Georg-August-Universität Göttingen
Mathematisches Institut
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07.11.2018

Introduction to Scattering Theory
Exercise Sheet 2

Exercise 3.
Let $A \in \mathcal{L}(\mathcal{H})$ with the spectral family $(E(\lambda))_{\lambda\in\mathbb{R}}$.

(1) Show that $\sum_{k=0}^{\infty} \frac{1}{k!}(\mathrm{i}tA)^k$ converges for any $t \in \mathbb{R}$ in $\mathcal{L}(\mathcal{H})$ and that the convergence is uniform for $t \in [-R, R]$, for all $R > 0$.

(2) We denote by $\mathrm{e}^{\mathrm{i}At} := \int \mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}E(\lambda)$ the unitary operator obtained from Theorem 2.2. Show that

$$\mathrm{e}^{\mathrm{i}tA} = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathrm{i}tA)^k.$$

Exercise 4.
Let A be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and let $M \subset \mathcal{H}$ be a closed subspace of $\mathcal{H}$ with orthogonal projection P. Show that:

$$\mathrm{e}^{\mathrm{i}tA}P = P\mathrm{e}^{\mathrm{i}tA} \quad \Longleftrightarrow \quad PA \subset AP.$$

Exercise 5.
Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces and let A be a self-adjoint operator in $\mathcal{H}$. Let $U\colon \mathcal{H} \to \mathcal{H}'$ be unitary and let $B := UAU^{-1}$; then B is a self-adjoint operator in $\mathcal{H}'$. Show that:

(1) The spectral families $(E(\lambda))_{\lambda\in\mathbb{R}}$ and $(E'(\lambda))_{\lambda'\in\mathbb{R}}$ of A and B satisfy $E'(\lambda) = UE(\lambda)U^{-1}$ for all $\lambda \in \mathbb{R}$.

(2) Let $f\colon \mathbb{R} \to \mathbb{C}$ be bounded and continuous. Then

$$f(B) = Uf(A)U^{-1}.$$

(3) If A is purely absolutely continuous (i.e. $\mathcal{H}_{\mathrm{ac}}(A) = \mathcal{H}$), then B is also purely absolutely continuous.

The solutions will be discussed in the tutorial on 14.11.2018.