Interplay between polynomial and transcendental entire dynamics

L. Rempe

Definitions and Motivation
Motivation
Basic definitions

Dynamic rays
The Eremenko-Lyubich class
Adam's favorite example

Conjugacies near infinity

Outlook: influences on polynomial dynamics

Department of Mathematical Sciences,
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Outline

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   - Adam's favorite example

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Definitions and Motivation
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- Basic definitions

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Conjugacies near infinity

Outlook: influences on polynomial dynamics
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Definitions and Motivation
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  - Directly applicable proofs/concepts.
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Transcendental dynamics

\[ f : \mathbb{C} \rightarrow \mathbb{C} \text{ entire, transcendental; } \]
\[ f^n := f \circ f \circ \cdots \circ f \]
\[ n \text{ times} \]

- **Fatou set** \( F(f) \): regular set.
  (Set of equicontinuity of the family \((f^n)\).)

- **Julia set**: ‘chaotic’ set.
  \( J(f) := \mathbb{C} \setminus F(f) \).

- **Escaping set**
  \[ I(f) := \{ z \in \mathbb{C} : \lim_{n \to \infty} f^n(z) = \infty \} \]
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Definitions and Motivation

Motivation

Basic definitions

Dynamic rays

The Eremenko-Lyubich class

Adam's favorite example

Conjugacies near infinity

Outlook: influences on polynomial dynamics

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Interplay between polynomial and transcendental entire dynamics

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Definitions and Motivation
Motivation
Basic definitions

Dynamic rays
The Eremenko-Lyubich class
Adam’s favorite example

Conjugacies near infinity

Outlook: influences on polynomial dynamics

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Definitions and Motivation
Motivation
Basic definitions

Dynamic rays
The Eremenko-Lyubich class
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Conjugacies near infinity
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$z \mapsto \exp(z) - 2$
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- $J$ is an uncountable union of Jordan arcs $g : [0, \infty) \to \mathbb{C}$ with $g(t) \to \infty$. We call $g((0, \infty))$ a ray and $g(0)$ the endpoint of that ray.

- The set $E$ of all endpoints has Hausdorff dimension 2, but the union $R$ of all rays has Hausdorff dimension 1.

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Rays of exponential maps

Curves of escaping points for exponential maps will belong to the Julia set, but nonetheless provide a natural generalization of external rays of polynomials.

We will give a proof of the existence of these curves for the simple case $f(z) = \exp(z) - 2$ on the board.

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**Question**

Let $f$ is a transcendental entire function and $z \in \mathcal{I}(f)$. Can $z$ be connected to infinity by a curve of escaping points of $f$?

(There is also a related question about the existence of such curves, due to Fatou.)

**Theorem 1 (Schleicher,Zimmer)**

If $f(z) = \exp(z) + \kappa$, $\kappa \in \mathbb{C}$, then the answer is yes.
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Definitions and Motivation
Motivation
Basic definitions
Dynamic rays
The Eremenko-Lyubich class
Adam's favorite example
Conjugacies near infinity
Outlook: influences on polynomial dynamics
Singular values

The set $\text{sing}(f^{-1})$ contains all values in which some branch of $f^{-1}$ cannot be defined. There are two types of such points:

- $c$ is a **critical value** if $c = f(w), \ f'(w) = 0$.
- $a$ is an **asymptotic value** if there is a curve $\gamma : (0, 1] \rightarrow \mathbb{C}$ such that $\lim_{t \to 0} |\gamma(t)| = \infty$ and $\lim_{t \to 0} f(\gamma(t)) = a$.

$S(f) := \text{sing}(f^{-1})$: singular values.

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The Eremenko-Lyubich class $\mathcal{B}$

A class of entire functions which is particularly interesting for our considerations was introduced by Eremenko and Lyubich in 1986:

$$\mathcal{B} := \{ f \text{ transcendental, entire : } \text{sing}(f^{-1}) \text{ is bounded}\}.$$ 

If $f \in \mathcal{B}$, then $I(f) \subset J(f)$.

In the following, we will mainly restrict ourselves to the Eremenko-Lyubich class.
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An entire function \( f \) has **finite order** if

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\log \log |f(z)| = O(\log |z|) \quad (z \to \infty).
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**Theorem 2** (Rottenfußer, Rückert, R., Schleicher)

Suppose that \( f \in B \) has finite order. Then every escaping point can be connected to infinity by a curve of escaping points.

(The theorem holds more generally for finite compositions of such functions.)

Barański independently proved the same result for hyperbolic \( f \) with a single completely invariant Fatou component.
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**Theorem 3 (RRRS)**

*There exists a hyperbolic function* $f \in B$ *such that every path-connected component of* $J(f)$ *is bounded.*

The function $f$ can be chosen such that

$$\log \log |f(z)| = (\log |z|)^{1+\varepsilon}.$$

There are even hyperbolic functions $f \in B$ such that every path-connected component of $J(f)$ is a point (but $J(f) \cup \infty$ is a compact connected set).
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Let $p$ be a polynomial of degree $\geq 2$ with a repelling fixed point at 0, and $\mu := p'(0)$.

Let $\psi$ be the (inverse of the) Kœnigs linearizing coordinate, defined near 0:

$$\psi(\mu \cdot z) = p(\psi(z)).$$

Let $\Psi : \mathbb{C} \to \mathbb{C}$ be the analytic extension of $\psi$ to the complex plane (using the functional relation).

($\Psi$ is called a Poincaré function.)
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An analog of Böttcher’s theorem

We say that $f, g \in B$ are quasiconformally equivalent near $\infty$ if there are quasiconformal maps $\phi, \psi : \mathbb{C} \to \mathbb{C}$ such that

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whenever $f(z)$ or $g(z)$ is sufficiently large.

**Theorem 4 (R., 2005)**

Suppose that $f$ and $g$ are quasiconformally equivalent near $\infty$. Then there is $R > 0$ and a quasiconformal map $\theta : \mathbb{C} \to \mathbb{C}$ such that $\theta \circ f = g \circ \theta$ on

$$J_R(f) := \{z \in \mathbb{C} : |f^n(z)| \geq R \text{ for all } n \geq 1\}.$$ 

Furthermore, the complex dilatation of $\theta$ on $I(f) \cap J_R(f)$ is zero.
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Definitions and Motivation

Motivation
Basic definitions

Dynamic rays

The Eremenko-Lyubich class
Adam's favorite example

Conjugacies near infinity

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It gives rise to the hope that escaping sets can also be used successfully to study the Julia sets of such wild functions. It also gives some explanation as to why the Julia sets of different explicit entire functions often look remarkably similar.
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Definitions and Motivation

Motivation

Basic definitions

Dynamic rays

The Eremenko-Lyubich class

Adam's favorite example

Conjugacies near infinity

Outlook: influences on polynomial dynamics

Similarities in Julia sets

\[ z \mapsto 2(\exp(z) - 1) \quad z \mapsto \lambda \sinh(z) \quad z \mapsto (z+1) \exp(z)-1 \]
Rigidity results

The conjugacy $\theta$ from the preceding theorem is “essentially unique” (up to countably many choices).

**Theorem 5 (R., 2005)**

*Let $f \in B$. Then there are no invariant line fields on $l(f)$.*

This and other rigidity results on the escaping set are used in work with van Strien on rigidity and density of hyperbolicity in families of real transcendental entire functions.
Rigidity results

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Transcendental polynomials?

Aspects of transcendental dynamics have started to become apparent in renormalization phenomena (i.e. when the degree gets large).

- Features of exponential dynamics appearing in parabolic renormalization (Shishikura).
- Trancendental aspects of measurable dynamics (Urbanski-Zdunik) appearing in work of Avila-Lyubich on Feigenbaum quadratics.
Cantor bouquets and hedgehogs

The proofs of the above results on existence of curves and conjugacy can be adapted to deal with model hedgehogs and (in the first case) actual hedgehogs. (This is ongoing work by Shishikura / Buff-Chéritat-Inou-R.)
Interplay between polynomial and transcendental entire dynamics

L. Rempe

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