Introduction to Scattering Theory

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Chapter 1

Classical particle scattering

Scattering occurs in a variety of physical situations. It normally involves a comparison of two different dynamics for the same system: the given dynamics and a "free" dynamics. It is hard to give a precise definition of "free dynamics" but important characteristics of a free dynamical system are that it is simpler than the given dynamics and that it conserves the momentum of the "individual constituents" of the physical system.

The simplest system with which to illustrate the ideas of scattering theory is the classical mechanics of a single particle moving in an external force field $f(x)$, $x \in \mathbb{R}^3$. This theory is equivalent to the scattering of two particles interacting with each other through a force field $f(x_1 - x_2)$ because the center of mass motion of such a two-body system separates from the motion of $x_{12} = x_1 - x_2$. The states of such a single particle system are points in *phase space* $\mathbb{R}^3 \times \mathbb{R}^3$, i.e. pairs $u(t) = (x(t), \dot{x}(t)) \in \mathbb{R}^6$ representing the position and the velocity of the particle. The evolution is given by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = F(u(t)). \tag{1.1}$$

The force field is obtained from a *potential* $V(x)$ and equals $-\mathrm{grad}\ V(x)$. The right-hand side of the evolution equation (1.1) thus reads

$$F(u(t)) = F \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} \dot{x}(t) \\ -\frac{1}{m}\,\mathrm{grad}\ V(x(t)) \end{pmatrix}.$$

Let us assume for simplicity that V has compact support and that the particle moves outside the support of V (and hence outside of the corresponding force field) for large $|t|$. We can then expect that

$$\begin{aligned} x(t) &= x_- + tv_-, \quad t \to -\infty, \\ x(t) &= x_+ + tv_+, \quad t \to +\infty. \end{aligned}$$

Conservation of the energy E implies that $|v_+| = |v_-|$. Furthermore, integrating the conservation law $\frac{1}{2}m\dot{x}(t)^2 = E - V(x(t))$ with a given initial condition $(x(t_0), \dot{x}(t_0)) =$

$(x_- + t_0 v_-, v_-)$, for t_0 sufficiently near $-\infty$ so that $\{x_- + tv_-, t < t_0\} \cap \operatorname{supp} V = \emptyset$, we observe that x_+ and v_+ are functions of x_- and v_-. This motivates to define the *scattering map*

$$S\colon \mathbb{R}^6 \to \mathbb{R}^6, \quad \begin{pmatrix} x_- \\ v_- \end{pmatrix} \mapsto \begin{pmatrix} x_+ \\ v_+ \end{pmatrix}.$$

Let us consider a particle moving in one spatial direction and assume that V is bounded, $V(x) \le E_0 = \max V$. For energies $E < E_0$, the particle will be reflected by the potential; its velocity will change sign (and vary temporarily while the particle is moving inside the support of V). The scattering map thus is of the form

$$\begin{pmatrix} x_- \\ v_- \end{pmatrix} \mapsto \begin{pmatrix} x_+(x_-, v_-) \\ -v_- \end{pmatrix}.$$

For $E > E_0$ the particle moves through and we expect a time delay compared with the free dynamics and again a temporary change of the velocity such that finally $v_+ = v_-$. The scattering map now has the form

$$\begin{pmatrix} x_- \\ v_- \end{pmatrix} \mapsto \begin{pmatrix} x_+(x_-, v_-) \\ v_- \end{pmatrix}.$$

In the case $E = E_0$ the particle stops at x_0 with $V(x_0) = E_0$ (if this point is reached in finite time).

Next, let us suppose that $\operatorname{supp} V$ is not compact but that V and $\operatorname{grad} V$ are sufficiently small for $|x| \to \infty$. Then we expect that the position of the particle will not exactly but asymptotically be of the form $x_\pm + tv_\pm$, i.e.

$$\exists (x_\pm, v_\pm) \in \mathbb{R}^6\colon |x(t) - x_\pm - tv_\pm| \to 0, \quad t \to \pm\infty. \tag{1.2}$$

Observe that the potential must indeed be very small at $\pm\infty$; even for the Coulomb potential, the particle will not be asymptotically free.

Let us now consider a one-dimensional particle with positive energy E in a force field with the potential

$$V(x) = C(1 + |x|)^{-\alpha}, \quad \alpha > 0, \quad C \ne 0.$$

We show that the particle moves asymptotically free in the sense of (1.2) if $\alpha > 1$. For simplicity, let $m = 2$. It suffices to consider the case $t \to +\infty$. We can choose an initial condition such that $x(t) \to \infty$, $t \to \infty$. By conservation of energy, $\dot{x}(t)^2 = E - V(x(t))$ so that

$$\dot{x}(t) = \sqrt{E - V(x(t))} \to \sqrt{E}, \quad t \to +\infty. \tag{1.3}$$

In particular, $\dot{x}(t)$ does not change sign for large t and hence $\dot{x}(t) > 0$.

(1) Let $\alpha > 1$; here, it suffices to suppose that $|V(x)| \le C(1+|x|)^{-\alpha}$. By (1.3), there are constants $t_0 \in \mathbb{R}$, $x_1 \in \mathbb{R}$ and $v_1 > 0$ such that for $t \ge t_0$

$$\begin{aligned} v_1 &\le \dot{x}(t) \text{ and} \\ x_1 + v_1 t &\le x(t). \end{aligned}$$

For any $y > 0$, the mean value theorem ensures the existence of $\eta \in (0, y)$ such that

$$\sqrt{E-y} = \sqrt{E-0} + y\left[\frac{\mathrm{d}}{\mathrm{d}s}\sqrt{E-s}\right]_{s=\eta} = \sqrt{E} - \frac{y}{2\sqrt{E-\eta}}.$$

For sufficiently large t, $E - V(x(t))$ is bounded away from zero so that there is a function $\eta(t)$ satisfying $|\eta(t)| \le c(1+t)^{-\alpha}$ for some constant $c > 0$ and

$$\begin{aligned} \dot{x}(t) &= \sqrt{E - V(x(t))} = \sqrt{E} + \eta(t) \text{ and} \\ x(t) &= x_0 + \sqrt{E}\, t + \int_{t_0}^{t} \eta(\tau)\,\mathrm{d}\tau \\ &= x_0 + \sqrt{E}\, t + \int_{t_0}^{\infty} \eta(\tau)\,\mathrm{d}\tau - \int_{t}^{\infty} \eta(\tau)\,\mathrm{d}\tau \\ &= \tilde{x}_0 + \sqrt{E}\, t + \tilde{\eta}(t) \end{aligned}$$

with $|\tilde{\eta}(t)| \le \tilde{c}\, t^{1-\alpha}$. This is the desired result with $x_+ = \tilde{x}_0$ and $v_+ = \sqrt{E}$.

(2) Assume that $\alpha \le 1$. By (1.3) there are $c_1, c_2 > 0$ (for $C > 0$) or $c_1, c_2 < 0$ (for $C < 0$) so that for $\alpha < 1$ and large t

$$\begin{aligned} \dot{x}(t) &= \sqrt{E - V(x(t))} \begin{cases} \le \sqrt{E} - c_1(1+t)^{-\alpha} \\ \ge \sqrt{E} - c_2(1+t)^{-\alpha} \end{cases} \quad \text{and} \\ x(t) &\begin{cases} \le \sqrt{E}\, t + c_3 - \frac{c_1}{1-\alpha}(1+t)^{1-\alpha} \\ \ge \sqrt{E}\, t + c_4 - \frac{c_2}{1-\alpha}(1+t)^{1-\alpha}. \end{cases} \end{aligned}$$

For $\alpha = 1$, the term $\frac{1}{1-\alpha}(1+t)^{1-\alpha}$ has to be replaced by $\ln(1+t)$. Let us assume for a contradiction that (1.2) holds true. Let $\alpha < 1$. Then, for sufficiently large t,

$$\sqrt{E}\, t + c_4 - \frac{c_2}{1-\alpha}(1+t)^{1-\alpha} - 1 \le x_+ + v_+ t \le \sqrt{E}\, t + c_3 - \frac{c_1}{1-\alpha}(1+t)^{1-\alpha} + 1.$$

Dividing by t and computing the limit $t \to \infty$, this implies that $\sqrt{E} \le v_+ \le \sqrt{E}$, i.e. $v_+ = \sqrt{E}$. Hence $c_1 = 0$, a contradiction. Similarly, a contradiction is obtained in the case $\alpha = 1$.

Assume that the particle is asymptotically free in the sense of (1.2). The maps

$$\Omega_\pm \colon (x_\pm, v_\pm) \mapsto (x(t), v(t))$$

have an important analogy in the quantum mechanical setting where they are called the *(Møller)-wave operators.* The scattering map thus satisfies

$$S = \Omega_+^{-1}\Omega_-.$$

It describes the process of scattering without comprising the time-dependent details of the event. One of the most important and most difficult problems in scattering theory yet is the *inverse problem*: given the scattering map S what can we say about the scattering center or the potential respectively?

Concerning *Coulomb scattering*, we will observe a similar phenomenon in the quantum mechanical setting: the wave operators exist in general only for potentials that decay faster than the Coulomb potential.

Finally, to make contact with physical experiments, we comment briefly on the notions *cross section* and *scattering angles*: A beam of constant energy is sent towards a target. The beam has a wide spread and an approximately uniform density ρ of particles per unit area of the plane $\mathbb{R}^2$ orthogonal to the beam. A detector sits at some scattering angle (ϑ, φ) far away from the target and collects (and counts) all particles that leave the target within some angular region of size $\Delta\Omega$ about (ϑ, φ). The measured quantity is

$$\frac{\text{number of particles hitting the detector}}{(\Delta\Omega)\rho}.$$

If $\Delta\Omega$ is very small and the detector and source of particles are very far from the target, this quantity is called the *differential cross section.* The integral of the differential cross section over all spatial directions yields the *total cross section.*

Chapter 2

Basic principles of scattering in Hilbert spaces

We begin with a brief overview about some relevant aspects of spectral theory in Hilbert spaces. For more details, we refer to [K-I, K-II] and of course [RS-I].

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of bounded operators on the Hilbert space $\mathcal{H}$, $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$, and let $A \in \mathcal{L}(\mathcal{H})$ be given. We say:

$$\begin{aligned} A_n \to A \textit{ weakly} \quad &:\Longleftrightarrow \quad \langle A_n f, g\rangle \to \langle Af, g\rangle\,,\ \forall f, g \in \mathcal{H}, \\ A_n \to A \textit{ strongly} \quad &:\Longleftrightarrow \quad A_n f \to Af,\ \forall f \in \mathcal{H}, \\ A_n \to A \textit{ in norm} \quad &:\Longleftrightarrow \quad \|A_n - A\| = \sup\{\|A_n f - Af\|\,;\ \|f\| \le 1\} \to 0. \end{aligned}$$

Clearly, norm convergence $\Longrightarrow$ strong convergence $\Longrightarrow$ weak convergence. For the purposes of scattering theory, we will see that strong convergence is the appropriate notion.

Lemma 2.1. *Assume that $A_n, B_n, A, B \in \mathcal{L}(\mathcal{H})$, $n \in \mathbb{N}$, and that $A_n \to A$ strongly and $B_n \to B$ strongly. Then $A_nB_n \to AB$ strongly.*

Proof. By the Uniform Boundedness Principle, there exists a constant $c \ge 0$ with $\|A_n\| \le c$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned} \|A_nB_nf - ABf\| &\le \|A_nBf - ABf\| + \|A_nB_nf - A_nBf\| \\ &\le \|(A_n - A)Bf\| + \|A_n\|\,\|(B_n - B)f\| \\ &\to 0 \end{aligned}$$

as $n \to \infty$. □

Let $\mathcal{H}$ be a Hilbert space and let $A\colon D(A) \to \mathcal{H}$ be a self-adjoint operator. By the spectral theorem, there exists a unique spectral family $(E(\lambda))_{\lambda\in\mathbb{R}}$ such that $A = \int_{\mathbb{R}} \lambda \,\mathrm{d}E(\lambda)$. The operators $E(\lambda)$ are bounded, $E(\lambda) \in \mathcal{L}(\mathcal{H})$ for any $\lambda \in \mathbb{R}$, and projections, $E(\lambda)^2 = E(\lambda) = E(\lambda)^*$ for any $\lambda \in \mathbb{R}$, and satisfy the following properties:

(i) Monotonicity: $\lambda \leq \mu \implies E(\lambda) \leq E(\mu)$.

(ii) Strong right continuity: $\forall \lambda \in \mathbb{R}\, \forall f \in \mathcal{H}\colon E(\lambda + \varepsilon)f \to E(\lambda)f$, $\varepsilon \downarrow 0$.

(iii) For all $f \in \mathcal{H}$, we have that $E(\lambda)f \to f$, $\lambda \to \infty$, and $E(\lambda)f \to 0$, $\lambda \to -\infty$.

For any $\varphi, \psi \in \mathcal{H}$, the sesquilinear form $\langle A\varphi, \psi\rangle$ is the Riemann-Stieltjes integral

$$\langle A\varphi, \psi\rangle = \int_{\mathbb{R}} \lambda \,\mathrm{d}\, \langle E(\lambda)\varphi, \psi\rangle \,. \tag{2.1}$$

The function $\lambda \mapsto \langle E(\lambda)\varphi, \varphi\rangle = \|E(\lambda)\varphi\|^2$ is the spectral measure μ_φ associated with the vector φ. The spectral theorem also says that given a spectral family $(E(\lambda))_{\lambda\in\mathbb{R}}$, there exists a unique self-adjoint operator A such that $A = \int_{\mathbb{R}} \lambda \,\mathrm{d}E(\lambda)$. In fact, the domain of integration in (2.1) is $\sigma(A) \subset \mathbb{R}$, the spectrum of A, as $E(\cdot)$ is locally constant on the resolvent set $\rho(A)$.

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