Georg-August-Universität Göttingen
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Introduction to Scattering Theory
Exercise Sheet 5

Exercise 11.
Let $\mathcal{F}\colon \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$, $d \geq 1$, be the Fourier transform. Show that, for $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$,

$$\mathcal{F}\left(\mathrm{e}^{-\frac{\gamma}{2}|x|^2}\right)(k) = \gamma^{-d/2}\mathrm{e}^{-\frac{|k|^2}{2\gamma}}.$$

Exercise 12.
For any $\gamma > 0$, let $\varphi_\gamma\colon \mathbb{R}^3 \to \mathbb{R}$ be defined by $\varphi_\gamma(x) := \gamma^{3/4}\mathrm{e}^{-\gamma|x|^2/2}$. Let $H_0 = \overline{-\Delta\restriction_{C_c^\infty(\mathbb{R}^3)}}$ in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^3)$ and

$$\mathcal{D} = \operatorname{span}\left\{\varphi_\gamma(\cdot - a);\, \gamma > 0,\, a \in \mathbb{R}^3\right\}.$$

Show that $\mathcal{D} \subset D(H_0)$ and $\overline{\mathcal{D}} = \mathcal{H}$.

Exercise 13.
Show that the free propagator $\mathrm{e}^{-\mathrm{i}tH_0}$ in $\mathbb{R}^3$ has a weak integral kernel

$$K_t(x,y) := \frac{1}{(4\pi\mathrm{i}t)^{3/2}}\mathrm{e}^{-\frac{|x-y|^2}{4\pi\mathrm{i}t}}, \quad x,y \in \mathbb{R}^3, \quad t \in \mathbb{R}\backslash\{0\},$$

in the sense that, for $f \in L_2(\mathbb{R}^3)$,

$$\left(\mathrm{e}^{-\mathrm{i}tH_0}f\right)(x) = L_2 - \lim_{R\to\infty}\int_{|y|<R} K_t(x,y)f(y)\,\mathrm{d}y, \text{ a.e.}$$

The solutions will be discussed in the tutorial on 05.12.2018.