Georg-August-Universität Göttingen
Mathematisches Institut
PD. Dr. Martin Kohlmann

Sommersemester 2019
22.05.2019

The Dislocation Problem in Hilbert Spaces
Exercise Sheet 4

Exercise 6.
Let $-\Delta_\Sigma$ denote the negative Laplacian in $L_2(\Sigma)$ and $-\Delta_{\Sigma;D}$ the negative Laplacian in $L_2(\Sigma)$ with an additional Dirichlet boundary condition at $x = 0$. Show that $(-\Delta_\Sigma + 1)^{-1} - (-\Delta_{\Sigma;D} + 1)^{-1}$ is a Hilbert-Schmidt operator.

Exercise 7.
A function $A(\cdot)$ from a measure space M to the (not necessarily bounded) self-adjoint operators on a Hilbert space $\mathcal{H}'$ is called measurable if and only if the function $(A(\cdot) + \mathrm{i})^{-1}$ is measurable. Given such a function, we define an operator A on $\mathcal{H} = \int_M^\oplus \mathcal{H}' \, \mathrm{d}\mu$ with domain

$$D(A) = \left\{ \psi \in \mathcal{H};\ \psi(m) \in D(A(m)) \text{ a.e.}, \int_M \|A(m)\psi(m)\|_{\mathcal{H}'}^2 \, \mathrm{d}\mu(m) < \infty \right\}$$

by

$$(A\psi)(m) = A(m)\psi(m).$$

We write $A = \int_M^\oplus A(m) \, \mathrm{d}\mu$. Show that the operator A is self-adjoint and that

$$\lambda \in \sigma(A) \quad \Longleftrightarrow \quad \forall \varepsilon > 0 : \mu\left(\{m;\ \sigma(A(m)) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset\}\right) > 0.$$

Hint: Use that, for any bounded Borel function F on $\mathbb{R}$,

$$F(A) = \int_M^\oplus F(A(m)) \, \mathrm{d}\mu.$$

The solutions will be discussed in the tutorial on 29.05.2019.