Georg-August-Universität Göttingen
Mathematisches Institut
PD. Dr. Martin Kohlmann

Sommersemester 2020
Mittwoch, 17.00-18.30 Uhr, HS 1
Beginn: 15.04.20

Seminar on
Geometric Mechanics and Symmetry

Classical mechanics, one of the oldest branches of science, has undergone a long evolution, developing hand in hand with many areas of mathematics, including calculus, differential geometry and the theory of Lie groups and Lie algebras. The modern formulations of Lagrangian and Hamiltonian mechanics, in the coordinate-free language of differential geometry, are elegant and general. They provide a unifying general framework for many seemingly disparate physical systems as rigid bodies and fluid flow. Symmetries of mechanical systems are represented mathematically by Lie group actions. The presence of a symmetry allows for a reduction in the number of dimensions of a finite-dimensional conservative mechanical system in two basic ways: by grouping together equivalent states and by exploiting conserved quantities (momentum maps) associated with the symmetry. The rigid body motion recast as a geodesic flow on the rotation group will serve as a main example to illustrate the results of the first part of the seminar.

The second part of the seminar treats what might be considered the infinite-dimensional version of the rigid body by a suitable change of the underlying configuration space. The key idea goes back to Arnold who observed in 1966 in a famous paper [1] that Euler ideal fluid motion may be identified with geodesic flow on the volume-preserving diffeomorphisms with a metric defined by the fluid's kinetic energy. This approach had later been developed in a rigorous analytical setting by Ebin and Marsden [2] in 1970. Roughly speaking, passing from finite to infinite dimensions in geometric mechanics means replacing matrix multiplication by the composition of smooth and invertible functions. Nonlinear partial differential equations as the Camassa-Holm (CH) and Degasperis-Procesi (DP) equations [3, 4] are some of the recently discussed models for shallow water flow which are suitable for the geometric approach presented here so that the seminar will lead to an active area of mathematical research.

Schedule:

1. Introduction to Lagrangian and Hamiltonian mechanics: [5, Ch. 1.1–1.4]
2. Recap of manifolds and matrix groups: [5, Ch. 2]
3. Riemannian and symplectic geometry on manifolds: [5, Ch. 3]
4. Lagrangian and Hamiltonian mechanics on manifolds: [5, Ch. 4]
5. Lie groups, Lie algebras and group actions: [5, Ch. 5–6.2]
6. Momentum maps: [5, Ch. 8]
7. The rigid body motion as a geodesic flow on $SO(3)$: [5, Ch. 1.5, 7.1], [6, Ch. 1, Par. 1–4]

8. The geometric setting of ideal continuum motion: [5, Ch. 11.1–11.2], [6, Ch. 1, Par. 5–6], [7], [8, Ch. 1–2]

9. The Burgers equation in the geometric picture: [8, Ch. 3]

10. The Camassa-Holm equation as a geodesic flow on the diffeomorphism group: [8, Ch. 4], [9]

11. The Camassa-Holm equation is bi-Hamiltonian and isospectral: [3], [5, Ch. 13], [10]

12. The geodesic flow for the periodic b-equation: [11]

The topic assignment together with a short introduction into the issues of the talks will be carried out during a **pre-meeting on Wednesday, 05.02.2020, 17.30 Uhr (s.t.) in Hörsaal 1**. If you are prevented from attending the pre-meeting, you can also apply for a talk by e-mail (`mkohlma@gwdg.de`). Please understand that I cannot offer talks after Wednesday, 08.04.20 for organisational reasons as the seminar will begin on Wednesday, 15.04.20.

Bibliography

[1] V.I. Arnold: Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. Ann. Inst. Fourier (Grenoble) **16**, 319–361 (1966)

[2] D.G. Ebin and J.E. Marsden: Groups of diffeomorphisms and the motion of an incompressible fluid. Ann. of Math. **92**, no. 2, 102–163 (1970)

[3] R. Camassa and D.D. Holm: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. **71**, no. 11, 1661–1664 (1993)

[4] A. Degasperis, D.D. Holm, and A.N.W. Hone: A new integrable equation with peakon solutions. Teoret. Mat. Fiz. **133**, no. 2, 170–183 (2002)

[5] D.D. Holm, T. Schmah, S. Stoica: Geometric Mechanics and Symmetry. From Finite to Infinite Dimensions. Oxford Texts in Applied and Engineering Mathematics. Oxford University Press 2009

[6] V.I. Arnold, B.A. Keshin: Topological Methods in Hydrodynamics. Applied Mathematical Sciences Vol. 125. Springer, 1998

[7] B. Kolev: Lie Groups and Mechanics: An Introduction. J. Nonlinear Math. Phys. **11**, no. 4, 480–498 (2004)

[8] A. Constantin, B. Kolev: On the geometric approach to the motion of inertial mechanical systems. J. Phys. A: Math. Gen. **35** (2002) R51–R79

[9] J. Lenells: Riemannian geometry on the diffeomorphism group of the circle. Ark. Mat. **45**, 297–325 (2007)

[10] A. Constantin: The Hamiltonian structure of the Camassa-Holm equation. Expo. Math. **15**, 053–085 (1997)

[11] J. Escher, J. Seiler: The periodic b-equation and Euler equations on the circle. J. Math. Phys. **51**(5) 053101, 6 (2010)