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Introduction to Scattering Theory
Exercise Sheet 1

Exercise 1.
Let $\mathcal{H}$ be a Hilbert space. An operator $V\colon \mathcal{H} \to \mathcal{H}$ is called isometric if $\|Vu\| = \|u\|$ for all $u \in \mathcal{H}$. An operator $U\colon \mathcal{H} \to \mathcal{H}$ is called unitary if U is isometric and surjective.

(1) Let $(V_n)_{n\in\mathbb{N}}$ be a sequence of isometric operators that converges strongly to a bounded operator V. Show that V is isometric.

(2) According to (1), the strong limit of a sequence of unitary operators is isometric but it is not necessarily unitary: Consider the unitary operators $\ell_2 \to \ell_2$ given by

$$U_n[(f_k)_{k\in\mathbb{N}}] \coloneqq (f_n, f_1, f_2, \ldots, f_{n-1}, f_{n+1}, f_{n+2}, \ldots),$$

for $f = (f_k)_{k\in\mathbb{N}} \in \ell_2$, and show that $(U_n)_{n\in\mathbb{N}}$ converges strongly to

$$U_0[(f_k)_{k\in\mathbb{N}}] \coloneqq (0, f_1, f_2, f_3, \ldots).$$

Exercise 2.
Let $\mathcal{H}$ be a Hilbert space and let $A \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator with $A = \int \lambda \, \mathrm{d}E(\lambda)$. Let $B \in \mathcal{L}(\mathcal{H})$. Show that:

$$[A, B] = 0 \quad \Longleftrightarrow \quad [B, E(\lambda)] = 0, \, \forall \lambda \in \mathbb{R}.$$

Hint. For the proof of "$\Longrightarrow$", show first that for any $\lambda_0 \in \mathbb{R}$ there exists a sequence of polynomials $p_n\colon \mathbb{R} \to \mathbb{R}$ such that $p_n(A)f \to E(\lambda_0)f$, $n \to \infty$, for all $f \in \mathcal{H}$.

The solutions will be discussed in the tutorial on 07.11.2018.