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Introduction to Scattering Theory
Exercise Sheet 9

Exercise 20. (Birman-Schwinger principle)
Let $A\colon D(A) \to \mathcal{H}$ be a self-adjoint operator in the Hilbert space $\mathcal{H}$ and assume that $B \geq 0$ is a bounded, symmetric operator in $\mathcal{H}$ such that the operator $B(A-z)^{-1}$ is compact for one (or all) $z \in \rho(A)$. Show that, for $E \in \rho(A) \cap \mathbb{R}$ and $m \in \mathbb{N}$, the following statements are equivalent:

(1) E is an eigenvalue of $A - B$ with multiplicity m.

(2) 1 is an eigenvalue of $B^{1/2}(A-E)^{-1}B^{1/2}$ with multiplicity m.

Exercise 21.
Let $f \in L_2(\mathbb{R})$ with $(\mathcal{F}f)(x) = 0$ for $x < 0$ and define $\psi_t(x) := \left(\mathrm{e}^{-\mathrm{i}t(-\Delta)} f\right)(x)$. Show that, for any $b \in \mathbb{R}$,

$$\int_{-\infty}^{b} |\psi_t(x)|^2 \,\mathrm{d}x \to 0, \quad t \to \infty,$$

i.e. $\psi_t(x)$ represents a wave travelling to the right. (Similarly, $\psi_t(\cdot)$ represents a wave travelling to the left if $(\mathcal{F}f)(x) = 0$ for $x > 0$.)

Hint. Apply [W-II, Thm. 21.16]: If $\mathcal{F}f \in C_c^\infty(\mathbb{R}^d)$ and if $G \subset \mathbb{R}^d$ is open with supp $\mathcal{F}f \subset G$, then for any $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}_0^d$ there exists $C = C(f, n, \alpha, \beta)$ so that

$$\left|x^\beta D^\alpha \mathrm{e}^{-\mathrm{i}t(-\Delta)} f(x)\right| \leq C(1+|t|)^{-n}, \quad x \in \mathbb{R}^d \backslash 2tG.$$

The solutions will be discussed in the tutorial on 23.01.2018.