INFINITESIMAL IDEAL SYSTEMS AND THE ATIYAH CLASS

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Abstract. This short note gives a geometric interpretation of the Atiyah class of a Lie pair. It proves that it vanishes if the subalgebroid is the kernel of a fibration of Lie algebroids. In other words, the Atiyah class of a Lie pair vanishes if the subalgebroid is the fiber of an ideal system in the Lie algebroid. In order to prove this, a new characterisation of the ideal condition for an infinitesimal ideal system is found.

Keywords: Lie algebroids, Lie pairs, ideals, infinitesimal ideal systems, representations, linear connections, fibrations of Lie algebroids, Atiyah class.

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Contents

1. Introduction

Let $A \to M$ be a Lie algebroid and $J \subseteq A$ a subalgebroid. The pair (A, J) is called a *Lie pair* [4]. Lie pairs arise e.g. in Lie theory, complex geometry and foliation theory. For instance, a smooth foliation F in the tangent algebroid TM of a smooth manifold M defines a Lie pair (TM, F).

Given a Lie pair, the quotient A/J always carries a flat representation of J, the *Bott connection* [1, 2]

$$\nabla^J : \Gamma(J) \times \Gamma(A/J) \to \Gamma(A/J), \quad \nabla^J_j \bar{a} = \overline{[j, a]}$$

for $j \in \Gamma(J)$ and $a \in \Gamma(A)$. An *extension* of the Bott connection ∇^J is a linear A-connection on A, that preserves J (in the second argument) and induces so a quotient connection $\Gamma(A) \times \Gamma(A/J) \to \Gamma(A/J)$ that restricts to ∇^J in the first argument. The *Atiyah class* [4] of the Lie pair (A, J) is a cohomology class $\alpha_J \in H^1(J, \mathrm{Hom}(A/J, \mathrm{End}(A/J)))$ that vanishes if and only if ∇^J has a extension ∇ that is *transverse* to it: For $a, b \in \Gamma(A)$ two ∇^J-flat sections, $\nabla_a b$ is again ∇^J-flat – see also [12, 11] for the more classical setting of foliations. However, from a more elementary geometric point of view, it is not completely clear what the existence of a transverse connection means for a Lie pair. In the case of a Lie pair (TM, F) as above, the Atiyah class vanishes for instance if F is simple, i.e. if the space of leaves M/F

is a smooth manifold. A transverse connection is in this case a projectable connection – such a connection always exists for a surjective submersion $M \to M'$ with connected fibers.

This short note generalises this fact to give a geometric interpretation of the Atiyah class of a Lie pair (A, J). More precisely, if there exists a fibration of Lie algebroids $A \to A'$ over a smooth surjective submersion $M \to M'$ with connected fibers, such that J is the kernel of the fibration, then its Atiyah class vanishes.

Such a fibration of Lie algebroids is always encoded in an infinitesimal ideal structure on the Lie pair (A, J). Such an *infinitesimal ideal system* is a triple (F_M, J, ∇), where $F_M \subseteq TM$ is an involutive subbundle such that $\rho(J) \subseteq F_M$ and ∇ is a flat F_M-connection on A/J with the following properties:

(iis1) If $a \in \Gamma(A)$ is ∇-parallel, then $[a, j] \in \Gamma(J)$ for all $j \in \Gamma(J)$.
(iis2) If $a, b \in \Gamma(A)$ are ∇-parallel, then $[a, b]$ is also ∇-parallel.
(iis3) If $a \in \Gamma(A)$ is ∇-parallel, then $\rho(a)$ is ∇^{F_M}-parallel, where $\nabla^{F_M} : \Gamma(F_M) \times \Gamma(TM/F_M) \to \Gamma(TM/F_M)$ is the Bott connection.

The definition is due to [9], but the structure already appeared in [7] in geometric quantisation as the infinitesimal version of polarisations on groupoids, and the case $F_M = TM$ has been studied independently in [5] in relation with a modern approach to Cartan's work on pseudogroups.

For simplicity, *ideal* is here short for *infinitesimal ideal system*. Ideals were called "IM-foliations" in an early version of [9] and in [13], in analogy to the "IM-2-forms" of [3]. The new terminology is more adequate, since an infinitesimal ideal system is an infinitesimal version of the *ideal systems* in [10, 8]. Let A be a Lie algebroid over M. An ideal system of A is a triple (R, J, θ) where $J \subseteq A$ is a Lie subalgebroid, $R = R(f)$ is a closed, embedded, wide, Lie subgroupoid of the pair groupoid $M \times M \rightrightarrows M$ corresponding to a surjective submersion $f : M \to \bar{M}$:

$$R(f) = \{(p, q) \in M \times M \mid f(p) = f(q)\},$$

and where θ is a linear action of $R(f)$ on the vector bundle $A/J \to M$, such that:

(1) $\rho(J) \subseteq T^f M$;
(2) if $a, b \in \Gamma(A)$ are θ-stable;

$$\theta((p, q), \bar{a}(q)) = \bar{a}(p) \text{ for all } (p, q) \in R$$

(and similarly for b), then $[a, b]$ is θ-stable;
(3) if $a \in \Gamma(A)$ is θ-stable and $j \in \Gamma(J)$, then $[a, j] \in \Gamma(J)$;
(4) the induced map $\bar{\rho} : A/J \to TM/T^f M$ is $R(f)$-equivariant with respect to θ and the canonical action θ_0 of $R(f)$ on $TM/T^f M$.

Here, the canonical action θ_0 is the action transported from the pullback bundle by the isomorphism $TM/T^f M \simeq f!T\bar{M}$.

An infinitesimal ideal system (F_M, J, ∇) with F_M simple and ∇ with trivial holonomy always integrates to an ideal system: $f : M \to M/F_M$ is the surjective submersion, so $F_M = T^f M$, and θ is given by parallel transport along the leaves of F_M, see [9]. The conditions on F_M and ∇ are equivalent to the quotient $(A/J)/\nabla \simeq (A/J)/\theta \to M/F_M$ being a vector bundle, which inherits then a Lie algebroid structure 'for free' such that the quotient map is a fibration of Lie algebroids [9]. Ideal systems were in fact defined as the kernels of fibrations of Lie algebroids. If such an infinitesimal ideal system (F_M, J, ∇) exists, then A *is reducible by the fiber* J, since J becomes the kernel of a fibration of Lie algebroids.

In general, a Lie pair will not carry an infinitesimal ideal structure, so the Lie pairs (A, J) which do must be considered special. The author's general goal is to find obstructions to Lie pairs carrying infinitesimal ideal system structures.

This paper solves a slightly different problem and proves that if a Lie algebroid A is reducible by the fiber J, then the Atiyah class of the Lie pair (A, J) is zero.

Theorem 1.1. *Let A be a Lie algebroid on M and J a subalgebroid. If there exists an infinitesimal ideal system (F_M, J, ∇^i) in A, such that the quotient $(A/J)/\nabla^i \to M/F_M$ exists and is smooth, then the Atiyah class of the Lie pair (A, J) vanishes.*

Since the infinitesimal ideal systems (F_M, J, ∇^i) such that the quotient Lie algebroid $(A/J)/\nabla^i \to M/F_M$ exists are exactly the infinitesimal ideal systems that integrate to ideal systems [9], the theorem can be reformulated as follows.

Theorem 1.2. *Let A be a Lie algebroid on M and J a subalgebroid. If there exists an ideal system (R, J, θ) in A, then the Atiyah class of the Lie pair (A, J) vanishes.*

Outline of the paper. Section 2.1 considers fibrations of vector bundles and infinitesimal ideal systems in vector bundles (i.e. Abelian Lie algebroids). It shows that the Atiyah class of such an infinitesimal ideal system vanishes if the infinitesimal ideal system defines a fibration of the vector bundle. Section 2.2 proves a new formulation of the ideal condition (iis1) in the definition of an infinitesimal ideal system. It further explains fibrations of Lie algebroids versus infinitesimal ideal systems [9], and defines the Atiyah class of an infinitesimal ideal system. Section 3 proves the main theorem (Theorem 1.1).

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2. Preliminaries

2.1. **Flat connections and fibrations of vector bundles.** Consider a fibration of vector bundles

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E' \\ {\scriptstyle q_E}\downarrow & & \downarrow{\scriptstyle q_{E'}} \\ M & \xrightarrow[\varphi_0]{} & M' \end{array}$$

That is, the map φ_0 is a surjective submersion with connected fibres and $\varphi^! \colon E \to \varphi_0^! E'$ is a surjective vector bundle morphism over the identity on M.

Proposition 2.1. *Let $\varphi\colon E \to E'$ be a fibration of vector bundles over $\varphi_0\colon M \to M'$. For any linear connection $\nabla'\colon \mathfrak{X}(M') \times \Gamma(E') \to \Gamma(E')$ there exists a linear connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ such that if $X \sim_{\varphi_0} X'$ and $e \sim_\varphi e'$ then $\nabla_X e \sim_\varphi \nabla'_{X'} e'$.*

Proof. Let $K \subseteq E$ be the kernel of the map φ; K is a subbundle of E. The inclusion of K in E is denoted by ι. Set $F_M = T^{\varphi_0} M$, an involutive and simple subbundle of TM. Choose a splitting $j\colon \varphi_0^! E' \to E$ of the short exact sequence $0 \to K \to E \to \varphi_0^! E' \to 0$ of vector bundles over M. Recall that each smooth section e' of $E' \to M'$ defines a smooth section $e'_!$ of $\varphi_0^! E' \to M$ by $e'_!(m) = (m, e'(\varphi_0(m)))$ for all $m \in M$, and that the space of sections of $\varphi_0^! E'$ is generated as a $C^\infty(M)$-module by these sections. Further, $\varphi \circ j \circ e'_! = e' \circ \varphi_0$, or in other words $(j \circ e'_!) \sim_\varphi e'$.

The connection ∇' defines as follows a connection $\varphi^! \nabla' \colon \mathfrak{X}(M) \times \Gamma(\varphi_0^! E') \to \Gamma(\varphi_0^! E')$: since ∇^{F_M} is flat, it is sufficient to define $\varphi^! \nabla'$ on vector fields $X \in \mathfrak{X}(M)$ that are φ_0-projectable to vector fields on M', i.e. such that the class of X in $\Gamma(TM/F_M)$ is ∇^{F_M}-flat. Set

$$(\varphi^! \nabla')_X e'_! = (\nabla'_{X'} e')_!$$

where $X' \in \mathfrak{X}(M')$ is the projection of X; $X \sim_{\varphi_0} X'$. In particular, $(\varphi^! \nabla')_X e'_! = 0$ for $X \in \Gamma(F_M)$.

Choose any connection $\tilde{\nabla}\colon \mathfrak{X}(M) \times \Gamma(K) \to \Gamma(K)$ and define a connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ by

$$\nabla_X(\iota(k) + j(e')) = \iota(\tilde{\nabla}_X k) + j((\varphi^! \nabla')_X e')$$

for all $e' \in \Gamma(\varphi_0^! E')$ and all $k \in \Gamma(K)$. It is now easy to check that if $X \sim_{\varphi_0} X'$ and $e \sim_\varphi e'$ then $\nabla_X e \sim_\varphi \nabla'_{X'} e'$. □

Consider here also a fibration of vector bundles $\varphi\colon E \to E'$ over $\varphi_0\colon M \to M'$ and set $K := \ker(\varphi) \subseteq E$ and $F_M := T^{\varphi_0}M \subseteq TM$, an involutive subbundle. Here and in the following, $\bar{e}$ always denotes the projection of $e \in \Gamma(E)$ to a section of $\Gamma(E/K)$. Define a flat connection $\nabla^\varphi\colon \Gamma(F_M) \times \Gamma(E/K) \to \Gamma(E/K)$ by setting $\nabla^\varphi_X \bar{e} = 0$ for all sections $e \in \Gamma(E)$ that are φ-related to some section $e' \in \Gamma(E')$, i.e. such that $\varphi \circ e = e' \circ \varphi_0$; see [9].

Assume that a linear connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ projects to a connection $\nabla'\colon \mathfrak{X}(M') \times \Gamma(E') \to \Gamma(E')$. Then for $X \sim_{\varphi_0} X'$ and $e \sim_\varphi e'$, the sections $\nabla_X e$ and $\nabla'_{X'} e'$ are φ-related; $\nabla_X e \sim_\varphi \nabla'_{X'} e'$. In other words, if $\bar{X}$ is a flat vector field for the F_M-Bott connection on TM/F_M and $\bar{e}$ is ∇^φ-flat, then $\overline{\nabla_X e}$ must be ∇^φ-flat and project to $\nabla'_{X'} e'$.

In particular if $X \sim_{\varphi_0} X'$ and $e \sim_\varphi 0$, i.e. $e \in \Gamma(K)$, then again $\nabla_X e$ must be a section of K. This shows that ∇ restricts to a connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(K) \to \Gamma(K)$ and defines so a quotient connection $\bar{\nabla}\colon \mathfrak{X}(M) \times \Gamma(E/K) \to \Gamma(E/K)$.

Now if $X \in \Gamma(F_M)$, then $X \sim_{\varphi_0} 0$ and so $\nabla_X e \sim_\varphi 0$ for all ∇^φ-flat[1] sections $e \in \Gamma(E)$. This means that $\nabla_X e \in \Gamma(K)$ for those sections. Then $\bar{\nabla}_X \bar{e} = 0 = \nabla^\varphi_X \bar{e}$, and since the ∇^φ-flat sections of E/K generate all sections of E/K as a $C^\infty(M)$-module, this implies that the restriction of $\bar{\nabla}$ to $\Gamma(F_M) \times \Gamma(E/K) \to \Gamma(E/K)$ equals the connection ∇^φ. This yields the following proposition.

Proposition 2.2. *Let $\varphi\colon E \to E'$ a fibration of vector bundles over $\varphi_0\colon M \to M'$ and consider ∇^φ as above. Take a linear connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ that projects to a linear connection $\nabla'\colon \mathfrak{X}(M') \times \Gamma(E') \to \Gamma(E')$ as in Proposition 2.1. Then ∇ is an extension of ∇^φ in the sense of the following definition.*

Definition 2.3. *Let $E \to M$ be a vector bundle, $K \subseteq E$ a subbundle and $F_M \subseteq TM$ an involutive subbundle, with a flat connection*

$$\nabla^p\colon \Gamma(F_M) \times \Gamma(E/K) \to \Gamma(E/K).$$

Then a connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ is an extension of ∇^p *if*

1. *∇ restricts to a connection $\mathfrak{X}(M) \times \Gamma(K) \to \Gamma(K)$ and*
2. *the induced (quotient) connection $\bar{\nabla}\colon \mathfrak{X}(M) \times \Gamma(E/K) \to \Gamma(E/K)$ satisfies $\bar{\nabla}_Y = \nabla^p_Y$ for all $Y \in \Gamma(F_M)$.*

In general, ∇^p as in Definition 2.3 does not need to come from a vector bundle fibration. If F_M is simple and ∇^p has no holonomy, then it defines a fibration of vector bundles [9]:

$$\begin{array}{ccc} E & \longrightarrow & (E/K)/\nabla^p \\ {\scriptstyle q_E}\downarrow & & \downarrow{\scriptstyle [q_E]} \\ M & \longrightarrow & M/F_M \end{array}$$

The space $(E/K)/\nabla^p$ is the quotient of E/K by the equivalence relation generated by parallel transport along paths in the leaves of F_M. That is, $\bar{e}_m$ and $\bar{e}'_{m'}$ are equivalent if there exists a ∇^p-flat section $\bar{e}$ of E/K with $\bar{e}(m) = \bar{e}_m$ and $\bar{e}(m') = \bar{e}'_{m'}$. The triple (F_M, K, ∇^p) is in fact an infinitesimal ideal system in the (Abelian) Lie algebroid $(E \to M, \rho = 0, [\cdot\,,\cdot] = 0)$.

The upper index p in Definition 2.3 means to recall that in general, ∇^p is what can be seen as an infinitesimal vector bundle "projection" – i.e. the right candidate for defining a fibration of vector bundles – even if M/F_M is not a smooth manifold, or ∇^p has non trivial holonomy. The remainder of this section shows that if ∇^p is defined by a fibration, then its *Atiyah class* must vanish. The first step towards the construction of the Atiyah class of ∇^p is the choice of an extension of ∇^p. This is possible due to the following lemma.

[1]By abuse of notation, say that $e \in \Gamma(E)$ is ∇^φ-flat if $\bar{e} \in \Gamma(E/K)$ is ∇^φ-flat.

Lemma 2.4. *Let $F_M \subseteq TM$ be an involutive subbundle of the tangent of a smooth manifold M. Let $E \to M$ be a smooth subbundle, and $K \subseteq E$ a subbundle. There always exists an extension of a linear connection*

$$\nabla^p\colon \Gamma(F_M) \times \Gamma(E/K) \to \Gamma(E/K).$$

Two extensions differ by a form $\phi \in \Gamma(T^*M \otimes E^* \otimes E)$ that induces a well-defined form $\bar\phi \in \Gamma(\mathrm{Hom}(TM/F_M, \mathrm{End}(E/K)))$ by $\bar\phi(\bar X)(\bar e) = \overline{\phi(X, e)}$ for all $X \in \mathfrak{X}(M)$, $e \in \Gamma(E)$.

Proof. Choose any TM-connection ∇^K on K. Choose any smooth complement H of F_M in TM and an arbitrary connection $\nabla^H\colon \Gamma(H) \times \Gamma(E/K) \to \Gamma(E/K)$. Then define $\bar\nabla\colon \mathfrak{X}(M) \times \Gamma(E/K) \to \Gamma(E/K)$ by

$$\bar\nabla_{X+Y}\bar e = \nabla^p_X \bar e + \nabla^H_Y \bar e$$

for $X \in \Gamma(F_M)$, $Y \in \Gamma(H)$ and $\bar e \in \Gamma(E/K)$. A choice of splitting $\sigma\colon E/K \to E$ of the short exact sequence

$$0 \to K \to E \overset{\mathrm{pr}}{\to} E/K \to 0$$

gives an isomorphism $E \simeq K \oplus E/K$, $E \ni e \simeq (e - \sigma(\bar e)) + \bar e \in K \oplus E/K$. Set $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, $\nabla_X e = \nabla^K_X(e - \sigma(\bar e)) + \sigma(\bar\nabla_X \bar e)$ for $X \in \mathfrak{X}(M)$ and $e \in \Gamma(E)$. The linear connection ∇ is an extension of ∇^p.

Assume that $\nabla'\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ is a second extension of ∇^p. Then as usual $\nabla - \nabla' =: \phi$ is a section of $T^*M \otimes E^* \otimes E$. By definition of an extension, $\phi(X, k) = \nabla_X k - \nabla'_X k \in \Gamma(K)$ for all $X \in \mathfrak{X}(M)$ and $k \in \Gamma(K)$. Hence there is an induced $\tilde\phi \in \Gamma(T^*M \otimes (E/K)^* \otimes E/K)$, $\tilde\phi(X, \bar e) = \overline{\phi(X, e)} = \overline{\nabla_X e - \nabla'_X e}$ for all $X \in \mathfrak{X}(M)$ and $e \in \Gamma(E)$. Now $\overline{\nabla_X e - \nabla'_X e} = \nabla^p_X \bar e - \nabla^p_X \bar e = 0$ for all $X \in \Gamma(F_M)$ and $e \in \Gamma(E)$ yields the existence of $\bar\phi$ as in the statement. □

Return to the situation of Proposition 2.1. Choose again $X \in \mathfrak{X}(M)$ a ∇^{F_M}-flat vector field, i.e. a vector field such that $X \sim_{\varphi_0} X'$ for some $X' \in \mathfrak{X}(M')$; see [9]. Choose also Y a section of F_M and $\bar e \in \Gamma(E/K)$ a ∇^φ-flat section, i.e. such that $e \sim_\varphi e'$ for some $e' \in \Gamma(E')$.

Then on the one hand $[Y, X] \in \Gamma(F_M)$ by definition of the Bott connection, and so

$$\bar\nabla_X \nabla^\varphi_Y \bar e + \bar\nabla_{[Y,X]}\bar e = \bar\nabla_X(0) + \nabla^\varphi_{[Y,X]}\bar e = 0.$$

On the other hand,

$$\nabla^\varphi_Y(\bar\nabla_X \bar e) = \nabla^\varphi_Y \overline{\nabla_X e} = 0$$

since ∇ projects under φ to ∇', so $\nabla_X e \sim_\varphi \nabla'_{X'} e'$ and $\overline{\nabla_X e}$ is therefore necessarily ∇^φ-flat. Hence, the condition that ∇ projects to a connection ∇' reads as follows.

Lemma 2.5. *Let $\varphi\colon E \to E'$ be a fibration of vector bundles over $\varphi_0\colon M \to M'$. If a connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ projects to a connection $\nabla'\colon \mathfrak{X}(M') \times \Gamma(E') \to \Gamma(E')$, then ∇ is an extension of ∇^φ and*

$$\nabla^\varphi_Y(\bar\nabla_X \bar e) - \bar\nabla_X \nabla^\varphi_Y \bar e - \bar\nabla_{[Y,X]}\bar e = 0 \tag{2.1}$$

for all ∇^{F_M}-flat vector fields $X \in \mathfrak{X}(M)$, for all $Y \in \Gamma(F_M)$ and all ∇^φ-flat sections $\bar e \in \Gamma(E/K)$.

Write $\omega(Y, X, \bar e)$ for the left-hand side of this equation. An easy computation shows that $\omega(Y, fX, \bar e) = f\omega(Y, X, \bar e)$ for all $f \in C^\infty(M)$. Since ∇^{F_M}-flat vector fields span $\mathfrak{X}(M)$ as $C^\infty(M)$-module, the section $\omega(Y, X, \bar e)$ must consequently be 0 for *all* $X \in \mathfrak{X}(M)$. In the same manner, then $\omega(Y, X, f\bar e) = f\omega(Y, X, \bar e)$ and (2.1) must actually be true for all $Y \in \Gamma(F_M)$, $X \in \mathfrak{X}(M)$ and $e \in \Gamma(E)$. Furthermore, $\omega(Y_1, Y_2, \bar e) = R_{\nabla^\varphi}(Y_1, Y_2)\bar e = 0$ for $Y_1, Y_2 \in \Gamma(F_M)$ shows that ω can be seen as an element ω_∇ of $\Omega^1(F_M, \mathrm{Hom}(TM/F_M, \mathrm{End}(E/K)))$:

$$\omega_\nabla(Y)(\bar X)\bar e = \overline{R_\nabla(Y, X)e}. \tag{2.2}$$

Note that ∇^{F_M} and ∇^{φ} (or ∇^p) induce a flat F_M-connection on $\operatorname{Hom}(TM/F_M, \operatorname{End}(E/K))$, which is given by

$$(\nabla^{\mathrm{Hom}}_Y \phi)(\bar{X})(\bar{e}) = \nabla^{\varphi}_Y(\phi(\bar{X})(\bar{e})) - \phi(\nabla^{F_M}_Y \bar{X})(\bar{e}) - \phi(\bar{X})(\nabla^{\varphi}_Y \bar{e}).$$

For simplicity, the induced differential operator $\mathbf{d}_{\nabla^{\mathrm{Hom}}}$ on $\Omega^\bullet(F_M, \operatorname{Hom}(TM/F_M, \operatorname{End}(E/K)))$ is denoted by $\mathbf{d}_{\nabla^p}$.

Proposition 2.6. *Let ∇^p be a flat F_M-connection on E/K and choose an extension $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ of ∇^p. Define ω_∇ as in (2.2). Then*

$$\mathbf{d}_{\nabla^p}\omega_\nabla = 0 \tag{2.3}$$

and the class of ω_∇ in $H^1_{\mathbf{d}_{\nabla^p}}(F_M, \operatorname{Hom}(TM/F_M, \operatorname{End}(E/K)))$ does not depend on the choice of the extension ∇.

Proof. For $Y_1, Y_2 \in \Gamma(F_M)$ the value of $(\mathbf{d}_{\nabla^p}\omega_\nabla)(Y_1, Y_2)$ at $\bar{X} \in \Gamma(TM/F_M)$ and $\bar{e} \in \Gamma(E/K)$ is

$$\begin{aligned}
&(\nabla^{\mathrm{Hom}}_{Y_1}(\omega_\nabla(Y_2)))(\bar{X})(\bar{e}) - (\nabla^{\mathrm{Hom}}_{Y_2}(\omega_\nabla(Y_1)))(\bar{X})(\bar{e}) - (\omega_\nabla[Y_1, Y_2])(\bar{X})(\bar{e}) \\
=&\nabla^p_{Y_1}(\omega_\nabla(Y_2)(\bar{X})(\bar{e})) - \omega_\nabla(Y_2)(\bar{X})(\nabla^p_{Y_1}\bar{e}) - \omega_\nabla(Y_2)(\nabla^{F_M}_{Y_1}\bar{X})(\bar{e}) \\
&- \nabla^p_{Y_2}(\omega_\nabla(Y_1)(\bar{X})(\bar{e})) + \omega_\nabla(Y_1)(\bar{X})(\nabla^p_{Y_2}\bar{e}) + \omega_\nabla(Y_1)(\nabla^{F_M}_{Y_2}\bar{X})(\bar{e}) - (\omega_\nabla[Y_1, Y_2])(\bar{X})(\bar{e}) \\
=&\nabla^p_{Y_1}\nabla^p_{Y_2}\bar{\nabla}_X\bar{e} - \cancel{\nabla^p_{Y_1}\bar{\nabla}_X\nabla^p_{Y_2}\bar{e}} - \cancel{\nabla^p_{Y_1}\bar{\nabla}_{[Y_2,X]}\bar{e}} - \cancel{\nabla^p_{Y_2}\bar{\nabla}_X\nabla^p_{Y_1}\bar{e}} + \bar{\nabla}_X\nabla^p_{Y_2}\nabla^p_{Y_1}\bar{e} + \cancel{\bar{\nabla}_{[Y_2,X]}\nabla^p_{Y_1}\bar{e}} \\
&- \cancel{\nabla^p_{Y_2}\bar{\nabla}_{[Y_1,X]}\bar{e}} + \cancel{\bar{\nabla}_{[Y_1,X]}\nabla^p_{Y_2}\bar{e}} + \bar{\nabla}_{[Y_2,[Y_1,X]]}\bar{e} \\
&- \nabla^p_{Y_2}\nabla^p_{Y_1}\bar{\nabla}_X\bar{e} + \cancel{\nabla^p_{Y_2}\bar{\nabla}_X\nabla^p_{Y_1}\bar{e}} + \cancel{\nabla^p_{Y_2}\bar{\nabla}_{[Y_1,X]}\bar{e}} + \cancel{\nabla^p_{Y_1}\bar{\nabla}_X\nabla^p_{Y_2}\bar{e}} - \bar{\nabla}_X\nabla^p_{Y_1}\nabla^p_{Y_2}\bar{e} - \cancel{\bar{\nabla}_{[Y_1,X]}\nabla^p_{Y_2}\bar{e}} \\
&+ \cancel{\nabla^p_{Y_1}\bar{\nabla}_{[Y_2,X]}\bar{e}} - \cancel{\bar{\nabla}_{[Y_2,X]}\nabla^p_{Y_1}\bar{e}} - \bar{\nabla}_{[Y_1,[Y_2,X]]}\bar{e} \\
&- \nabla^p_{[Y_1,Y_2]}\bar{\nabla}_X\bar{e} + \bar{\nabla}_X\nabla^p_{[Y_1,Y_2]}\bar{e} + \bar{\nabla}_{[[Y_1,Y_2],X]}\bar{e} \\
=&R_{\nabla^p}(Y_1, Y_2)(\bar{\nabla}_X\bar{e}) - \bar{\nabla}_X(R_{\nabla^p}(Y_1, Y_2)\bar{e}) + \bar{\nabla}_{[Y_2,[Y_1,X]]+[[Y_1,Y_2],X]-[Y_1,[Y_2,X]]}\bar{e} = 0.
\end{aligned}$$

The last equality uses the flatness of ∇^p and the Jacobi identity. The second statement follows from the second part of Lemma 2.4: assume that ∇' is a second extension of ∇^p, then the reader is invited to check that

$$\omega_\nabla = \omega_{\nabla'} + \mathbf{d}_{\nabla^p}\bar{\phi},$$

with $\bar{\phi} \in \Gamma(\operatorname{Hom}(TM/F_M, \operatorname{End}(E/K))) = \Omega^0(F_M, \operatorname{Hom}(TM/F_M, \operatorname{End}(E/K)))$ the form induced as in Lemma 2.4 by the difference $\phi = \nabla - \nabla'$. □

By Proposition 2.6, the cohomology class of ω_∇ is an invariant of the flat connection ∇^p.

Definition 2.7. *Let ∇^p be a flat F_M-connection on E/K. Choose an extension $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. Then the* Atiyah class of ∇^p *is the class*

$$\alpha_{\nabla^p} = [\omega_\nabla] \in H^1_{\mathbf{d}_{\nabla^p}}(F_M, \operatorname{Hom}(TM/F_M, \operatorname{End}(E/K))).$$

Alternatively, this class is also the Atiyah class of the infinitesimal ideal system (F_M, K, ∇^p) in the vector bundle $E \to M$.

The triple (F_M, K, ∇^p) defines an involutive, linear subbundle $F_E \subseteq TE$ [9], and so a right-invariant foliation on the principal bundle of frames of E. If $K = \{0\}$ then it is horizontal and the Atiyah class above coincide with Molino's Atiyah class of a foliated bundle [11].

The definition of the Atiyah class of a flat partial connection is motivated by the following theorem, the proof of which is now easy to complete using Proposition 2.1 and Lemma 2.5.

Theorem 2.8. *Assume that $\varphi\colon E \to E'$ is a fibration of vector bundles over $\varphi_0\colon M \to M'$. Consider the flat connection ∇^φ defined by the fibration as in Proposition 2.2. Then α_{∇^φ} vanishes.*

This yields immediately the following obstructions to a flat F_M-connection on E/K defining a fibration.

Corollary 2.9. *Let $E \to M$ be a smooth vector bundle, $K \subseteq E$ a subbundle and $F_M \subseteq TM$ an involutive subbundle. Let $\nabla^p \colon \Gamma(F_M) \times \Gamma(E/K) \to \Gamma(E/K)$ be a flat connection. If ∇^p induces a fibration of vector bundles $E \to (E/K)/\nabla^p$ over $M \to M/F_M$, then $\alpha_{\nabla^p} = 0$.*

Corollary 2.10. [11] *Let M be a smooth manifold and F_M an involutive subbundle of TM. If F_M is simple, then the Atiyah class of ∇^{F_M} vanishes.*

2.2. **Infinitesimal ideal systems.** This subsection discusses the notion of infinitesimal ideal system in a general Lie algebroid, and defines the Atiyah class of such an ideal.

2.2.1. *Main properties of infinitesimal ideal systems.* Consider an ideal (F_M, J, ∇) in a Lie algebroid $A \to M$. Since for $j \in \Gamma(J)$, the class $\bar{j} = 0 \in \Gamma(A/J)$, (iis1) implies that $[j, j'] \in \Gamma(J)$ for all $j' \in \Gamma(J)$, i.e. J is automatically a Lie subalgebroid of A. The following proposition reformulates the ideal condition (iis1) in the definition of an infinitesimal ideal system.

Proposition 2.11. *Let $A \to M$ be a Lie algebroid, $F_M \subseteq TM$ an involutive subbundle and $J \subseteq A$ a subalgebroid with $\rho(J) \subseteq F_M$. Let $\nabla \colon \Gamma(F_M) \times \Gamma(A/J) \to \Gamma(A/J)$ be a flat connection. Then the following are equivalent.*
(iis1) *$[a, j] \in \Gamma(J)$ for all $j \in \Gamma(J)$ and $a \in \Gamma(A)$ ∇-parallel;*
(iis1') *$\nabla^J_j \bar{a} = \nabla_{\rho(j)} \bar{a}$ for all $j \in \Gamma(J)$ and $a \in \Gamma(A)$.*

Proof. First assume that $[a, j] \in \Gamma(J)$ for all $j \in \Gamma(J)$ and $a \in \Gamma(A)$ ∇-parallel. Then for all ∇-parallel sections $\bar{a}$ of A/J:
$$\nabla^J_j \bar{a} = \overline{[j, a]} = 0$$
for all $j \in \Gamma(J)$. That is, a ∇-parallel section of A is also ∇^J-parallel. Since $\Gamma(A/J)$ has local basis frames consisting of ∇-parallel sections of A/J, see [9], conclude as follows. Take $a \in \Gamma(A)$ defined on a neighbourhood of $p \in M$. Then there is an open set $U \subseteq M$, $p \in U$, smooth functions $f_1, \ldots, f_r \in C^\infty(U)$ and $\bar{a}_1, \ldots, \bar{a}_r \in \Gamma(A/J)^\nabla$ such that $\bar{a} = \sum_{i=1} f_i \bar{a}_i$. Compute then for $j \in \Gamma(J)$:
$$\nabla^J_j \bar{a} = \sum_{i=1} \pounds_{\rho(j)}(f_i)\bar{a}_i = \nabla_{\rho(j)} \bar{a}.$$

Conversely, assume that $\nabla^J_j \bar{a} = \nabla_{\rho(j)} \bar{a}$ for all $j \in \Gamma(J)$ and $a \in \Gamma(A)$. Then for a ∇-flat section $a \in \Gamma(A)$ the class $\overline{[j, a]}$ equals $\nabla^J_j \bar{a} = \nabla_{\rho(j)} \bar{a} = 0$ and so $[j, a] \in \Gamma(J)$ for all $j \in \Gamma(J)$. □

This implies a simplification of the definition of an ideal in the case $F_M = \rho(J)$.

Proposition 2.12. *Let $A \to M$ be a Lie algebroid Let $A \to M$ be a Lie algebroid, $F_M \subseteq TM$ an involutive subbundle and $J \subseteq A$ a subbundle with $\rho(J) = F_M$. Let $\nabla \colon \Gamma(F_M) \times \Gamma(A/J) \to \Gamma(A/J)$ be a flat connection. Then (iis2) and (iis3) in the definition of an ideal follow from (iis1).*

Proof. First take two ∇-flat sections $a, b \in \Gamma(A)$ and take any $X \in \Gamma(F_M)$. Then there exists $j \in \Gamma(J)$ with $\rho(j) = X$ and compute
$$\nabla_X \overline{[a, b]} = \nabla_{\rho(j)} \overline{[a, b]} = \nabla^J_j \overline{[a, b]} = \overline{[j, [a, b]]} = \overline{[[j, a], b]} + \overline{[a, [j, b]]}.$$
Since a, b are ∇-flat, the brackets $[j, a], [j, b] \in \Gamma(J)$, and again since a, b are ∇-flat, $[[j, a], b] + [a, [j, b]] \in \Gamma(J)$. Therefore $\nabla_X \overline{[a, b]} = 0$, which proves (iis2).

In the same manner, take $a \in \Gamma(A)$ a ∇-flat section and take $X \in \Gamma(F_M)$. Then there exists $j \in \Gamma(J)$ such that $\rho(j) = X$ and compute $\nabla^{F_M}_X \overline{\rho(a)} = \overline{[X, \rho(a)]} = \overline{\rho[j, a]}$. Then $[j, a] \in \Gamma(J)$ implies $\rho[j, a] \in \Gamma(F_M)$ and so $\nabla^{F_M}_X \overline{\rho(a)} = 0$. □

2.2.2. *Fibrations of Lie algebroids, infinitesimal ideal systems and the Atiyah class.* The following theorem shows that infinitesimal ideal systems in Lie algebroids define quotients of Lie algebroids, up to some topological obstructions. The paper [6] proves that an infinitesimal ideal system defines a sub-representation (up to homotopy) of the adjoint representation of the Lie algebroid, after the choice of an extension of the infinitesimal ideal system connection. These two results suggest that indeed, an infinitesimal ideal systems is the right notion of ideal in a Lie algebroid.

Theorem 2.13. [9] *Let (F_M, J, ∇) be an ideal in a Lie algebroid A. Assume that $\bar{M} = M/F_M$ is a smooth manifold and that ∇ has trivial holonomy. Then the vector bundle $(A/J)/\nabla \to M/F_M$ carries a Lie algebroid structure such that the projection (π, π_M) is a Lie algebroid morphism.*

$$\begin{array}{ccc} A & \xrightarrow{\pi} & (A/J)/\nabla \\ {\scriptstyle q_A}\downarrow & & \downarrow{\scriptstyle [q_A]} \\ M & \xrightarrow[\pi_M]{} & M/F_M \end{array}$$

Conversely, let

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A' \\ {\scriptstyle q_A}\downarrow & & \downarrow{\scriptstyle q_{A'}} \\ M & \xrightarrow[f]{} & M' \end{array}$$

be a fibration of Lie algebroids, i.e. a Lie algebroid morphism such that (φ, φ_0) is a fibration of vector bundles. Then $J := \ker(\varphi) \subseteq A$ is a subalgebroid of A and $F_M = T^{\varphi_0}M \subseteq TM$ is an involutive subbundle. The equality $T\varphi_0 \circ \rho = \rho' \circ \varphi$ yields immediately $\rho(J) \subseteq F_M$.

Define as before the connection $\nabla^{\varphi} \colon \Gamma(F_M) \times \Gamma(A/J) \to \Gamma(A/J)$ by setting $\nabla^{\varphi}_X \bar{a} = 0$ for all sections $a \in \Gamma(A)$ that are φ-related to some section $a' \in \Gamma(A')$. Then the properties of the Lie algebroid morphism (φ, φ_0) imply that $(F_M, J, \nabla^{\varphi})$ is an infinitesimal ideal system in A, see [9].

Note that an ideal (F_M, J, ∇) is defined as above by the kernel of a fibration of Lie algebroids if and only if it integrates to an ideal system in the sense of Higgins and Mackenzie [8, 10], see [9].

Example 2.14 (Flat connections on vector bundles)**.** Let E be a vector bundle over M and K a subbundle. Then any flat connection of an involutive subbundle $F_M \subseteq TM$ on E/K as in §2.1 defines an infinitesimal ideal system (F_M, K, ∇) in the Lie algebroid $(E, \rho = 0, [\cdot\,,\cdot] = 0)$. Here, Theorem 1.1 is Corollary 2.9.

An infinitesimal ideal system in a Lie algebroid is, by forgetting the Lie algebroid structure, automatically an infinitesimal ideal system in the underlying vector bundle. The Atiyah class of a general infinitesimal ideal system is defined below as the Atiyah class of the infinitesimal ideal system in the underlying vector bundle.

Definition 2.15. *The* Atiyah class of an infinitesimal ideal system (F_M, J, ∇) *in a Lie algebroid $A \to M$ is the Atiyah class of the flat connection $\nabla \colon \Gamma(F_M) \times \Gamma(A/J) \to \Gamma(A/J)$.*

By the considerations above, the Atiyah class of an infinitesimal ideal system is really just the Atiyah class of the connection, and defines an obstruction to the ideal defining a fibration of vector bundles. Indeed, a review of the proof of Theorem 2.13 in [9] reveals that if the fibration of vector bundles is well-defined, then the Lie algebroid structure on the quotient comes 'for free' along. The following result follows hence immediately from Corollary 2.9.

Proposition 2.16. *Let (F_M, J, ∇) be an ideal in a Lie algebroid $A \to M$. If the quotient Lie algebroid $(A/J)/\nabla \to M/F_M$ as in Theorem 2.13 exists, then the Atiyah class*

$$\alpha_\nabla \in H^1(F_M, \mathrm{Hom}(TM/F_M, \mathrm{End}(A/J)))$$

of the ideal vanishes.

The remainder of this section describes two natural examples of ideals.

Example 2.17 (The usual naive notion of ideal in a Lie algebroid)**.** A (naive) ideal I in a Lie algebroid $A \to M$ is a subbundle over M such that $[a, i] \in \Gamma(I)$ for all $i \in \Gamma(I)$ and all $a \in \Gamma(A)$. The inclusion $I \subseteq \ker(\rho)$ follows immediately and shows that this definition of an ideal is very restrictive. These usual ideals correspond obviously to the infinitesimal ideal systems $(F_M = 0, J = I, \nabla = 0)$ in A. In particular, an ideal in a Lie algebra is an infinitesimal ideal system. In this case, the quotient Lie algebroid A/I over $M/F_M = M$ is always defined.

It is easy to check that the Atiyah class of the Lie pair (A, I) is zero, which coincides with the fact that the quotient Lie algebroid $A/I \to M$ is always defined.

Example 2.18 (The Bott connection and reduction by simple foliations)**.** A standard example of a Lie algebroid is the tangent space TM of a smooth manifold M, endowed with the usual Lie bracket of vector fields and the identity id_{TM} as anchor. Consider an involutive subbundle $F_M \subseteq TM$ and the Bott connection

$$\nabla^{F_M} : \Gamma(F_M) \times \Gamma(TM/F_M) \to \Gamma(TM/F_M)$$

associated to it. Then Propositions 2.11 and 2.12 show that (F_M, F_M, ∇^{F_M}) is an ideal in TM. If the quotient Lie algebroid exists, then it is isomorphic to $T(M/F_M) \to M/F_M$, see [9]. For this class of infinitesimal ideal system, Theorem 1.1 is Corollary 2.10.

3. Geometric interpretation of the Atiyah class of a Lie pair

This section proves Theorem 1.1. The first subsection recalls the construction of the Atiyah class of a Lie pair (A, J) [4]. The second subsection explains how it can be constructed from an extension of an infinitesimal ideal system structure with fiber J, and proves the theorem.

3.1. The Atiyah class of a Lie pair.

Consider a Lie algebroid A over a smooth manifold M, together with a subalgebroid $J \subseteq A$. The pair (A, J) is a *Lie pair* [4]. Recall that the Lie pair defines the flat *Bott connection*

$$\nabla^J : \Gamma(J) \times \Gamma(A/J) \to \Gamma(A/J), \quad \nabla^J_j \bar{a} = \overline{[a, j]}$$

for $j \in \Gamma(J)$, and $\bar{a} \in \Gamma(A/J)$ the class of $a \in \Gamma(A)$. This induces as usual a flat J-connection ∇^{Hom} on $\mathrm{Hom}(A/J, \mathrm{End}(A/J)) = (A/J)^* \otimes (A/J)^* \otimes (A/J)$:

$$(\nabla^{\mathrm{Hom}}_j \phi)(\bar{a}_1, \bar{a}_2) = \nabla^J_j(\phi(\bar{a}_1, \bar{a}_2)) - \phi(\nabla^J_j \bar{a}_1, \bar{a}_2) - \phi(\bar{a}_1, \nabla^J_j \bar{a}_2).$$

For simplicity, $\mathbf{d}_{\nabla^J}$ denotes the Koszul differential on $\Omega^\bullet(J, (A/J)^* \otimes (A/J)^* \otimes (A/J))$ defined by this connection.

Choose an extension $\nabla^A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ of ∇^J. That is, $\nabla^A_a j \in \Gamma(J)$ for all $a \in \Gamma(A)$ and $j \in \Gamma(J)$ and the induced quotient connection $\overline{\nabla^A} : \Gamma(A) \times \Gamma(A/J) \to \Gamma(A/J)$ restricts to ∇^J on sections of J in the first argument. Then $\omega_{\nabla^A} \in \Omega^1(J, \mathrm{Hom}(A/J, \mathrm{End}(A/J)))$ defined by

$$\omega_{\nabla^A}(j)(\bar{a})(\bar{b}) = \overline{R_{\nabla^A}(j, a)b}, \quad j \in \Gamma(J), a, b \in \Gamma(A),$$

satisfies $\mathbf{d}_{\nabla^J} \omega_{\nabla^A} = 0$ – this works as in Proposition 2.6, see [4]. The *Atiyah class of the Lie pair* (A, J) is the cohomology class

$$\alpha_J := [\omega_{\nabla^A}] \in H^1_{\mathbf{d}_{\nabla^J}}(J, \mathrm{Hom}(A/J, \mathrm{End}(A/J))),$$

see [4]. It does not depend on the choice of the extension ∇^A of the Bott connection ∇^J.

3.2. **Proof of Theorem 1.1.** Consider now an infinitesimal ideal system (F_M, J, ∇^i) in A. Choose an extension $\nabla\colon \mathfrak{X}(M)\times\Gamma(A)\to\Gamma(A)$ of ∇^i. That is, ∇ preserves sections of J and quotients to a connection $\bar\nabla\colon \mathfrak{X}(M)\times\Gamma(A/J)\to\Gamma(A/J)$ that restricts to ∇^i on sections of F_M.

Consider the basic connection $\nabla^{\rm bas}\colon \Gamma(A)\times\Gamma(A)\to\Gamma(A)$ induced by ∇. It is defined by $\nabla^{\rm bas}_{a_1}a_2=[a_1,a_2]+\nabla_{\rho(a_2)}a_1$. Then a section $a\in\Gamma(A)$ such that $\bar a\in\Gamma(A/J)$ is ∇^i-flat satisfies

$$\nabla^{\rm bas}_a j=[a,j]+\nabla_{\rho(j)}a\in\Gamma(J) \tag{3.4}$$

for all $j\in\Gamma(J)$. To see this, recall that $[a,j]\in\Gamma(J)$ by the definition of an infinitesimal ideal system, and $\rho(j)\in\Gamma(F_M)$ induces $\overline{\nabla_{\rho(j)}a}=\bar\nabla_{\rho(j)}\bar a=\nabla^i_{\rho(j)}\bar a=0$ in $\Gamma(A/J)$. Since ∇^i-flat sections generate all sections of A, one finds that $\nabla^{\rm bas}_a j\in\Gamma(J)$ for all $j\in\Gamma(J)$ and *all* $a\in\Gamma(A)$. This proves the following lemma.

Lemma 3.1. *Consider a Lie algebroid A over a smooth manifold M, and an infinitesimal ideal system (F_M, J, ∇^i) in A. Choose an extension $\nabla\colon \mathfrak{X}(M)\times\Gamma(A)\to\Gamma(A)$ of ∇^i. Then the basic connection $\nabla^{\rm bas}\colon\Gamma(A)\times\Gamma(A)\to\Gamma(A)$ defined by ∇ satisfies $\nabla^{\rm bas}_a j\in\Gamma(J)$ for all $a\in\Gamma(A)$ and all $j\in\Gamma(J)$.*

As a consequence, there is a quotient connection

$$\overline{\nabla^{\rm bas}}\colon \Gamma(A)\times\Gamma(A/J)\to\Gamma(A/J),\quad \overline{\nabla^{\rm bas}}_{a_1}\bar a_2=\overline{\nabla^{\rm bas}_{a_1}a_2}.$$

The following proposition shows that $\nabla^{\rm bas}$ is an extension of the Bott connection $\nabla^J\colon\Gamma(J)\times\Gamma(A/J)\to\Gamma(A/J)$.

Proposition 3.2. *Consider a Lie algebroid A over a smooth manifold M, and an infinitesimal ideal system (F_M, J, ∇^i) in A. Choose an extension $\nabla\colon \mathfrak{X}(M)\times\Gamma(A)\to\Gamma(A)$ of ∇^i. Then the quotient connection $\overline{\nabla^{\rm bas}}$ restricts on sections of J to the Bott connection $\nabla^J\colon\Gamma(J)\times\Gamma(A/J)\to\Gamma(A/J)$, $\nabla^J_j\bar a=\overline{[j,a]}\in\Gamma(A/J)$.*

Proof. Choose $j\in\Gamma(J)$ and $a\in\Gamma(A)$. Then

$$\overline{\nabla^{\rm bas}}_j\bar a=\overline{\nabla^{\rm bas}_j a}=\overline{[j,a]+\nabla_{\rho(a)}j}=\overline{[j,a]}=\nabla^J_j\bar a. \tag{3.5}$$

In the third equation, $\nabla_X j\in\Gamma(J)$ for all $X\in\mathfrak{X}(M)$ and $j\in\Gamma(J)$ is given by the choice of the connection ∇. □

Hence, $\nabla^{\rm bas}$ can be used to construct the Atiyah class α_J of the Lie pair (A,J):

$$\alpha_J=[\omega_{\nabla^{\rm bas}}]\in H^1(J,\operatorname{Hom}(A/J,\operatorname{End}(A/J)))$$

for $\nabla^{\rm bas}$ defined by an extension ∇ of the infinitesimal ideal system connection ∇^i.

The remainder of this section proves Theorem 1.1. The key to this proof is the following: an infinitesimal ideal system (F_M, J, ∇^i) in A makes the anchor ρ into a chain map

$$\rho^\star\colon\ (\Omega^\bullet(F_M,\operatorname{Hom}(TM/F_M,\operatorname{End}(A/J))),\mathbf{d}_{\nabla^i})\to(\Omega^\bullet(J,\operatorname{Hom}(A/J,\operatorname{End}(A/J))),\mathbf{d}_{\nabla^J}).$$

Lemma 3.3. *Let (J,A) be a Lie pair over a smooth manifold M. Let $F_M\subseteq TM$ be an involutive subbundle with $\rho(J)\subseteq F_M$ and let $\nabla\colon\Gamma(F_M)\times\Gamma(A/J)\to\Gamma(A/J)$ be a linear connection. Then*

$$\rho^\star\colon\Omega^\bullet(F_M,\operatorname{Hom}(TM/F_M,\operatorname{End}(A/J)))\to\Omega^\bullet(J,\operatorname{Hom}(A/J,\operatorname{End}(A/J))),$$
$$(\rho^\star\omega)(j_1,\ldots,j_l)(\overline{a_1},\overline{a_2})=\omega(\rho(j_1),\ldots,\rho(j_l))(\overline{\rho(a_2)})(\overline{a_1})$$

is a well-defined degree-preserving morphism of $C^\infty(M)$-modules and a morphism of modules over $\rho^\star\colon\Omega^\bullet(F_M)\to\Omega^\bullet(J)$.

Proof. The map $\rho^\star$ is well-defined because $\rho(J)\subseteq F_M$, and $\bar a=0\in A/J$ if and only if $a\in\Gamma(J)$, which implies $\rho(a)\in\Gamma(F_M)$ and so $\overline{\rho(a)}=0$ in TM/F_M. □

Lemma 3.4. *Let $A \to M$ be a Lie algebroid and let (F_M, J, ∇) be an infinitesimal ideal system in A. Then $\rho^\star$ satisfies $\mathbf{d}_{\nabla^J} \circ \rho^\star = \rho^\star \circ \mathbf{d}_{\nabla^i}$.*

Remark 3.5. Note that conversely, $\mathbf{d}_{\nabla^J} \circ \rho^\star = \rho^\star \circ \mathbf{d}_{\nabla^i}$ does not imply that (F_M, J, ∇) is an infinitesimal ideal system. In general, the following result holds. Let $\phi \in \Omega^1(J, \operatorname{End}(A/J))$ be defined by $\phi(j)\bar{a} = \nabla_{\rho(j)}\bar{a} - \nabla^J_j \bar{a}$ for all $j \in \Gamma(J)$ and all $a \in \Gamma(A)$. Consider the open subset $U \subseteq M$ defined by $U := \{p \in M \mid \rho(A(p)) \not\subseteq F_M(p)\}$. Then $\rho^\star$ satisfies $\mathbf{d}_{\nabla^J} \circ \rho^\star = \rho^\star \circ \mathbf{d}_{\nabla^i}$ if and only if there exists $\lambda \in \Gamma_U(J^*)$ such that $\phi|_U = \lambda \cdot \operatorname{id}_{A/J}$.

Proof of Lemma 3.4. Choose $\omega \in \Omega^l_U(F_M, \operatorname{Hom}(TM/F_M, \operatorname{End}(A/J)))$. Then on the one hand,

$$
\begin{aligned}
&\rho^\star(\mathbf{d}_{\nabla^i}\omega)(j_1, \ldots, j_{l+1})(\overline{a_1}, \overline{a_2}) = (\mathbf{d}_{\nabla^i}\omega)(\rho(j_1), \ldots, \rho(j_{l+1}))(\overline{\rho(a_2)})(\overline{a_1}) \\
&= \sum_{i=1}^{l+1} (-1)^{i+1} \nabla^{\mathrm{Hom}}_{\rho(j_i)}(\omega(\rho(j_1), \ldots, \hat{i}, \ldots, \rho(j_{l+1})))(\overline{\rho(a_2)})(\overline{a_1}) \\
&\quad + \sum_{i<l} (-1)^{i+l} \omega([\rho(j_i), \rho(j_l)], \rho(j_1), \ldots, \hat{i}, \ldots, \hat{l}, \ldots \rho(j_{l+1}))(\overline{\rho(a_2)})(\overline{a_1}).
\end{aligned}
\tag{3.6}
$$

On the other hand

$$
\begin{aligned}
&\mathbf{d}_{\nabla^J}(\rho^\star\omega)(j_1, \ldots, j_{l+1})(\overline{a_1}, \overline{a_2}) \\
&= \sum_{i=1}^{l+1} (-1)^{i+1} \nabla^{\mathrm{Hom}}_{j_i}(\rho^\star\omega(j_1, \ldots, \hat{i}, \ldots, j_{l+1}))(\overline{a_1}, \overline{a_2}) \\
&\quad + \sum_{i<l} (-1)^{i+l} \rho^\star\omega([j_i, j_l], j_1, \ldots, \hat{i}, \ldots, \hat{l}, \ldots j_{l+1})(\overline{a_1}, \overline{a_2}) \\
&= \sum_{i=1}^{l+1} (-1)^{i+1} \nabla^{\mathrm{Hom}}_{j_i}(\rho^\star\omega(j_1, \ldots, \hat{i}, \ldots, j_{l+1}))(\overline{a_1}, \overline{a_2}) \\
&\quad + \sum_{i<l} (-1)^{i+l} \omega(\rho[j_i, j_l], \rho(j_1), \ldots, \hat{i}, \ldots, \hat{l}, \ldots \rho(j_{l+1}))(\overline{\rho(a_2)})(\overline{a_1}).
\end{aligned}
\tag{3.7}
$$

Proposition 2.11 and the compatibility of the Lie bracket with the anchor give for all $j, j_1, \ldots, j_l \in \Gamma(J)$ and $a_1, a_2 \in \Gamma(A)$:

$$
\begin{aligned}
&\nabla^{\mathrm{Hom}}_j(\rho^\star\omega(j_1, \ldots, j_l))(\overline{a_1})(\overline{a_2}) \\
&= \nabla^J_j \left(\omega(\rho(j_1), \ldots, \rho(j_l))(\overline{\rho(a_2)})(\overline{a_1})\right) - \omega(\rho(j_1), \ldots, \rho(j_l))(\overline{\rho(a_2)})(\nabla^J_j \overline{a_1}) \\
&\quad - \omega(\rho(j_1), \ldots, \rho(j_l))(\overline{\rho([j, a_2])})(\overline{a_1}) \\
&= \nabla^i_{\rho(j)} \left(\omega(\rho(j_1), \ldots, \rho(j_l))(\overline{\rho(a_2)})(\overline{a_1})\right) - \omega(\rho(j_1), \ldots, \rho(j_l))(\overline{\rho(a_2)})(\nabla^i_{\rho(j)} \overline{a_1}) \\
&- \omega(\rho(j_1), \ldots, \rho(j_l))(\nabla^{F_M}_{\rho(j)} \overline{\rho(a_2)})(\overline{a_1}) \\
&= \nabla^{\mathrm{Hom}}_{\rho(j)}(\omega(\rho(j_1), \ldots, \rho(j_l)))(\overline{\rho(a_2)})(\overline{a_1}),
\end{aligned}
$$

Hence, by (3.6) and (3.7), $\rho^\star \circ \mathbf{d}_{\nabla^i} = \mathbf{d}_{\nabla^J} \circ \rho^\star$ and the proof is complete. □

As a consequence, if the Lie pair (A, J) carries an ideal structure (F_M, J, ∇^i), then $\rho^\star$ induces a map in cohomology. The following theorem shows that the Atiyah class of the Lie pair is then the image under this map of the Atiyah class of the infinitesimal ideal system.

Theorem 3.6. *Let $A \to M$ be a Lie algebroid and let (F_M, J, ∇^i) be an ideal in A. The image under $\rho^\star$ of the Atiyah class*

$$\alpha_{\nabla^i} \in H^1_{\mathbf{d}_{\nabla^i}}(F_M, \operatorname{Hom}(TM/F_M, \operatorname{End}(A/J)))$$

of the ideal is the Atiyah class of the Lie pair (A,J)

$$\alpha_J \in H^1_{\mathbf{d}_{\nabla^J}}(J, \operatorname{Hom}(A/J, \operatorname{End}(A/J))).$$

Proof. Take an extension ∇ of ∇^i, and consider the associated basic connection $\nabla^{\rm bas}$. By Proposition 3.2, it is an extension of the Bott connection ∇^J and so the Atiyah class α_J is the cohomology class of $\omega_{\nabla^{\rm bas}}$. It suffices therefore to show that $\rho^\star\omega_\nabla = \omega_{\nabla^{\rm bas}}$. This is a simple computation:

$$\begin{aligned}
\omega_{\nabla^{\rm bas}}(j,\overline{a_1})(\overline{a_2}) &= \nabla^J_j \overline{\nabla^{\rm bas}}_{a_1}\overline{a_2} - \overline{\nabla^{\rm bas}}_{a_1}\nabla^J_j\overline{a_2} - \overline{\nabla^{\rm bas}}_{[j,a_1]}(\overline{a_2})\\
&= \nabla^J_j\overline{[a_1,a_2]+\nabla_{\rho(a_2)}a_1} - \overline{\nabla^{\rm bas}}_{a_1}\overline{[j,a_2]} - \overline{[[j,a_1],a_2]+\nabla_{\rho(a_2)}[j,a_1]}\\
&= \overline{[j,[a_1,a_2]+\nabla_{\rho(a_2)}a_1]} - \overline{[a_1,[j,a_2]]+\nabla_{\rho[j,a_2]}a_1} - \overline{[[j,a_1],a_2]+\nabla_{\rho(a_2)}[j,a_1]}\\
&= \overline{[j,\nabla_{\rho(a_2)}a_1]} - \overline{\nabla_{\rho[j,a_2]}a_1} - \overline{\nabla_{\rho(a_2)}[j,a_1]}\\
&= \nabla^J_j\bar\nabla_{\rho(a_2)}\overline{a_1} - \bar\nabla_{\rho[j,a_2]}\overline{a_1} - \bar\nabla_{\rho(a_2)}\nabla^J_j\overline{a_1}\\
&= \nabla^i_{\rho(j)}\bar\nabla_{\rho(a_2)}\overline{a_1} - \bar\nabla_{[\rho(j),\rho(a_2)]}\overline{a_1} - \bar\nabla_{\rho(a_2)}\nabla^i_{\rho(j)}\overline{a_1} = \omega_\nabla(\rho(j),\overline{\rho(a_2)})(\overline{a_1})\\
&= \rho^\star\omega_\nabla(j,\overline{a_1})(\overline{a_2})
\end{aligned}$$

for $j\in\Gamma(J)$ and $a_1,a_2\in\Gamma(A)$. □

Theorem 3.6 and Proposition 2.16 now induce Theorem 1.1.

References

1. R. Bott, *On a topological obstruction to integrability*, Global Analysis (Proc. Sympos. Pure Math., Vol. XVI, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 127–131. MR 0266248
2. ———, *Lectures on characteristic classes and foliations. Notes by Lawrence Conlon. Appendices by J. Stasheff.*, Lectures algebraic diff. Topology, Lect. Notes Math. 279, 1-94 (1972)., 1972.
3. H. Bursztyn, M. Crainic, A. Weinstein, and C. Zhu, *Integration of twisted Dirac brackets*, Duke Math. J. **123** (2004), no. 3, 549–607.
4. Z. Chen, M. Stiénon, and P. Xu, *From Atiyah classes to homotopy Leibniz algebras*, Comm. Math. Phys. **341** (2016), no. 1, 309–349. MR 3439229
5. M. Crainic, M. A. Salazar, and I. Struchiner, *Multiplicative forms and Spencer operators*, Math. Z. **279** (2015), no. 3-4, 939–979. MR 3318255
6. T. Drummond, M. Jotz Lean, and C. Ortiz, *VB-algebroid morphisms and representations up to homotopy*, Differential Geometry and its Applications **40** (2015), 332–357.
7. E. Hawkins, *A groupoid approach to quantization*, J. Symplectic Geom. **6** (2008), no. 1, 61–125.
8. P. J. Higgins and K. Mackenzie, *Algebraic constructions in the category of Lie algebroids*, J. Algebra **129** (1990), no. 1, 194–230. MR 1037400 (92e:58241)
9. M. Jotz Lean and C. Ortiz, *Foliated groupoids and infinitesimal ideal systems*, Indag. Math. (N.S.) **25** (2014), no. 5, 1019–1053.
10. K. C. H. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, London Mathematical Society Lecture Note Series, vol. 213, Cambridge University Press, Cambridge, 2005.
11. P. Molino, *Classe d'Atiyah d'un feuilletage et connexions transverses projetables.*, C. R. Acad. Sci. Paris Sér. A-B **272** (1971), A779–A781. MR 0281224
12. ———, *Classes caractéristiques et obstruction d'Atiyah pour les fibrés principaux feuilletés*, C. R. Acad. Sci. Paris Sér. A-B **272** (1971), A1376–A1378. MR 0287571
13. M. Zambon and C. Zhu, *Distributions and quotients on degree 1 NQ-manifolds and Lie algebroids*, J. Geom. Mech. **4** (2012), no. 4, 469–485. MR 3011897

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