Georg-August-Universität Göttingen
Mathematisches Institut
PD. Dr. Martin Kohlmann

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Introduction to Scattering Theory
Exercise Sheet 8

Exercise 18.
Let $A = \int \lambda \, \mathrm{d}E(\lambda)$ be a self-adjoint operator on the Hilbert space $\mathcal{H}$. Let $\mathcal{M}(A)$ denote the set of all $\varphi \in \mathcal{H}$ such that $\mathrm{d}\,\langle \varphi, E(\lambda)\varphi \rangle = |f(\lambda)|^2 \, \mathrm{d}\lambda$ where $f \in L_\infty(\mathbb{R})$. Let $|||\varphi||| = \|f\|_\infty$. Show that $|||\cdot|||$ is a norm and $\mathcal{M}(A)$ is dense (in the $\mathcal{H}$-norm) in $\mathcal{H}_{\mathrm{ac}}(A)$.

Exercise 19.

(1) Let $H_0 \coloneqq \overline{-\Delta \restriction_{C_c^\infty(\mathbb{R}^3)}}$ and let $u \in L_2(\mathbb{R}^3)$. Show that for any $R \geq 0$,

$$\left\| \chi_{B_R} \mathrm{e}^{-\mathrm{i}tH_0} u \right\| \to 0, \quad t \to \infty.$$

Hint. Apply the estimate of Corollary 3.15.

(2) Let $H = H_0 + V$ with $V \in L_2(\mathbb{R}^3)$ bounded. Show that for $u \in \mathcal{H}_\pm = R(\Omega_\pm(H, H_0))$,

$$\left\| \chi_{B_R} \mathrm{e}^{-\mathrm{i}tH} u \right\| \to 0, \quad t \to \infty.$$

(3) Comment on the behavior of $\left\| \chi_{B_R} \mathrm{e}^{-\mathrm{i}tH} u \right\|$ for $t \to \pm\infty$, if u is an eigenfunction of H.

(4) Give a physical interpretation of the results in (1)–(3).

The solutions will be discussed in the tutorial on 16.01.2019.