Applications of near-parabolic renormalization

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Trends and Developments in Complex Dynamics
Oberwolfach
October 24, 2008
Plan

Want to understand the dynamics of a quadratic polynomial $f$ when it has an irrational indifferent fixed point of high type

$$f(0) = 0, \quad f'(0) = e^{2\pi i \alpha}, \quad \alpha = \frac{1}{a_1 \pm \frac{1}{a_2 \pm \frac{1}{\ldots}}} \quad (a_i \in \mathbb{N} \text{ large})$$

Near-parabolic renormalization $f \mapsto Rf$

Inou-S. “uniform lower bound on the nonlinearity of $R^n f$”

Reconstructing $f$ from $Rf, R^2 f, \ldots$

Rigidity, hairs in hedgehogs
Return map and renormalization

\[ \mathcal{R} f = ( \text{first return map of } f ) \text{ after rescaling} \]
\[ = g \circ f^k \circ g^{-1} \quad (\text{if return time } \equiv k) \]

**Renormalization**

- high iterates of \( f \) \leftrightarrow fewer iterates of \( \mathcal{R} f \)
- fine orbit structure for \( f \) \leftrightarrow large scale orbit structure for \( \mathcal{R} f \)

Successive construction of \( \mathcal{R} f, \mathcal{R}^2 f, \ldots \), helps to understand the dynamics of \( f \) (orbits, invariant sets, rigidity, bifurcation, \ldots)

Our case:

- return time and first return map may be discontinuous
- glue the boundary curves
Understanding irrationally indifferent fixed points (or their hedgehogs) via renormalization

Cylinder/Near-parabolic renormalization

\[ \mathbb{C}^* = \mathbb{C} \setminus \{0\} \]

The first return map \( \mathcal{R}f \) can be defined when \( f(z) = e^{2\pi i \alpha}z + \ldots \) is a small perturbation of \( z + a_2z^2 + \ldots \) (\( a_2 \neq 0 \)) and \(|\arg \alpha| < \pi/4\).
Douady-Hubbard theory of parabolic implosion

\[ f_0(z) = z + a_2 z^2 + \ldots \quad (a_2 \neq 0) \]

\[ f'(0) = e^{2\pi i \alpha}, \quad \alpha \text{ small} \quad |\arg \alpha| < \frac{\pi}{4} \]

\[ E_{f_0} \]

\[ E_f \]

\[ \tilde{R} f = \chi_f \circ E_f \]

first return map

\( E_f \) depends continuously on \( f \)

(after a suitable normalization)
Basic checkerboard pattern for parabolic map

\[ f_0(z) = z + z^2 \]

Color points according to the integer part of Fatou coordinate and the sign of imaginary part.

Observation: if a parabolic basin contains only one simple critical point, then the checkerboard pattern (and the dynamics) in the basin is the same.
Basic checkerboard pattern for parabolic map 2

\[ f_0(z) = z + z^2 + \ldots \]

change coordinate  
\[ w = -1/z \]

\[ F_0(w) = w + 1 + \ldots \]

\[ \infty = \text{parabolic fixed pt} \]

Fatou coord. conjugates  
\[ F_0 \text{ to } T : w \mapsto w + 1 \]
Horn map $E_{f_0}$ seen via checkerboard pattern

repelling Fatou coordinate

$\infty = \text{parabolic fixed pt}$

attracting Fatou coordinate

$\infty \text{ to } 1 \text{ branched cover with only one critical value on the cylinder}$
Checkerboard after a perturbation

\[ f'(0) = e^{2\pi i \alpha}, \quad \alpha \text{ small} \quad |\arg \alpha| < \frac{\pi}{4} \]

Checkerboard pattern can be defined by Fatou coordinate of perturbed attracting cylinder

But some portion of the pattern will be lost because of recurrent orbits, crit. pts etc. (conflicting colors, more ramifications, ...)

\[ E_{f_0} \]
Truncated checkerboard pattern

Truncated portions $D_{-n}, D'_{-n}, D''_{-n}$ are likely to remain after a perturbation.
Truncated pattern induces a cubic-like covering

\[ P(z) = z(1 + z)^2 \]

\( V \subset \subset V' \)

can be interpreted as a covering pattern
Inou-S. defined a class $\mathcal{F}_1$ of maps with the truncated checkerboard pattern or cubic-like covering property

$$\mathcal{F}_1 \ni f = P \circ \varphi^{-1}, \text{ where } \varphi : V \to \mathbb{C} \text{ univalent }$$

with $f(0) = 0$, $f'(0) = 1$, $f$ has a unique critical point

If $f = z + z^2$ or $f \in \mathcal{F}_1$ and $\alpha$ is of sufficiently high type, then $\mathcal{R}^n(e^{2\pi i \alpha} f) \in \mathcal{F}_1$ ($n = 1, 2, \ldots$).

**Idea of proof** (first work with parabolic, then perturb to near-parabolic)

1. $\mathcal{F}_1 \ni f$
2. trunc. pattern & estimates & trunc. pattern
3. or covering property & on $f$ and Fatou coord. & in Fatou coordinate
4. analytic estimates via theory of univalent functions (including Golusin inequality) and numerical estimates involving values of elementary functions

Moreover if $f, g \in \mathcal{F}_1$ and $\alpha$ of suff. high type, then $d(\mathcal{R}^n(e^{2\pi i \alpha} f), \mathcal{R}^n(e^{2\pi i \alpha} g)) \to 0$ exponentially fast ($n \to \infty$).

proved via Teichmüller theory
Rigidity problem or “smoothness” of conjugacy

Differentiable functions

In small scale...
homeomorphism: can do anything
quasi-symmetric, quasi-conformal: bounded ratio

asymptotically conformal: ratio $\to 1$

$C^{1+\alpha}$ : ratio $\to 1$ “fast”

For conjugacies between dynamical systems...
compare orbits
to see details, need to iterate many times
Renormalization and Rigidity (an oversimplified view)

Suppose $f$ and $\tilde{f}$ have “the same combinatorial type” and admit successive construction of renormalizations.

$f_0 = f$
$f_1 = Rf_0$
$f_2 = Rf_1$
$f_3 = Rf_2$

\[ \tilde{f}_0 = \tilde{f} \]
\[ \tilde{f}_1 = R\tilde{f}_0 \]
\[ \tilde{f}_2 = R\tilde{f}_1 \]
\[ \tilde{f}_3 = R\tilde{f}_2 \]

\{h_n\} “bounded” \[ \rightarrow \] $f$ and $\tilde{f}$ quasi-conformally conjugate

$d(f_n, \tilde{f}_n) \rightarrow 0$ \[ \rightarrow \] $h_n \rightarrow \text{linear}$ \[ \rightarrow \] conjugacy is asymptotically conformal or smooth, etc.
Various Renormalizations

Feigenbaum

Circle map

Near-parabolic

proper subintervals
-> Cantor set

partition of interval

covering by sector or croissant-like domains

gluing/identification needed to define the renormalization
Reconstructing $f$ from $Rf$

want to get an information (e.g. approx. conjugacy) on $f$ from that of $Rf$
a difficulty comes from gluing (of overlapping domains) in the construction of near-parabolic renormalization

Compare: Yoccoz renormalization for univalent germs

He was able to switch the roles of the dynamics and the gluing ------ (“non-linear continued fractions”)
Not possible with our $f$ which has a critical point and non-univalent

How did the dynamics of $Rf$ appear within the dynamics of $f$
(or in the dynamics of earlier generations)?
A Zen question:

What is your “existence” before your parents were born?
Need to understand what the dynamics $f$ really is

$F_{can}$ on $\Omega_{can}$
canonical map trunc. pattern

$\psi_f$
gluing which commutes with $F_{can}$

Inou-S.: This region is enough for the construction of next renormalization $\mathcal{R}f$
Relationship between $f$ and $g = Rf$

$\psi_f$ and $\psi_g = \psi Rf$

Fatou coordinate of $f$

$dynamical plane of g$

$z \mapsto e^{2\pi iz}$

$F_{can}$
How did the dynamics of $g = \mathcal{R}f$ appear within the dynamics of $f$?

**gluing:**

1. well-defined after gluing
2. return map is $F_{can}$ modulo $\psi_g$
3. this picture embeds into $f$
Theorem 1 (structure): For \( f = e^{2\pi i \alpha} h \) (\( h \in \mathcal{F}_1 \) or \( h = z + z^2 \), \( \alpha \) high type), there exist domains \( \Omega_f^{(0)} \supset \Omega_f^{(1)} \supset \Omega_f^{(2)} \supset \ldots \) which correspond to renormalizations \( f, \mathcal{R}f, \mathcal{R}^2f, \ldots \), each \( \Omega_f^{(k)} \) is a union of open sets \( \Omega_{n_1,n_2,\ldots,n_k}^{(k)} \) and \( \{0\} \), which is isomorphic to the truncated checkerboard pattern cut off at certain width, and \( \Omega_{n_1,n_2,\ldots,n_{k-1}}^{(k-1)} \supset \Omega_{n_1,n_2,\ldots,n_k}^{(k)} \). The intersection \( \Omega_f^{(\infty)} = \cap_{k=0}^{\infty} \Omega_f^{(k)} \) is a closed set containing 0 and the forward critical orbit. \( f \) is injective on this set.

Remark: \( \Omega_f^{(k)} \) and \( \Omega_{n_1,n_2,\ldots,n_k}^{(k)} \) are not canonical. We allow two sequences \( n_1, \ldots, n_k \) and \( n'_1, \ldots, n'_k \) to represent more or less the same regions \( \Omega_{n_1,n_2,\ldots,n_k}^{(k)} \) and \( \Omega_{n'_1,n'_2,\ldots,n'_k}^{(k)} \) due to “overlapping.”
Applications

Theorem 2 (rigidity): Let $f$ and $\hat{f}$ be two maps as in Theorem 1, then there exists a quasiconformal map from a neighborhood of $\Omega_f^{(\infty)}$ to a neighborhood of $\Omega_{\hat{f}}^{(\infty)}$, which conjugates $f$ to $\hat{f}$ on $\Omega_f^{(\infty)}$, asymptotically conformal at $\Omega_f^{(\infty)}$ and conformally differentiable at the critical orbit.

Theorem 3 (hairs): Let $f$ and $\Omega^{(k)}$, $\Omega_{n_1,n_2,\ldots,n_k}$ be as in Theorem 1. For an “allowable” sequence $n_1, n_2, \ldots$, the intersection $\bigcap_{k=1}^{\infty} \Omega^{(k)}_{n_1,n_2,\ldots,n_k}$ is either empty or an arc tending to 0 (closed arc when 0 is added). The set of these arcs are cyclically permuted by $f$. In particular, there is an arc in $\Omega_f^{(\infty)}$ from the critical point to 0.

by Ricardo Perez-Marco

by Lasse Rempe
Danke Schön!
Thank you
The distance moved within $\Omega_f$ is uniformly bounded.
Successive homotopies

If the cut-off $\to \infty \implies \Omega_f \cap \Omega'_{f,n} \cap \Omega''_{f,n,m} \cap \ldots$ is empty

If the cut-off stays bounded $\implies \Omega_f \cap \Omega'_{f,n} \cap \Omega''_{f,n,m} \cap \ldots$ is an arc

“phase-space Brjuno-Yoccoz function”