *(Costello)*

2D (dg) 2 tier TFTs are (roughly) in one-one correspondence to non-commutative Calabi-Yau varieties, i.e. to dg categories that look like $D^b(CohX)$, for X Calabi-Yau. There is some kind of "volume form" on it, in particular there is a trace map $Hom(A, A) \to \mathbb{C}[-]$ for each object A.

Talk 5: Geometric Hecke operators (Heinloth)

1.9 Goals of the talk

In this talk, some of the constructions described by Toni Pantev are repeated. We also want to motivate the formulation of the Geometric Langlands conjecture.

1.10 Notation

Let G be a reductive group (think of Gl_n). Let $T \subset G$ be a maximal torus, containd in a Borel subgroup B, which also contains the nilpotent radical N (in case of Gl_N, T consists of the diagonal matrices, B of the upper triangular ones, and N of the upper triangular ones with 1s on the diagonal).

Recall: Yesterday we looked at the biquotient $k(C)^* \backslash \prod_{x \in C} K_x^* / O_x^*$.

Langlands considered $Fun(G(k(C)) \backslash \prod'_{x \in C} G(K_x) = G((t))(k))$. Decompose this space as representation of $\prod' G(K_x) =: G(A_{k(C)})$.

Simplify: Look at

$$Fun(\underbrace{G(k(C)) \backslash \prod{}' G(K_x) / \prod G(O_x)}_{DQ}).$$

What is left of the representation of $\prod_{x \in C} G(K_x)$?

We still have an action of $Fun_c(\underbrace{G(O_x)\backslash G(K_x)/G(O_x)}_{=':H})$. If ϕ comes from here, and f of the functions on DQ, then

$$(f * \phi)(g) = \int_{G(K_x)} f(gh^{-1})\phi(h)\, dh.$$

Note: $DQ \cong Bun_G(k)$, the moduli space of G-bundles on C. The correspondence is given for $g_x \in G(K_x)$ by gluing the trivial G-bundle on $C - x$ and on a disc at x using g_x.

1.15 Remark. Fix isomorphism $K_x = k((t))$, $O_x = k[[t]]$. Then

$$G(K_x) = G(k((t))\,) = \amalg_{\mu in\, \mathrm{Hom}(G_m,T)^+} G(k[[t]])\mu(t)G(k[[t]]).$$

(μ runs through the dominant weigts).

For Gl_n, $G(k((t])\,) = \amalg_{d_1 \geq \cdots \geq d_n \in \mathbb{Z}} \operatorname{diag}(d+1, \ldots, d_n)G(k[[t]])$.

In particular, H has a basis given by $1_{G[[t]]\mu(t)G[[t]]}$.

1.16 Theorem. *(Satake)*
As algebras:

$$H \cong \mathbb{C}[cochar(T)]^W \quad (= R(^LG)).$$

In particular, H is commutative.

Proof. Satake's trick: consider $Fun(\prod O_x N(K_x)\backslash G(k((t))\,)/G[[t]])$. This is a $\mathbb{C}[cochar(t)] \times H$-module, and free of rank one for the part $\mathbb{C}[cochar(t)]$.

Note: $G(K_x) = \amalg_{\mu \in cochar(T)} N(K_x)\mu(t)G(O_x)$

For Gl_n this means: every element of $G(k((t)))$ is of the form $A \operatorname{diag}(t^{d_1}, \ldots, t^{d_n})B$ with $A \in N$ and $B \in G(k[[t]])$.

We now get the map $H \to \mathbb{C}[cochar]$ by $h \mapsto \mu$, where μ is defined by $1 * h = \mu \cdot 1$. □

1.11 Geometrization of the construction

Note:

- $G[[t]]$ is a group scheme/k, namely $G[[t]](R) := G(R[[t]])$. For Gl_n: $A^1[[t]] \cong A^\infty$ is the infinite dimensional affine space over k.)
- $G((t))$ is a group-ind-scheme/k. For Gl_n: $A^1((t))$.
- The affine Grassmannian $Gr_G := G((t))/G[[t]]$ is an infinite dimensional ind-schme/k, which is formally smooth. The latter means that $k[\epsilon]/\epsilon^n$ points lift to $k[[epsilon]]$.

Recall: $Gr_mu := G[[t]]\mu(t)G[[t]]/G[[t]] \subset Gr_G$ is finite dimensional.
Denote by $\bar{G}r^\mu \subset Gr_G$ the closure of Gr_μ.

1.17 Theorem. *(Geometric Satake isomorphism, due to Lusztig, Ginzburg, Mirkovic-Vilou, Drinfeld,...)*

$$Perv_{G[[t]]}(Gr_G) \cong Rep(^LG)$$

as tensor categories (Tannaka). The representations are finite dimensional over $\mathbb{C}$. The tensor structure on Perv comes from convolution of functions, Perv should be a replacement of the functions, (equivalently one can use D-modules).

1.12 Perverse sheaves

X any variety, then $Perv(X) \subset D^b_{etale}(X)$ with marvelous properties:

(1) If $j\colon Z \to \text{smooth}\bar{Z} \hookrightarrow X$, L a local system on Z. Then $IC(L) = j_{!*}L \in Perv(X)$ and $Perv(X)$ is generated by these. The functor $j_{!*}$ lies "in between" $j_!$ and j_*.

(2) $Perv(X)$ is abelian, $H^*(X, j_{!*}L)$ is Poincare dual to $H^*(X, j_{!*}\check{L})$.

(3) $f\colon X \to Y$ proper, then $Rf_*(IC(X)) = \bigoplus_i Perv[i] \in D^b(X)$. Here $IC(X) = j_{!*}Q$ for $j\colon X^{smooth} \hookrightarrow X$.

   (Compare this to the spectral sequence for cohomology which degenerated in the smooth case: $H^*(X, \bar{Q}) \cong H^*(Y, R^i f_* Q)$).

1.13 Correspondences

Pantev introduced yesterday: $Hecke^\mu_x = \{(V, v', \phi\colon V|_{C-x} \cong V'_{C-x})$ with $q^\mu_x, p^\mu_x\colon Hecke^\mu_x \to Bun_G$.

If $V = G \times C$ is trivial then V' differs from V only on disc at x, i.e. V' can be obtained from cocycle in $G(K_x)/G(O_x) = Gr$. The fibers of p^μ_x, q^μ_x are isomorphic to $\bar{Gr}^\mu$.

Now the Geometric Satake corresponence is given as follows:

To $IC(\bar{Gr}^\mu)$ corresponds the finite dimensional representation V^μ.

Talk 6: TFT II (Ben-Zvi)

Recall: an n-dimensional (dg) TFT Z assigns:

$$\begin{aligned} M^n &\mapsto Z(M) \in \mathbb{C} \\ N^{n-1} &\mapsto Z(N) \in \mathbb{C} - Vect \\ Y^{n-2} &\mapsto Z(Y) \in \mathbb{C} - Cat. \end{aligned}$$

(for closed manifolds).

For manifols with boundary/corners: adding labels (with the appropriate rules) corresponds to "closing off" and therefore fits in the pattern of the table.

In $2D$: to the interval with endpoints labelled by A and B we assign a $\mathbb{C}$-vector space $\mathrm{Hom}(A, B)$.

To a triangle with vertices labelled by A, B, C we assign an element in $\mathrm{Hom}(A, B)\,tensor\,\mathrm{Hom}(B, C)$. The triangle is often draw "streched out", where the vertices are themselves replaced by edges (with label running constantly along the edge).

1.18 Theorem. *(Moore-Segal, Costella)*
2-tiered $2D$ TFTs with label set Λ are in one to one correspondence to non-commutative Calabi-Yau categories with Λ the set of objects (e.g. $D^b(Coh(X))$ for a Calabi-Yau variety X).

(Recall that 1-tier $2D$ TFTs are in 1-1 correspondence to commutative Frobenius algebras.)

Recall: $Z(S^{n-1})$ in an nD TFT is a (commutative) algebra.

Similarly, $Z(S^{n-2})$ in an nD TFT is a monoidal category, with the same picture as in the earlier case.

We have $Z(S^0)$ is monoidal (and not more), $Z(S^1)$ is braided, $Z(S^2)$ is symmetric.

1.14 Physics

To a Calabi-Yau X we assign two 2D TFTs, the A-model (using only the symplectic structure of X) and the B-model (using only the complex structure of X).

They are special cases of σ-models, i.e. made out of maps $2d$-surfaces$\to X$. These give rise to category of boundary conditions (D-branes)

In the B-model, this category (of B-branes) is $D^b(X)$, the derived category of coherent sheaves. In the A-model, we get $Fuk(X)$, (a version of) the Fukaya category of X. Objects are Lagranians $M \subset X$ together with L a local system on M. This is a non-commutative CY category.

The morphism spaces are obtained by moving the two Lagrangians in transverse position, then $\mathrm{Hom}((M_1, L_1), (M_2, L_2)) := \oplus_{M_1 \cap M_2} L_{x_1} \otimes L_{x_2}$.

This is slightly oversimplified, because we really get an A^∞-category; higher Homs are obtained e.g. by counting pseudoholomorphic discs with marked points with boundary on the union of the Lagrangians.

1.15 Electric-Magnetic Duality in $4d$ gauge theory

These ideas are due to Kapustin-Witten.

Let G be a complex reductive group, G_c the maximal compact subgroup.

Then there is a 4d quantum field theory, called "$N = 4$ SUSY Yang-Mills with gauge group G_c". It is a gauge theory: fields include connections on G_c-bundles on $\mathbb{R}^4$ together with $\phi_1, \ldots, \phi_6$ sections of $ad(P)$ (6 bosonic fields).

To put this in perspective: there are $N = 1$ SUSY Yang-Mills in $10D$ with A_{10} on $\mathbb{R}^{10}$. We then reduce to $4D$ and get the connection A_4 and $\phi_1, \ldots, \phi_6$ sections valued in adjoint bundle, to be thought of as the remaining components of the connection.

The theory has a complex parameter $\tau \in H$ (the upper half plane), the coupling constant. Thenthe action is

$$\frac{4\pi i}{\tau^2} \int F \wedge *F + \frac{\theta}{2\pi} \int F \wedge F$$

where F is the curvature.

S-duality: Physics predicts that the theory for the group G and parameter τ is isomorphic to the theory for ${}^L G_c$ with parameter $-1/\tau$.

(There is a whole $Sl_2(\mathbb{Z})$-symmetry in the parameter; $\tau \mapsto \tau + 1$ preserves the group).

1.16 Topological twists

We have to make sure that the expressions (for the action) which are formulated above for $\mathbb{R}^4$ are well defined independend of a metric, to give rise to topological field theories.

It turns out that to achieve this, we change the transformation rules for the twists: instead of being scalars we think of $\phi_1, \dots, \phi_4$ as adjoint 1-forms, i.e. think of $\mathbb{R}^{10} = T^*M^4 \times \mathbb{R}^2$, with $M = \mathbb{R}^4$.

Therefore, our fields $(A, \phi = (\phi_1, \dots, \phi_4))$ are considered to be connections on $G = (G_c)_{\mathbb{C}}$-bundle.

The resulting field theory has two odd symmetries Q_1, Q_2 with $Q_i^2 = 0$.

Cohomology of either symmetries is a TFT. This actually gives a P^1 worth of TFTs, corresponding to $Q = aQ_1 + bQ_2$.

The resulting TFT depends on a single parameter $\Psi \in P^1$, and for the action we get

$$S = \frac{i\Psi}{4\pi} \int F \wedge F + Expr(Q),$$

where the second summand is exact.

The idea now is that one still should have a duality

$$\{G, \Psi\} \cong \{{}^L G, -\frac{1}{\Psi}\}.$$

1.17 Kapustin-Witten's connection to geometric Langlands

Given a 4D TFT and a surface C gives by dimension reduction a $2D$ TFT Z_C with $Z_C(\Sigma) = Z(C \times \Sigma)$.

In our case Kapustin-Witten "work out":

$$Z_{C,G,\Psi} \text{ corresponds to NC Calabi-Yau,}$$

with in particular: for $\Psi = 0$ be get $D^b(Bun_G, D)$, for $\Psi = \infty$ we get $D^b(Loc_G, O)$. Choosing G and ${}^L G$, the duality above gives the Langlands duality

$$D^b(Bun_G, D) \cong D^b(Loc_{{}^L G}, O).$$

We finally want to identify the Hecke operator and multiplication operators:

1.18 Hecke operators

On $Z_{C,G,0}$ acts $Z_{C,G,0}(S^2)$ (a symmetric monoidal category).

On $Z_{C,{}^L G,\infty}$ acts $Z_{C,{}^L G,\infty}(S^2)$. The latter category is identified as the coherent sheaves on $Loc_{{}^L G}(S^2) = */{}^L G = Rep({}^L G)$.

Using geometric Satake duality, $Z_{C,G,0}(S^2)$ also is isomorphic to $Rep({}^L G)$.

1.19 Gauge theory to TFT

Let a 4D gauge theory be given, with connection A and fields ϕ. Then

$Z(M^4) \in \mathbb{C}$ is the number of solutions of field equations on M^4.

$Z(N^3) \in \mathbb{C} - Vect$ is H^*(moduli space of solutions to "monopole" equations on N^3).

$Z(N^2) \in \mathbb{C} - Cat$ is the category of branes on the moduli spaces of solutionson C. More precisely, we consider $C \times \Sigma$ (with C very small). Solutions to field equations should (in approximation) cooorespond to maps from Σ to $\{$solutions of eqns on $C\}$. Note that here we get to a σ-model. Depending on $\Psi \in \{0, \infty\}$, the σ-model will be the A-model or the B-model (worked out by Kapustin-Witten).

The space of solutions on C is the Hitchin moduli space (of Higgs bundels on C). We get a correspondence {hyperkähler manifolds} $\to Loc_G C =$ coherent sheaves (if $\Psi = 0$) and {hyperkähler} $\to T^*Bun_G C = D_{Bun}$, if $\Psi = \infty$.

Note: none of this has been made rigorous so far, a lot of technical problems have to be overcome (e.g.: moduli spaces are non-compact, what is the right cohomology? perhaps some kind of Floer theory?).

Talk 7: Abelianization and the Hitchen System (Pantev)

1.19 Conjecture. *There is an equivalence of categories*

$$c\colon D^b(Loc_G, O) \xrightarrow{\sim} D^b(Bun_{^LG}, D)$$

Under this equivalence, the structure sheaves of points go to automorphic D-modules, i.e. $H^\mu(c(O_{\mathbb{V}})) = c(O_{\mathbb{V}}) boxtensor \mathbb{V}[-d^\mu]$.

Note that the structure sheaves of points in a certain sense form an orthonormal basis of $D^b(Loc_G, O)$.

Attention: The way the conjecture is formulated above, it is completely wrong; it has to be motified suitably.

Problem: The categories above are of completely different flavor. E.g.: $Bun_{^LG}$ is disconnected, $\pi_0(Bun_{^LG}) = H^2(C, \pi_1(^LG)) = \pi_1(^LG)$.

Therefore, $Bun_{^LG} = \amalg_{\gamma\in\pi_1(^LG)} Bun^\gamma_{^LG}$. This implies that

$$D^b(Bun^LG, D) = \prod_{\gamma\in\pi_1(^LG)} D^b(Bun^LG^\gamma, D).$$

1.20 Theorem. *(Simpson)*
Loc_G is an irreducible algebraic variety.

It follows that $D^b(Loc_G, O)$ does not split as a product of categories, it is indecomposable.

Fix: Replace Loc_G with the moduli stack of local systems, called $\mathcal{L}oc_G$.

Simplifying assumption (not necessary, just avoids technicalities): Restrict to $\mathcal{L}oc^{rs}_G$, the moduli stack of regularly stable local systems (by definition stable local systems with the minimal possible automorphisms). One has to replace Bun^LG accordingly, considering only stable bundles.

If one want to work with all (also non-stable) bundles, one replaces the moduli space Bun^LG by a quotient of the moduli stack (which has the same moduli space, and such the full moduli stack is a gerbe over the intermediate objct; this exists canonically in the algebraic situation). On the local systems side one uses the stack of all local systems.

We have a map $\mathcal{L}oc^{rs}_G \to Loc^{rs}_G$, this is a gerbe bounded by $Z(G)$.

Then

$$D^b(\mathcal{L}oc^{rs}_G, O) = \prod_{\gamma\in Z(G)\check{}} D^b(\mathcal{L}oc^{rs}_G, O).$$

But ^LG being the Langlands dual of G implies that $Z(G)\check{} = \pi_1(^LG)$.

1.21 Conjecture. *There is an equivalence of categories*

$$D^b(Bun_{{}^L G}, D) \overset{c}{\xleftarrow{\sim}} D^b(\mathcal{L}oc_G, O).$$

Under this equivalence, for all $\mathbb{V} \in Loc_G$, $H^\mu(c(O_{\mathbb{V}})) = c(O_{\mathbb{V}}) \boxtimes \mathbb{V}^\mu[-d^\mu]$.

The conjecture is known completely only in two cases: $G = \mathbb{C}^\times$ and $G = Sl_2(\mathbb{C})$.

Further partial results are known.

In some cases there are functors (several different ones) which diagonalizes Hecke correspondence on the "orthonormal basis" of sheaves O_V as in the addendum to the conjecture.

1.20 Classical limit

We have a difficult conjecture coming from quantum theory and involving quantization (in the picture of Kapustin-Witten). We want to simplify by passing ot a classical limit.

One strategy would be to do this, prove the classical limit statement, and then pass to the full conjecture by some kind of quantization (promising implementations of this program are on the way).

1.21 Deformations of the geometry

Loc_G is a natural "deformation" of a simpler space. Specifically, we have a specialization

$$\mathcal{L}oc_G \text{ specialializes to } T\mathcal{B}un^0_G,$$

where Bun^0_G is the stack of G-bundles on C which are topologically trivial (connected component of the "identity").

We have a family

$$\begin{array}{ccccc} \mathcal{L}oc_G \times \mathbb{C}^\times & \xrightarrow{\text{inclusion}} & H & \xleftarrow{\text{incl}} & T\mathcal{B}un^0_G \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}^\times & \xrightarrow{\text{incl}} & \mathbb{C} & \longleftarrow & \{0\}. \end{array}$$

Here H is the moduli stack of Z-connections (integral connections) on principal G-bundles.

Recall: If $V \to C$ is a principal G-bundle, we had the Atiyah sequence

$$0 \to ad(V) \to A(V) \xrightarrow{\sigma} T_C \to 0,$$

a connection is a splitting ∇ of this sequence.

1.22 Definition. A z-connection on V for $z \in \mathbb{C}$ is a splitting ∇ of

$$0 \to ad(V) \to A(V) \xrightarrow{z\sigma} T_C \to 0,$$

1.23 Example. $G = Gl_n$. Then V is the frame bundle of a vector bundle E. A connection on V is a connection on E, i.e. $E \xrightarrow{\nabla} E \otimes \Omega^1_C$ satisfying the Leibnitz rule $\nabla(fe) = df \cdot e + e\nabla(f)$.

A z-connection on V corresponds to $\nabla \colon E \to E \otimes \Omega^1_C$ with $\nabla(fe) = zdf \cdot e + f\nabla(e)$.

1.24 Remark. If $z \neq 0$ then $\frac{1}{z}\nabla$ is an ordinary connection.

If $z = 0$, then a z-connection is an O-linear map $T_C \xrightarrow{\nabla} ad(V)$ corresponding to an element $\theta \in H^0(C, ad(V) \otimes \Omega^1_C)$.

Note that ∇ is integrable if and only if $\theta \wedge \theta = 0$.

1.25 Definition. (V, θ) with the integrability condition (which is automatically satisfied on a surface) is called a *Higgs bundle.*

I.e.: 0-connections are just Higgs bundles.

1.26 Remark. It is true in general that on a Kähler manifold the stack of Higgs bundles is the tangent stack to the stack of bundles; easy for surfaces and hard in general.

This means:

$$T_V \mathcal{B}un^0_G = H^1(C, ad(V)) = H^0(C, ad(V) \otimes \Omega^1_C)\check{}$$

Therefore

$$T_V\check{\mathcal{B}}un^0_G = H^0(C, ad(V) \otimes \Omega^1_C).$$

So $H_0 = Higgs^0_G = T\check{\mathcal{B}}un^0_G$.

This is the first deformation, for the right hand side of the Langlands correspondence.

1.22 Deformation of D-modules

$D \to Bun^L G$ is a sheaf of filtered algebras (filtered by the order of the differential operator). In such a case, there is a natural deformation to the associated graded algebra, i.e. here

$$D \text{ specializes to } gr(D^\bullet) \cong S^\bullet T_{Bun^L G}.$$

Specifically, we have $R \to Bun^L G \times \mathbb{C}$ such that

$$R|_{Bun_{L_G} \times z} \cong \begin{cases} D; & z \neq 0 \\ gr(D^\bullet); & z = 0. \end{cases}$$

If p: $Bun^L G \times \mathbb{C} Bun^L G$ is the projection, then

$$R \subset p_1^* D := p^{-1} D \otimes O_{Bun^L G \times \mathbb{C}}$$

such that

$$R = \{\sum z^i P_i \mid P_i \in D^i\};$$

one always can construct such a “Rees ring”.

Alltogether: $D^b(\mathcal{L}oc_G, O)$ specializes to $D^b(\mathcal{H}iggs^0_G, O)$, and $D^b(Bun^L G, D)$ specializes to

$$D^b(Bun^L G, S^\bullet T) = D^b(T\check{}Bun^L G, O) = D^b(Higgs^L G, O).$$

Expectation: Classical limit conjecture: There should exist an equivalence

$$c^{cl} \colon D^b(\mathcal{H}iggs^0_G, O) \xrightarrow{\sim} D^b(Higgs^L G, O)$$

such that it sends structure sheaves of points to automorphic O-modules (defined via the classical limit of Hecke correspondences).

Observe that this looks rather symmetric, on both sides we see "parts" of the full moduli stack of Higgs bundles. The corresponding general classical conjecture is the following:

1.27 Conjecture. *there is an equivalence*

$$c\colon D^b(\mathcal{H}iggs_G, O) \xrightarrow{\sim} D^b(\mathcal{H}iggs^L G, O).$$

Together with a suitable Hecke correspondence "diagonalization".

Talk 8: Local correspondence, the geometric Satake isomorphism (Heinloth)

Recall: the geometric Satake isomorphism says

$$Perv_{G[[t]]}(Gr_G) \cong Rep(^L G)$$

(where we work with coefficients $\mathbb{C}$).

The proof will be given by Tannaka duality:

(1) Show the LHS is a (rigid) symmetric tensor category; rigid means for all x there is an $\check{x}$ s.t. $1 \to x \otimes \check{x}$, $x \otimes \check{x} \to 1$ satisfying suitable axioms.

(2) Construct an faithful exact tensor functor $Perv_{G[[t]]}(Gr_G) \to Vect_{\mathbb{C}}$

(3) These two properties imply (via Tannaka duality) that LHS is isomorphic to $Rep(H)$ for some group H.

(4) Identify H with $^L G$.

Main point: How to construct the tensor structure? There are two ways to do this.

First idea: Recall that $Gr_G = G((t))/G[[t]]$

$$\begin{array}{ccccc} G((t)) \times G((t))/G[[t]] & \longrightarrow & G((t))^{G[[t]]} \times G((t))/G[[t]] & \xrightarrow{m} & G((t))/G[[t]] \\ \downarrow p & & & & \\ & & Gr_G \times Gr_G & & \end{array}$$

where $m(g,h) = gh$ is the multiplication map.

Given $P_1, P_2 \in Perv(Gr_G)$, $p^*(P_1 \boxtimes P_2)$ descens to $P_1 \tilde{\boxtimes} P_2$ on the middle space; define

$$P_1 * P_2 := Rm_! P_1 \tilde{\boxtimes} P_2.$$

Problems: whey should $P_1 * P_2$ be perverse, why isomorphic to $P_2 * P_1$.

Second idea: Uses "nearby cycles".

1.23 Nearby cycles

Topologically:

$$\begin{array}{ccccc} X_0 & \xrightarrow{i} & X & \longleftarrow & X^\circ \\ \downarrow & & \downarrow \pi & & \downarrow \pi^\circ \\ \{0\} & \longrightarrow & [0,1] & \longleftarrow & (0,1] \end{array}$$

We assume π° is smooth, the $X^\circ \cong X_n \times (0,1]$ is trivial.

1.28 Definition. Given a complex K^* on X°, define $\Phi_\pi(K^*) := i^* Rj_* K_*$. $\Phi_\pi(K^*)$ is called *nearby cycle*.

Then: assume $K^* = pr^*_{X_h} K^*$ is constant, then $H^*(X_0, \Phi(K^*)) = H^*(X_n, K^*)$.

1.29 Theorem. *Φ maps perverse sheaves to perverse sheaves. Φ commutes with duality. Φ commutes with proper pushforward.*

Thre is a complex/algebraic analog (more complicated because of the non-trivial fundamental group of the punctured disc; therefore use its universal covering).

Here: take any curve C (but think of $C = A^1$). Recall $G((t))/G[[t]]$ can be considered as moduli space of G-bundles V on C together with $V|_{C-x} \xrightarrow{\cong}$ trivial bundle.

$$Gr_C = \{(V, x, \phi) \mid x \in C, \phi\colon V|_{C-x} \to G \times (C - x)\} \xrightarrow{p} C$$

The fibers are isomorphic to Gr_G.

Set

$$Gr_{C\times C} := \{(V, x, x', phi) \mid x, x' \in C, \phi\colon V|_{C-\{x,x'\}} \xrightarrow{\cong} G\times X - \{x, x'\}\} \to C\times C.$$

Note: fiber over (x, x) is isomorphic to Gr_G, but the fiber over (x, x') with $x \neq x'$ is isomorphic to $Gr_G \times Gr_G$.

Finally

$$\widetilde{Gr}_{C\times C} := \{(V, V', x, x', \phi, \phi') \mid \phi\colon V|_{C-x} \cong G\times(C-x), \phi'\colon V'|_{C-x'} \cong V|_{C-x'}\} \to C\times C.$$

Here all fibers are isomorphic to $Gr_G \times Gr_G$, this is a smooth locally trivial fibration.

There is an obvious map

$$p\colon \widetilde{Gr}_{C\times C} \to Gr_{C\times C}; (V, V', \phi, \phi') \mapsto (V', \phi \circ \phi').$$

1.30 Definition. $P_1 * P_2 := \Phi_f(P_1 \boxtimes P_2)$,

where $P_1, P_2 \in Perv(Gr_G)$. The map f comes from choice of local coordinates, it gives the following pullback diagram, and is then used in the nearby cycles construction.

$$\begin{array}{ccc} A^1 \times A^1 & \longleftarrow & Gr_{C\times C} \\ \uparrow (t,-t) & & \uparrow \\ A^1 & \xleftarrow{f} & Gr_{C\times C}|_\Delta \\ \uparrow & & \uparrow \\ \{0\} & \longleftarrow & Gr_G \end{array}$$

Now:

(1) $P_1 * P_2$ is perverse, because Phi preserves perverse sheaves.

(2) $P_1 * P_2 \cong P_2 * P_1$ functorially, same for associativity.

(3) This gives the same definition as before:

$$\begin{array}{ccc} \widetilde{Gr}_{C\times C} & \longleftarrow & \widetilde{Gr}_{C\times C-\Delta} \\ \downarrow & & \downarrow \\ Gr_{C\times C} & \longleftarrow & Gr_{C\times C-\Delta} \\ \downarrow & & \downarrow \\ C\times C & \longleftarrow & C\times C-\Delta. \end{array}$$

Let $\tilde{f}$ be the composition on the left.

Now: nearby cycles for the smooth fibration yields $\Phi_{\tilde{f}}(P_1 * P_2) = P_1 \tilde{\boxtimes} P_2$, and $p_0 = m$ in previous definition; now use that Φ_f commutes with push-forward.

(4) Φ_f commutes with proper push-forwards. This implies

$$H^*(Gr, P_1 * P_2) = H^*(Gr, P_1) \otimes H^*(Gr, P_2).$$

Note that this means that H^* is a tensor functor!

Problem: calculate $H^*(Gr, IC(Gr_\mu))$.
Recall that Satake looekd at $N(k((t)))$-orbits in $G((t))/G[[t]]$. Then

$$G((t)) = \amalg_{v\in X_*(T)} N(k((t)))v(t)G[[t]].$$

We should try to do the same.

1.31 Definition.

$$S_\nu := N(k((t)))\nu(t)G[[t]]/G[[t]] \subset Gr_G$$
$$T_\nu := N^-(k((t)))\nu(t)G[[t]]/G[[t]] \subset Gr_G$$

where N^- is the opposite nilpotent: the lower triangular matrices with 1 on the diagonal.

Note that both these are infinite dimensional affine subvarieties of Gr_G.

1.32 Theorem. *(1)*

$$S_\nu \cap Gr^\mu = \begin{cases} \emptyset; & \nu(t) \notin \bar{G}r^\mu \\ \textit{equidimensional of dim } \rho(\nu+\mu); \textit{ otherwise} & \end{cases}$$

Here μ is dominant, and ρ is have the sum of the positive roots.

(2) Given $P \in Perv_{G[[t]]}(Gr_G)$, then

$$H_c^*(S_\nu, i^*P) = \begin{cases} 0; & \text{unless } * = \rho(\nu+\mu) \\ H^*_{T_\nu}(Gr, P) & \end{cases}$$

1.33 Corollary. *Using a suitable spectral sequence,*

$$H^*(Gr_G, P) = \oplus_\nu H^*(S_\nu, i^*P)$$

is an exact and faithful functor.

Proof. Exactness also follows from the spectral sequence of the stratification by $N(k((t)))$-orbits. □

Talk 9: Real groups (Ben-Zvi)

1.24 Goals of this talk

Goal is to describe joint work of Ben-Zvi and Nadler, which apply ideas from Geometric Langlands to representation theory.

Here, geometric Langlands questions for $C = P^1$ is related to classical representation theory problems.

Main point with $C = P^1$ is, that there are not many bundles. More precisely

$$Bun_G P^1 = G[t^{-1}]\backslash G((t))/G[[t]].$$

As a set, this is in bijection with $G[[t]]\backslash G((t))/G[[t]]$.

Note that (for G semisimple) the set of bundles isomorphic to the trivial bundle is open dense in $Bun_G P^1$, this part is isomorphic to $*/G$.

1.25 Tamely ramified geometric Langlands

Consider $Bun_G(P^1, 0, \infty)$, the set of G-bundles with flags at 0 and at ∞. This contains as open subset the set of parabolic bundles which are trivial. This part is $B\backslash G/B$

From now on, write $Bun_G := Bun_G(P^1, 0, \infty)$.

$D^b(Gun_G, D)$ contains in some sense as open dense subcategory $D^b_B(G/B, D)$.

1.26 Geometric representation theory

Beilinson-Bernstein: Localization of representations.

If G acts on X, then $\mathfrak{g}$ acts by vector fields, therefore $U\mathfrak{g}$ acts by differential operators: $U\mathfrak{g} \mapsto \Gamma(D_X)$.

Now there are functors both ways between $D_X - mod$ and $U\mathfrak{g}$-mod, one being global sections Γ, the other Δ, with $\Delta M := M \otimes_{U\mathfrak{g}} D_X$ for an $U\mathfrak{g}$-module M.

Consider now $X = G/B$. Then Borel-Weil says that representations of G are found in functions on spaces related to G/B (rather $G/N \xrightarrow{H} G/B$).

Beilinson-Bernstein: $D - mod$ on G/B are basically representations of $U\mathfrak{g}$.

We consider $D - mod$ on $B\backslash G/B$ (i.e. on G/B with extra invariance properties): this is strongly related to the Bernstein-Gelfand-Gelfand category O, or to Harish-Chandra bimodules. Here we look as representations of G as a real group.

I.e. the right hand side of Geomtric Langlands, $D^b_B(G/B, D)$ is strongly related to representations of G as real Lie group.

Let $B_{\mathbb{R}}$ be a real form of G, i.e. with complexification G.

Then the category of Harish-Chandra modules $HC_{G_{\mathbb{R}}}$ is by Kashiwara-Schmid equivalent to $D^b(G_{\mathbb{R}}\backslash G/B)$.

1.34 Remark. Landlands (as part of the classical Langlands program) classified irreducibles in terms of ${}^LG$.

Actually, he doesn't describe representations of one real form $G_{\mathbb{R}}$ alone. Instead, if θ is a quasisplit real form of G (i.e. compatible with Borel subgroup)

This gives rise to an extension $1 = toG \to G^\theta \to \mathbb{Z}/2 \to 1$ (use the involution for the real form and take the semi-direct product). Set

$$S := \{\sigma \in G^\theta - G \mid \sigma^2 = id\}/G.$$

For each such σ there is a real form G_σ of G, and these are considered simulaneously.

Next question: describe the whole derived category $HC_{G_\sigma, \sigma\in S}$.

Adams-Barbasch-Vogan: take Langlands parameters; consider them as points of an algebraic variety with an action of $\check{G}$.

Soergel: HC is described by D-modules on ABV-parameter space. Observe that this is a complex algebraic variety associated to $\check{G}$.

1.27 Geometric Langlands on P^1 implies Soergel's conjecture

Recall: $Bun := Bun_G(P^1, 0, \infty)$ contains $B\backslash G/B$, the locus of trivial representations.

Set Bun_θ as G-bundles on $(P^1, 0, \infty)$ which are *real* for θ and antipodal. It contains as trivial-bundle locus the disjoint union

$$\amalg_{\sigma\in S} G_\sigma\backslash G/B.$$

For the latter one, D-modules correspond to HC (Harish-Chandra modules).

1.35 Remark. In this talk, the term D-module is used for categories of constructible sheaves;

1.28 Tamely ramified geometric Langlands conjecture

Consider $Bun_G(C, x)$, bundles with parabolic structure on x. Let $I \subset G[[t]]$ be the Iwahora subgroup, i.e. with constant term in B.

(Recall that $Perv(G[[t]]\backslash G((t))/G[[t]]) = Coh_{{}^LG}(\cdot) = Rep^L G$.)

Then $D^b(I\backslash G((t))/I)$ acts on $Bun_G(C, s)$, given the Hecke operators.

This derived category is identified as $D^b_{{}^LG}(St_{{}^LG}, O)$, where the Steinberg variety

$$St = \{(g, B_1, B_2) \mid B_1, B_2 \text{ Borel in } {}^LG, g \in B_1 \cap B_2\}.$$

Tamely ramified geomtric Langlands should now say that $Bun_G(C,x)$ is dual to $Loc_{^LG}(C,x)$ (in the latter space of local systems we allow as imple pole at x and additional we have a compatible flag at x.

1.36 Theorem.

$$D(Bun_G(P^1,0,\infty)) \xrightarrow{\sim} \overline{D(Coh(St_{^LG}))}.$$

To apply this to representation theory, we have to relate this to $D(B\backslash G/B,D)$.

Note that there is a rotation S^1-action on P^1 fixing $0,\infty$. It turns out that the locus of trivial bundle is exactly given by the fixed points of this action.

From this it follows that $D(B\backslash G/B,D)$ is obtained from $D(Bun_G,P^1,0,\infty),D)$ by S^1-localization (i.e. passing to S^1-fixed points and tensoring with $\mathbb{C}[u,u^{-1}]$, $u\in H^2(BS^1)$).

The (proved) Geometric Langlands in this situation says: we have to S^1-localize $\overline{D(Coh(^LG\backslash St_{^LG}))}$. But need to identify the S^1-action!

For a variety X, consider the loop space $X^{S^1}=T_X[-1]$ with its S^1-action, where we use derived algebraic geometry for the definition.

There is an isomorphism $(CohX^{S^1})$ and $D_X()$; a manifestation between the relation of cyclic cohomology and de Rham cohomology.

Finally, this implies that the S^1-localization we have to do, gives $D(^LB\backslash{}^LG/^LB)$

Talk 10: Cartier duality for group stacks (Pantev)

Reminder: Classical limit geometric Langlands conjecture:

There exists an equivalence of categories

$$c\colon D^b(\mathcal{H}iggs_G,O)\to D^b(\mathcal{H}iggs_{^LG},O)$$

so that by restriction we obtain an equivalence

$$c^{cl}\colon D^b(\mathcal{H}iggs^0_G,O)toD^b(Higgs^LG,O).$$

(Part 2 will be a statement about Hecke operations).

Note: The stack $\mathcal{H}iggs_G$ has connected components labelled by $\pi_1(G)$, and has stabilizers labelled by the center of G, $Z(G)$.

In particular,

$$D^b(\mathcal{H}iggs_G,O)=\prod_{\alpha\in\pi_1(G),\beta\in Z(G)^{\check{}}} D^b(Higgs^{\alpha}_G,O,\beta).$$

(the β stands for twisted sheaves).

Similarly,

$$D^b(\mathcal{H}iggs^LG,O)=\prod_{\beta\in\pi_1(^LG),\alpha\in Z(^LG)^{\check{}}} D^b(Higgs^{\beta}_{^LG},O,\alpha).$$

The subcategories are unions of certain components:

$$D^b(\mathcal{H}iggs^0_G,O)=\prod_{\beta\in Z(G)^{\check{}}} D^b;(Higgs^0_G,O,\beta),$$

$$D^b(Higgs^L G, O) = \prod_{\beta \in \pi_1(^L G)} D^b(Higgs^L G^\beta, O).$$

Note: All the spaces $Higgs^?_?$ are Calabi-Yaus, the extra twist ($\alpha \in Z(?)\check{}$ corresponds to an additional non-trivial B-field).

1.29 Classical limit of Hecke operators

Note that
$$D^b(Higgs^L G, O) = D^b(Bun^L G, S^\bullet T).$$
Another way to say what these objects are:

$$QCoh(Bun^L G, S^\bullet T) = \{(E, \phi) \mid E \in QCoh(Bun_{^L G}), \phi\colon E \to E \otimes \Omega^1_{Bun^L G}\}$$

where ϕ is O-linear, and $\phi \wedge \phi = 0$ to ensure commutativity relations.

Idea: Deform Hecke correspondences $(Hecke^\mu_x, IC)$ to correspondences for Rees modules on $Bun_{^L G} \times \mathbb{C}$ to get
$$(Hecke^\mu_x, I^{\mu,cl}),$$
where $I^{\mu,cl}$ is a Higgs heaf on $Hecke^\mu_x$.

We now have to define $I^{\mu,cl}$. Then use $I^{\mu,cl}$ as the kernel of an integral transform as follows:

Still, we have two maps $p^\mu_x, q^\mu_x\colon Hecke^\mu_x \to Bun_{^L G}$. Then we define classical Hecke
$$H^{\mu,cl}_x(E, \phi) := (q^\mu_x)_!((p^\mu_x)^*(E, \phi) \otimes I^{\mu,cl}).$$

1.37 Conjecture. *Part 2 of the classical limit conjecture:*
$$H^{\mu,cl}(c^{cl}(O_{(V,\theta)})) = c^{cl}(O_{(V,\theta)} \boxtimes (\rho^\mu(V), \rho^\mu(\theta))[-d^\mu])$$

Answer of construction for $I^{\mu,cl}$.

(1) Put a good filtration on IC and pass to the associated graded. Unfortunately, this gives the wrong answer.

In the case where μ is minuscle, the answer is right. So the program works in case the group has enough minuscle weigts.

(2) $IC_{Hecke^\mu_x}$ is a mixed Hodge module in the sense of Saito. It has a Hodge filtration.

Taking the Rees module for the Hodge filtration gives a specialization which should be $I^{\mu,cl}$. Problem here: we don't know how to compute this. This is related to the structures of the fibers of $p^{\mu'}_x$ and $q^{\mu'}_x$ simultaneously for all μ' bigger than μ.

(3) A third natural candidate for $I^{\mu,cl}$ that can be written explicitly, is obtained by looking at a local duality (which is a theorem). This is the one for which the classical limit conjecture can be proved! Unfortunately, it is not known that this $I^{\mu,cl}$ is obtained as the classical limit of the correct (global) object. Therefore we can't derive anything for the original geometric Langlands conjecture.

1.30 Abelianization

The duality equivalence c s a manifestation of Cartier duality for commutative group stacks.

This requires us to interpret $\mathcal{H}iggs_G$ as a commutative group stack.

Remember that $\mathcal{H}iggs_G$ is the stack of (V, θ), where V is a G-principal bundle, $\theta \in H^0(C, ad(V) \otimes \Omega^1_C)$ (with an integrability condition which automatically satisfied for a surface C).

The abelianization of $\mathcal{H}iggs_G$ is the Hitchin map:

If $p_1, \ldots, p_r$ (homogeneous polynomials on $\mathfrak{g}$ of degree $d_1, \ldots, d_r$ which are $ad(G)$-invariant) generate the ring of invariant polynomials $\mathbb{C}[\mathfrak{g}]^G$, then we get a map

$$h\colon \mathcal{H}iggs_G \to H^0(S^{d_1}\Omega^1_C) \oplus \cdots H^0(S^{d_r}\Omega^1_C); (V, \theta) \mapsto (p_1(\theta), \ldots, p_r(\theta)).$$

(Here $r = rk(G) = \dim(T)$).

Then the stacky dimension $\dim_{\mathbb{C}} \mathcal{H}iggs_G = 2 \dim H^0(S^{d_1}\Omega^1_C) + \cdots$.

From now on we assume that G is semisimple, to avoid negative dimensions.

1.38 Example. $G = Gl_n(\mathbb{C})$. Then choose p_i by $\delta(\theta - \lambda \,\mathrm{id}) = \sum (-1)^i \lambda^i p_i(\theta)$ (An alternative choice could be $p_i(\theta) := \mathrm{tr}(\theta^i)$, which we don't like here); this gives a basis of invariant polynomials in this case. Then

$$h\colon \mathcal{H}iggs_{Gl_n} \to H^0(\Omega^1) \oplus H^0((\Omega^1)^{\otimes 2}) \oplus \cdots \oplus H^0((\Omega^1)^{\otimes n}).$$

In this case, the fiber of h over a point $(\alpha_1, \ldots, \alpha_n)$ is the compactified Jacobian of the curve

$$\bar{c}_\alpha := \{\sum (-1)^i \lambda^i \alpha_i = 0\} \subset Tot(\Omega^1_C)$$

where $\lambda \in p^* \Omega^1_C$ tautological.

It is easy to see that we get an n-sheated cover $\pi\colon \bar{C}_\alpha \to C$, called the spectral curve (it describes the spectrum of θ).

Hitchin proved: $h^{-1}(\alpha) = \overline{J^0(\bar{C}_\alpha)}$, the compactification of the degree zero component of the Jacobian.

Note that we achieved that $\mathcal{H}iggs_{Gl_n}$ now looks like a family of abelian groups, the Jacobians.

We now want to rewrite the Hitchin map without a choice of basis $p_1, \ldots, p_r$.

1.31 Hitchin map without choices

Given (V, θ) we will construct a cover $\tilde{C}_{V,\theta} \to C$ which is a W-Galois cover, called the cameral cover of (V, θ), where W is the Weyl group of G.

Then the Hitchin base of G will be B_G, the moduli space of such covers, the Hitchin map is

$$h\colon \mathcal{H}iggs_G \to B_G; (V, \theta) \mapsto (\tilde{C}_{V,\theta} \to C).$$

The fibers of H will be Pryms.

To define $\tilde{V}_{V,\theta} \to V$ look at $q\colon \mathfrak{g} \to \mathfrak{g}//G$. Then we get a map of fiber bundles

$$ad(V) = V \times_{ad} \mathfrak{g} \to V \times_{q \circ ad} (\mathfrak{g}//G) = V \times \mathfrak{g}//G$$

Similarly, we get a map

$$\nu\colon ad(V)\otimes\Omega^1_C \to \Omega^1_C\otimes(\mathfrak{g}//G) = (\mathfrak{g}\otimes\Omega^1_C)//G.$$

Now

$$(\mathfrak{g}\otimes\Omega^1_C)//G \cong \Omega^1_C\otimes(t/W)$$

so we get a W-Galois cover

$$\Omega^1\otimes t \to \Omega^1\otimes(t/W)$$

The (V,θ) cameral cover is the defined to be the pullback

$$\begin{array}{ccc} \tilde{C} & \longrightarrow & Tot(\Omega^1_C\otimes t) \\ \downarrow & & \downarrow \\ C & \xrightarrow{\nu\circ\theta} & Tot(\Omega^1_C\otimes t/W) \end{array}$$

Note: $B_G := \Gamma(C,(\Omega^1\otimes t)/W)$.

Talk 11: Global correspondence in $char = p$ (Heinloth)

1.32 Goal of the talk

Give a proof of a related result (Braverman) —the closest to the original result which really is known.

1.33 A global correspondence in Charakteristic p

Fix C/k with $char(k) = p$, $G = Gl_n$, k perfect (or finite).

1.39 Definition. D-modules in characteristic p.
If X/k is a smooth variety. Set

$$Dj_X := < O_X, T_X > /(\partial\partial' - \partial'\partial = [\partial,\partial'], \partial f - f\partial = \partial(f)).$$

1.40 Example. $X = A^1$, then

$$D_{A^1} = k < x, \partial_x > = \bigoplus_{i\geq 0} O_{A^1}\partial_x^k.$$

1.41 Proposition. *Properties:*

(1) $D_X \to End_k(O_x)$ *is* not *injective. E.g.:* ∂_x^p *acts as* 0 *on* O_{A^1}.

(2) D_X *has a large center: for* $X = A^1$ *the center of* $k < x, \partial_x >$ *is* $k[x^p, partial_x^p] = H^0(T^*X^{(1)}, O_T^*X)$*, and* $k[x,\partial_x^p]$ *is a large commutative subalgebra.*

Thus any D*-module on* A^1 *is a sheaf on* $(T^*A^1)^{(1)}$.

In general: $X^{(1)} := k\times_{Frob,k} X$*, this way* $f\mapsto f^p$ *becomes a* k*-linear morphism.*

Always: $T_x \hookrightarrow D_x; \partial_x \mapsto \partial_x^p$.

In a coordinate free way: ∂ any derivation, then ∂^p acts on O_X as a derivation which we call $partial^{[p]}$ (giving an element of degree 1 in D_X). Set

$$\partial \mapsto i(\partial) := \partial^p - \partial^{[p]} \in D_X.$$

1.42 Example.

$$(x\partial_x)^p = x^p \partial_x^p + \cdots + \underbrace{x\partial_x}_{=(x\partial_x)^{[p]}}$$

This way, D_X is a sheaf of algebra on $T^*X^{(1)}$; any D-module is a modlue for a finite dimensional algebra on ${}^*X^{(1)}$, and $(O_X^p, T_X) = Z(D_X)$.

1.43 Theorem. *D_X is an Azumaya algebra on $T^*X^{(1)}$ (i.e. flat locally isomorphic to End(E), E a vector bundle).*

Proof. Idea: consider pullback

$$\begin{array}{ccc} X \times_{X^{(1)}} T^*X^{(1)} & \xrightarrow{p} & T^*X^{(1)} \\ \downarrow & & \downarrow \\ X \to X^{(1)} & & \end{array}$$

Note that D_X is already a sheaf on $X \times_{X^{(1)}} T^*X^{(1)}$.

Then $p^*D_X \to \mathrm{End}(D_X)$ is an isomorphism (check stalkwise).

In particular, D_X defines a gerbe $\widetilde{T^*X^{(1)}}$ on $T^*X^{(1)}$, the moduli space of E such that $D_X \cong \mathrm{End}(E)$.

And $(D_X - mod) = Coh(\widetilde{T^*X^{(1)}})$. □

1.44 Proposition. *$\pi\colon X \to Y$ a (smooth) morphism of smooth varieties. Consider the coderivative $d\pi\colon \pi^*TY^{(1)} \to T^*X^{(1)}$ and $p\colon p^*T^*Y^{(1)} \to T^*Y^{(1)}$; then the Azumaya algebras p^*D_Y and $(d\pi)^*D_X$ are canonically equivalent.*

This means that we get a corresponding diagram of maps of gerbes if we apply $\tilde{\cdot}$ above.

1.45 Remark. π^*D_Y is a $p^*D_Y - (d\pi)^*D_X$-bimodule; this gives a Morita equivalence.

1.46 Corollary. *Using this, it is easy to define derived push-forward:*

$$\pi_*\colon D(D_X - mod) \to D(D_Y - mod); M \mapsto Rp_*((equiv)(d\pi)^*M).$$

Similarly, one gets an alternative definition of $\pi^!$ and π^*.

1.47 Remark. There is a version of the whole story if X, Y are stacks (but it is not obvious how to define it).

1.34 Application to $Bun_n := Bun_{Gl_n}$

1.48 Definition. Loc_n := moduli space of local systems on C, i.e. vector bundles with connection: (E, ∇) with $\nabla\colon E \to E \otimes \Omega^1$ connection.

1.49 Theorem. *There is an equivalence*

$$\Phi\colon D^b(D^0_{Bun_n}) \xrightarrow{\sim} D^b(Loc^0_n).$$

which maps skyscrapers to automorphics, namely

$$\Phi Hecke^i \Phi^{-1} \cong \Lambda^i E_{univ} \otimes -,$$

where E_{univ} is the universal local system on $C \times Loc_n$.

Moreover, $Hecke^i = Hecke$ of $\operatorname{diag}(t,\dots,t,1,\dots,1)$.

1.50 Definition. We need to define the open subsets Bun^0_n,...

Recall the Hitchin map

$$h\colon T^*Bun_n \to Hitch = \bigoplus_{i=1}^n H^0(C,\Omega^i); (E,\theta\colon E \to E\otimes\Omega^1) \mapsto \operatorname{tr}(\Lambda^i\theta).$$

Inside $Hitch$ we have the smooth locus $Hitch^0$ on which $\tilde{C} \xrightarrow{n:1} C$, $\tilde{C} \subset T^*_c C$ is smooth.

Let T^*Bun^0 be the pullback of $Hitch^0$.

Now:

$$D^b(D^0_{Bun_n}) := D(Coh(\widetilde{T^*Bun^0_n}^{(1)})).$$

Loc^0_n: If (E,∇) is a vecotr bundle with connection, then ∇^p is O_X-linear, therefore

$$\nabla^p\colon E \to Frob^*_c\Omega_c \otimes E$$

is O_X-linear. In particular, we can associate a spectral curve

$$Loc_n \to Hitch^{(1)}; (E,\nabla) \mapsto \operatorname{tr}(\Lambda^i\nabla^p).$$

Loc^0_n is the pullback of $(Hitch^{(1)})^0$.

Now recall: The fibers of the Hitchin map h over $Hitch^0$ are just $Pic_{\tilde{C}}$. this is obtained as follows: If $L \to \tilde{C}$ is a line bundle, then π_*L is a rank n-vector bundle on C ($\pi\colon \tilde{C} \to C$). The $O_{\tilde{C}}$-module structure on L corresponds to $\theta\colon E \to E \otimes \Omega^1_C$.

Recall: $\tilde{C}^{(1)} \subset T^*C^{(1)} \to C^{(1)}$ and Loc^0_n is the set of splittings of $D_C|_{\tilde{C}^{(1)}}$.

Rest: $\widetilde{T^*Bun^0_n} = \widetilde{Pic_{\tilde{C}}}$ is a group stack.

Then by dualizing we come from the group stack to splittings of the associated Azumaya, i.e. to D_C.

Finally, Φ comes from $Hom(\widetilde{Pic_C}, BG_m) \cong Loc^0_n$, and then we se a Fourier-Mukay-Transform.

Talk 12: Quantization of Fourier-Mukay transforms (Pantev)

Recall: For $\mathcal{H}iggs_G$ we defined

$$B_G = \Gamma(C, (\Omega^1_C \otimes t)/W),$$

the Hitchin base, also sometimes called $Hitch$.

We had the Hitchin map

$$h\colon \mathcal{H}iggs_G \to B_G; (V,\theta) \mapsto (\tilde{C}_{V,\theta} \to C)$$

with pull-back

$$\begin{array}{ccc} \tilde{C} & \longrightarrow & Tot(\Omega^1 \otimes t) \\ \big\downarrow & & \big\downarrow \\ C & \xrightarrow{\nu\circ\theta} & Tot((\Omega^1 \otimes t)/W) \end{array}$$

For $B_G \times C$ we get a universal cameral cover $\tilde{C} \to B_G \times C$ over B_G, using all sectoins of $Tot((\Omega^1 \otimes t)/W)$.

1.51 Theorem. *(Donagi-Gaitsgory 2000)*
The universal caemral cover $\tilde{C} \to B_G \times C$ together with G determine an affine abelian group scheme $\mathcal{T} \to B_G \times C$ so that

(1) $h\colon \mathcal{H}iggs_G \to B_G$ is a principal homogeneous stack over the group commutative stack $Tors_{\mathcal{T}}$.

(2) Every choice of a spin structure/theta characteristic $\xi \in I^{g-1}(C)$ s.t. $\xi^{\otimes 2} = \Omega^1_C$ deterines a section of h.

1.52 Remark. (1) The first part of the theorem holds in any dimension, the second part (which was actually proved by Hitchin) is true only for curves.

(2) The statement is not correct: one has to replace $\mathcal{H}iggs_G$ by regularized Higgs

$$\mathcal{H}iggs_G^{reg} := \{(V,\theta,\Gamma) \mid \Gamma \subset ad(V)\}$$

V G-principal on C, $\theta \in ad(V)\otimes\Omega^1$, Γ a sheaf of regular centralizers (fiber over each point a centralizer of a regular element), and s.t. $\theta in \Gamma \otimes \Omega^1_C$.

If we restrict to $\mathcal{H}iggs^0_G \to B^0_G$, where B^0_G is the moduli space of cameral covers with a simple Galois ramification (slightly smaller than Heinloth's open subset).

On this open part, every thing is regular (with a unique Γ), so there $\mathcal{H}iggs_G^{reg,0} = \mathcal{H}iggs^0_G$; from now on we work here.

(3) The Hitchin section of h associated to $\xi \in I^{g-1}(C)$ is quite subte, it is a geomtric analog of Kostant's section s in Lie group theory of the projection

$$\mathfrak{g} \to \mathfrak{g}//G$$

s always hits the regular part of $\mathfrak{g}$, it is built out of an Sl_2-triple; i.e. if $e, f, h \in \mathfrak{g}$ generate $Lie(Sl_2)$ and e, f are regular nilpotent in $\mathfrak{g}$, then $s(\mathfrak{g}//G) = f + Z(e)$.

(For Gl_n,) Hitchin's section works in the same spirit; he fixed one Higgs bundle: $\xi^{-n} \oplus \xi^{-n+1} \oplus \cdots \oplus O_C$, and then constructs varies θ as "canonical" easy matrix with right characteristic polynomial/Higgs image.

In most cases $\mathcal{T}|_{\{\tilde{C}\}\times C}$ is given by $\tilde{C} \xrightarrow{\pi} C$ as

$$\mathcal{T}|_{\{\tilde{C}\}\times C} := \pi_*(char(G) \otimes O^\times_{\tilde{C}})^W.$$

(this is for all types except type $G = SO(2n+1)$).

Note: $H^1(C, \mathcal{T}|_{\{\tilde{C}\}\times C})$ is a Prym variety for $\tilde{C}$. (If $\tilde{C} \to C$ was etale, then $H^1(C, \mathcal{T}|_{\{\tilde{C}\}\times C}) = (char(G) \otimes Pic(\tilde{C}))^W$).

1.53 Theorem. *(Donagi-Pantev)*
Given G and ${}^LG$, hen B_G and $B_{{}^LG}$ depend only on the Lie algebras $\mathfrak{g}$, ${}^L\mathfrak{g}$.

And there is a natural isomrophism $B_G \xrightarrow{\cong} B_{{}^LG}$ mapping B_G^0 to $B_{{}^LG}^0$.

Under this isomorphism, the restrictions of $\mathcal{H}iggs_G$ and $\mathcal{H}iggs_{{}^LG}$ become Cartier dual commutative group stacks.

1.54 Definition. Given a commutative group stack $H \to B$, the Cartier dual is defined as

$$H^D := Hom_{Group}(H, BO_B^\times) = Ext^1(H, O_B^\times).$$

(Ext in the derived category of complexes of abelian sheaves on B).

1.55 Remark. In general, this is not an anti-involution; moreover, there is an issue of defining this dual: it might happen that Ext^1 is not representable.

If H is filtered with graded pieces of the following types:

(1) family of abelian varieties

(2) sheaf of finitely generated abelian groups

(3) affine tori

(4) classifying stacks of the last two

then the duality preserves this subclass, and is a duality there. (Other pieces can also be allowed).

Stacks with the described pieces are called *Beilinson 1-motives.*

1.56 Example.
- If $A \to B$ is a family of abelian varieties over B, then $A^D \to B$ is the dual family of abelian varieties.
- If $H \to B$ is a family of affine tori, then $H^D = B(char(H))$, the classifying stack of the family of character lattices.
- If $\Lambda \to B$ is a family of free abelian groups, then $\Lambda^D = B(\mathrm{Hom}(\Lambda, O^X))$.

If H, H^D are dual and $(H^D)^D = H$ then we have a universal extension $P \in Ext^1(H \otimes_{\mathbb{Z}} H^D, O^\times)$ which can be viewed as a sheaf $P \in Coh(H \times_B H^D)$. For a family of abelian varieties, this is the Poincare sheaf.

We get a Fourier-Mukay equivalence (with this kernel P)

$$D^b(H, O) \to D^b(H^D, O).$$

There are short exact sequences (distinguished triangles) of commutative group stacks

$$0 \to \mathcal{H}iggs_G^0 \to \mathcal{H}iggs_G \to \pi_1(G) \to 0 \tag{1.57}$$

$$0 \to BZ(G) \to \mathcal{H}iggs_G^0 \to Higgs_G^0 \to 0 \tag{1.58}$$

$$0 \to BZ({}^LG) \to \mathcal{H}iggs^LG \to Higgs_{{}^LG} \to 0 \tag{1.59}$$

$$0 \to Higgs_{{}^LG}^0 \to Higgs_{{}^LG} \to \pi_1({}^LG) \to 0 \tag{1.60}$$

Under Cartier duality, (**??**) goes to (**??**) and (**??**) goes to (**??**).

By the duality, $Higgs_B^0$ and $Higgs^L G^0$ over B are supposed to be dual plain abelian varieties over B. The key is to establish this; this is some kind of Poincare duality with suitable coefficients. Then one uses the Hecke correspondences and the self-duality of the Picard stack of a curve; puts everything together to get the result.