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The Dislocation Problem in Hilbert Spaces
Exercise Sheet 1

Exercise 1. (Existence of cut-off functions)
Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}$. We denote by

$$\operatorname{supp} f := \overline{\{x \in U;\ f(x) \neq 0\}}$$

the support of f. Let $C_c^\infty(U)$ be the set of all $f \in C^\infty(U)$ such that supp f is a compact subset of U.

(1) Show that the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) := \begin{cases} \mathrm{e}^{-1/t}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

satisfies $g \in C^\infty(\mathbb{R})$.
Hint: There are polynomials P_k such that $g^{(k)}(t) = P_k(\frac{1}{t})\mathrm{e}^{-1/t}$ and $P_{k+1}(x) = x^2(P_k(x) - P_k'(x))$ for $t > 0$. Look at $\lim_{t\to+0} g^{(k)}(t)$.

(2) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} \exp\left(-\frac{1}{1-x^2}\right), & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

Show that $f \in C_c^\infty(\mathbb{R})$.

Exercise 2. (IMS Localization Formula)
Let $(J_a)_{a\in A}$ be a partition of unity as in Definition 1.1 and let $H = h_0 + V$ for $h_0 = \overline{-\Delta\restriction_{C_c^\infty(\mathbb{R}^n)}}$ and a potential V belonging to the Kato class. Show that

$$H = \sum_{a\in A} J_a H J_a - \sum_{a\in A} |\nabla J_a|^2.$$

The solutions will be discussed in the tutorial on 08.05.2019.