Schrödinger Operators and their Spectra

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Preface

This lecture begins with a brief overview about the spectral theorem and its consequences for the spectrum of self-adjoint operators in Hilbert spaces. The key results are stated mainly without proofs to allow for a quick entry into the relevant aspects of spectral theory. Then our main goal is to study the spectrum of several classes of Schrödinger operators and to look at some important examples occurring in mathematical physics (e.g. the harmonic oscillator or the hydrogen atom). Searching for solutions of the IVP for the Schrödinger equation, we will discuss and prove Stone’s theorem on strongly continuous unitary one-parameter groups. Finally, we will look at spectral measures that allow for a characterization and a decomposition of the spectrum of self-adjoint operators and the Hilbert space itself. The lecture will end with an outlook concerning some aspects of quantum scattering theory.
Chapter 1

Overview: The spectral theorem and the spectrum of self-adjoint operators in Hilbert space

Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$. An operator $P \in \mathcal{L}(\mathcal{H})$ is called (orthogonal) projection if $P^2 = P = P^*$. For symmetric operators $A, B \in \mathcal{L}(\mathcal{H})$, we write $A \leq B$ if

$$\langle Au, u \rangle \leq \langle Bu, u \rangle, \quad \forall u \in \mathcal{H}.$$ 

For two projections $P$ and $Q$,

$$P \leq Q \iff R(P) \subset R(Q) \iff PQ =QP = P.$$ 

We als comment on different notions of convergence of bounded operators: Let $(A_n)_{n \in \mathbb{N}} \in \mathcal{L}(\mathcal{H})$ be a sequence of bounded operators and let $A \in \mathcal{L}(\mathcal{H})$.

(i) **Norm convergence:** $\|A_n - A\| \to 0$, $n \to \infty$, i.e.

$$\sup\{\|A_n f - Af\|; \|f\| \leq 1\} \to 0, \quad n \to \infty.$$ 

(ii) **Strong convergence:** $\forall f \in \mathcal{H}: A_n f \to Af$, $n \to \infty$.

(iii) **Weak convergence:** $\forall f, g \in \mathcal{H}: \langle A_n f, g \rangle \to \langle Af, g \rangle$, $n \to \infty$.

Note: Norm convergence $\Longrightarrow$ strong convergence $\Longrightarrow$ weak convergence.

**Definition 1.1.** Let $(E(\lambda))_{\lambda \in \mathbb{R}} \subset \mathcal{L}(\mathcal{H})$ be a family of projections with the following properties:

(i) **Monotonicity:** $\lambda \leq \mu \implies E(\lambda) \leq E(\mu)$.

(ii) **Strong right continuity:** $\forall \lambda \in \mathbb{R} \forall f \in \mathcal{H}: E(\lambda + \varepsilon)f \to E(\lambda)f$, $\varepsilon \downarrow 0$. 


(iii) For all $f \in \mathcal{H}$, we have that $E(\lambda)f \to f$, $\lambda \to \infty$, and $E(\lambda)f \to 0$, $\lambda \to -\infty$. Then $(E(\lambda))_{\lambda \in \mathbb{R}} \subset \mathcal{L}(\mathcal{H})$ is called a spectral family.

**Remark 1.2.** Why do we need strong convergence in (ii) and (iii)?

1. Weak convergence + monotonicity imply strong convergence.
2. Norm convergence + monotonicity of projections imply constance.

**Remark 1.3.** For any spectral family $(E(\lambda))_{\lambda \in \mathbb{R}}$ there also exists the strong limit from the left at $\lambda \in \mathbb{R}$,

$$E(\lambda - 0)f := \lim_{\varepsilon \downarrow 0} E(\lambda - \varepsilon)f, \quad \forall f \in \mathcal{H}.$$  

It is easy to see that $E(\lambda - 0)$ is a projection. It is possible that $E(\lambda - 0) \neq E(\lambda)$.

**Example 1.4.** Let $\mathcal{H} = L_2(\mathbb{R})$ and let $E(\lambda) = \chi_{(-\infty, \lambda]}(x)$ be multiplication with the characteristic function for the interval $(-\infty, \lambda]$. Then $(E(\lambda))_{\lambda \in \mathbb{R}}$ is a spectral family.

**Example 1.5.** Let $A \in \mathcal{L}(\mathcal{H})$ be symmetric and compact with $\text{dim } R(A) = 1$, the eigenvalues $\lambda_n \in \mathbb{R} \setminus \{0\}$ and an orthonormal basis $(u_n)_{n \in \mathbb{N}}$ of $R(A)$ with $Au_n = \lambda_n u_n$ for $n \in \mathbb{N}$. Let

$$E(\lambda) := \sum_{\lambda_n \leq \lambda} \langle \cdot, u_n \rangle u_n, \quad \lambda < 0,$$

$$E(\lambda) := P_{N(A)} + \sum_{\lambda_n \leq \lambda} \langle \cdot, u_n \rangle u_n, \quad \lambda \geq 0.$$  

Then $(E(\lambda))_{\lambda \in \mathbb{R}}$ is a spectral family.

Let $m: \mathbb{R} \to \mathbb{R}$ be monotonically increasing and right continuous. For $\varphi \in C_c(\mathbb{R})$ (i.e. $\varphi$ is continuous and $\text{supp } \varphi$ is compact, $\text{supp } \varphi \subset (-R, R)$ for some $R > 0$), we define the Riemann-Stieltjes integral

$$\int_{-\infty}^{\infty} \varphi(x) \, dm(x) := \lim_{n \to \infty} \sum_{i=1}^{n} \varphi(x_i)[m(x_{i+1}) - m(x_i)];$$

the points $x_i, i = 1, \ldots, n+1$, are an equidistant partition of $(-R, R)$ with $x_i < x_{i+1}$ and $x_1 = -R$, $x_{n+1} = R$.

For any fixed $f \in \mathcal{H}$, the function

$$\mathbb{R} \to [0, \infty), \quad \lambda \mapsto \langle E(\lambda)f, f \rangle$$

is monotonically non-decreasing and right continuous. For $\varphi \in C_c(\mathbb{R})$ with $\text{supp } \varphi \subset (-R, R)$, the limit

$$\int_{\mathbb{R}} \varphi(\lambda) \, d\langle E(\lambda)f, f \rangle := \lim_{n \to \infty} \sum_{j=1}^{n} \varphi(\lambda_j)[\langle E(\lambda_{j+1})f, f \rangle - \langle E(\lambda_j)f, f \rangle]$$

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exists; again the points \( \lambda_j, j = 1, \ldots, n + 1 \), are an equidistant partition of \((-R, R)\) with \( \lambda_j < \lambda_{j+1} \) and \( \lambda_1 = -R, \lambda_{n+1} = R \). For this Riemann-Stieltjes integral, we use the notation
\[
\int \varphi(\lambda) \, d\mu_f(\lambda) := \int \varphi(\lambda) \, d(\langle E(\lambda)f, f \rangle).
\]
We also say that the function \( \lambda \mapsto \langle E(\lambda)f, f \rangle \) generates the Riemann-Stieltjes measure (or Lebesgue-Stieltjes measure) \( \mu_f \).

Given a spectral family \( (E(\lambda))_{\lambda \in \mathbb{R}} \), we now look for a self-adjoint operator \( H \) so that
\[
H = \int \lambda \, dE(\lambda)
\]
in a suitable sense. For this purpose, we first define the domain
\[
\mathcal{D} := \left\{ f \in \mathcal{H}; \int \lambda^2 \, d\mu_f(\lambda) < \infty \right\} = \left\{ f \in \mathcal{H}; \lim_{R \to \infty} \int_{-R}^{R} \lambda^2 \, d\langle E(\lambda)f, f \rangle < \infty \right\}.
\]
(1.1)

For \( f \in \mathcal{D} \) and \( g \in \mathcal{H} \) one shows that
\[
\left| \int_{-\infty}^{\infty} \lambda \, d\langle E(\lambda)f, g \rangle \right|^2 \leq C_f \| g \|^2
\]
with a constant \( C_f \geq 0 \). For all \( f \in \mathcal{D} \),
\[
\mathcal{H} \to \mathbb{C}, \quad g \mapsto \int \lambda \, d\langle E(\lambda)f, g \rangle
\]
is a continuous anti-linear functional on \( \mathcal{H} \). By the Riesz representation theorem, there is \( w \in \mathcal{H}, w = w_f \), such that
\[
\langle w, g \rangle = \int \lambda \, d\langle E(\lambda)f, g \rangle, \quad \forall g \in \mathcal{H}.
\]
We now define
\[
Hf := w_f, \quad \forall f \in \mathcal{D},
\]
i.e. \( H : \mathcal{D} \to \mathcal{H} \) is linear and
\[
\langle Hf, g \rangle = \int \lambda \, d\langle E(\lambda)f, g \rangle, \quad \forall g \in \mathcal{H}.
\]
(1.2)

One shows that:

(1) \( \mathcal{D} \subset \mathcal{H} \) is dense.

(2) \( H : \mathcal{D} \to \mathcal{H} \) is symmetric.

(3) \( H \pm i : \mathcal{D} \to \mathcal{H} \) is surjective.
This provides a proof of the following theorem.

**Theorem 1.6.** Given a spectral family \((E(\lambda))_{\lambda \in \mathbb{R}}\) there exists a unique self-adjoint operator \(H\) such that
\[
H = \int \lambda \, dE(\lambda)
\]
in the sense of (1.1) and (1.2).

Contrariwise but much more difficult to prove we note the following theorem.

**Theorem 1.7.** Let \(H : D(H) \to \mathcal{H}\) be a self-adjoint operator in the Hilbert space \(\mathcal{H}\). Then there is a unique spectral family \((E(\lambda))_{\lambda \in \mathbb{R}}\) such that
\[
H = \int \lambda \, dE(\lambda),
\]
i.e. the operator obtained for \((E(\lambda))_{\lambda \in \mathbb{R}}\) in Theorem 1.6 equals \(H\).

**Remark 1.8.** Theorem 1.6 and Theorem 1.7 are the Spectral Theorem for self-adjoint operators in Hilbert space. This yields a “diagonalization” of \(H\), in analogy to the principal axis transformation for symmetric matrices.

**Definition 1.9.** Let \(T : D(T) \to \mathcal{H}\) be densely defined and let \(A \in \mathcal{L}(H)\). We say that \(A\) commutes with \(T\) if \(Au \in D(T)\) for all \(u \in D(T)\) and if
\[
[A, T]u := ATu - TAu = 0, \quad \forall u \in D(T).
\]

**Theorem 1.10.** Let \(H : D(H) \to \mathcal{H}\) be self-adjoint, let \((E(\lambda))_{\lambda \in \mathbb{R}}\) be the associated spectral family and let \(A \in \mathcal{L}(H)\). Then:
\[
[A, H] = 0 \iff [A, E(\lambda)] = 0, \quad \forall \lambda \in \mathbb{R}.
\]

**Theorem 1.11.** Let \(H : D(H) \to \mathcal{H}\) be self-adjoint and let \(M \subset \mathcal{H}\) be a closed subspace with projection \(P\). We assume that \([P, H] = 0\) and that there is \(\lambda_0 \in \mathbb{R}\) such that \(\langle Hu, u \rangle \leq \lambda_0 \|u\|^2\) for all \(u \in M \cap D(H)\) and \(\langle Hu, u \rangle > \lambda_0 \|u\|^2\) for all \(0 \neq u \in M^\perp \cap D(H)\). Then \(P = E(\lambda_0)\).

An important application of the spectral theorem for self-adjoint operators in Hilbert space is the option to study functions of operators: For certain classes of functions \(f\), one studies
\[
f(H) := \int f(\lambda) \, dE(\lambda)
\]
with the domain
\[
D(f(H)) := \left\{ u \in \mathcal{H}; \int |f(\lambda)|^2 \, d \langle E(\lambda)u, u \rangle < \infty \right\}.
\]
We will see that \( e^{-itH}, \ t \in \mathbb{R}, \) generates a strongly continuous group of unitary operators and that \( u(t) := e^{-itH}u_0 \) solves the Schrödinger equation provided \( H = -\Delta + V \) is self-adjoint. On the other hand, \( e^{-itH}, \ t \geq 0, \) \( H \geq 0, \) is a strongly continuous semi-group of operators and \( v(t) := e^{-itH}v_0 \) is a solution to the heat equation provided \( H \) is a self-adjoint extension of \( -\Delta. \) Characteristic functions \( \chi_{(a,b)}(H) = E((a,b)) = E(b) - E(a) \) yield spectral projections associated with intervals. Another application is the square root of a non-negative operator.

**Theorem 1.12.** Let \( H \geq 0 \) be self-adjoint with the spectral family \( (E(\lambda))_{\lambda \in \mathbb{R}}. \) We define an operator \( T \) by setting

\[
D(T) := \left\{ u \in \mathcal{H}; \int_0^\infty \lambda \, d \langle E(\lambda)u, v \rangle < \infty \right\}
\]

and

\[
T := \int_0^\infty \sqrt{\lambda} \, dE(\lambda),
\]

i.e.

\[
\langle Tu, v \rangle := \int_0^\infty \sqrt{\lambda} \, d \langle E(\lambda)u, v \rangle, \quad \forall u \in D(T), \forall v \in \mathcal{H}.
\]

Then \( T \) is a non-negative self-adjoint operator with \( T^2 = H \) and \( T \) is a square root of \( H, \) denoted as \( T = \sqrt{H}. \) The (non-negative) square root of \( H \) is unique.

Given a self-adjoint operator \( H \) in the Hilbert space \( \mathcal{H} \) with the associated spectral family \( (E(\lambda))_{\lambda \in \mathbb{R}}, \) we now focus on the characterization of the spectrum \( \sigma(H) \) with the aid of the properties of the \( E(\lambda). \) First of all we recall the definition of the spectrum of some closed operator.

(1) **Spectrum and resolvent set.** Let \( T: D(T) \to \mathcal{H} \) be closed. We define the resolvent set \( \rho(T) \) by

\[
\rho(T) := \left\{ z \in \mathbb{C}; (T-z): D(T) \to \mathcal{H} \text{ bijective}, (T-z)^{-1} \in \mathcal{L}(\mathcal{H}) \right\}
\]

\[
= \left\{ z \in \mathbb{C}; (T-z): D(T) \to \mathcal{H} \text{ bijective} \right\}.
\]

For a closed operator \( T: D(T) \to \mathcal{H}, \) we call

\[
\sigma(T) := \mathbb{C} \setminus \rho(T)
\]

the spectrum of \( T. \)

(2) **Point spectrum and continuous spectrum.** Let \( \sigma_p(T) \) be the point spectrum of \( T \) given by the set of eigenvalues of \( T, \) i.e.

\[
\lambda \in \sigma_p(T) \iff N(T - \lambda) \neq \{0\},
\]
and let \( \sigma_{\text{cont}}(T) = \sigma(T) \setminus \sigma_p(T) \) be the \textit{continuous spectrum of} \( T \). Trivially,

\[
\sigma(T) = \sigma_p(T) \cup \sigma_{\text{cont}}(T) \quad \text{(disjoint union)}.
\]

A decomposition of this type holds in particular for self-adjoint operators, as for self-adjoint operators the residual spectrum is empty.

(3) \textbf{Discrete spectrum and essential spectrum.} Let \( H : D(H) \to \mathcal{H} \) be self-adjoint. We define \( \sigma_{\text{disc}}(H) \), the \textit{discrete spectrum of} \( H \), as the set of eigenvalues of \( H \) having finite multiplicity and being isolated points of the spectrum. In other words, \( \lambda \in \sigma_{\text{disc}}(H) \) if and only if \( 0 < \dim N(H - \lambda) < \infty \) and if there is \( \varepsilon > 0 \) with the property \( \sigma(H) \cap (\lambda - \varepsilon, \lambda + \varepsilon) = \{ \lambda \} \). We define \( \sigma_{\text{ess}}(H) \), the \textit{essential spectrum of} \( H \), by

\[
\sigma_{\text{ess}}(H) := \sigma(H) \setminus \sigma_{\text{disc}}(H).
\]

We thus have the disjoint decomposition

\[
\sigma(H) = \sigma_{\text{disc}}(H) \cup \sigma_{\text{ess}}(H).
\]

Obviously, \( \sigma_{\text{ess}}(H) \) consists of all accumulation points of \( \sigma(H) \) and all eigenvalues of infinite multiplicity. In particular, \( \sigma_{\text{ess}}(H) \) is a closed subset of \( \mathbb{R} \) whereas \( \sigma_{\text{cont}}(H) \) is not necessarily closed. We will show later that \( \sigma_{\text{ess}}(H) \) is invariant under perturbations by symmetric and compact operators.

\textbf{Theorem 1.13.} Let \( H \) be a self-adjoint operator in the Hilbert space \( \mathcal{H} \) with the spectral family \( (E(\lambda))_{\lambda \in \mathbb{R}} \).

1. For \( \zeta \in \mathbb{R} \),

\[
\zeta \in \rho(H) \iff \exists \varepsilon > 0 : E(\zeta - \varepsilon) = E(\zeta + \varepsilon).
\]

2. For \( \zeta \in \rho(H) \),

\[
\| (H - \zeta)^{-1} \| = \frac{1}{\text{dist}(\zeta, \sigma(H))}.
\]

3. We have that

\[
H \geq 0 \iff E(\lambda) = 0, \quad \forall \lambda < 0.
\]

\textbf{Proof.} To prove “\( \iff \)” in (1), let \( \varepsilon > 0 \) with \( E(\zeta - \varepsilon) = E(\zeta + \varepsilon) \). Then

\[
R_\zeta := \int_{-\infty}^{\infty} (\lambda - \zeta)^{-1} dE(\lambda) \in \mathcal{L}(\mathcal{H})
\]

with \( \| R_\zeta \| \leq \varepsilon^{-1} \). It is easy to see that \( (H - \zeta)R_\zeta = I \) and \( R_\zeta(H - \zeta) = I\big|_{D(H)} \).

“\( \Longrightarrow \)” : We assume that \( E(\zeta - \varepsilon) \neq E(\zeta + \varepsilon) \) for any \( \varepsilon > 0 \) and choose for any \( \varepsilon > 0 \)
a function \( u_\varepsilon \in R(E(\zeta + \varepsilon) - E(\zeta - \varepsilon)) = R(E(\zeta + \varepsilon)) \cap R(E(\zeta - \varepsilon)) \) with \( \|u_\varepsilon\| = 1 \). Then \( u_\varepsilon \in D(H) \) with
\[
\|(H - \zeta)u_\varepsilon\|^2 = \int_{\zeta - \varepsilon}^{\zeta + \varepsilon} |\lambda - \zeta|^2 \, d\langle E(\lambda)u_\varepsilon, u_\varepsilon \rangle \leq \varepsilon^2 \|u_\varepsilon\|^2.
\]
Hence \( H - \zeta \) cannot be inverted continuously so that \( \zeta \notin \rho(H) \). To prove (2), we use that
\[
\|(H - \zeta)^{-1}f\|^2 = \int_{-\infty}^{\infty} |\lambda - \zeta|^{-2} \, d\langle E(\lambda)f, f \rangle, \quad \forall f \in \mathcal{H},
\]
and conclude
\[
\|(H - \zeta)^{-1}\| \leq \frac{1}{\text{dist}(\zeta, \sigma(H))}.
\]
As \( \sigma(H) \) is closed, given \( \zeta \in \rho(H) \), we can find some \( \lambda_0 \in \sigma(H) \) such that
\[
|\lambda_0 - \zeta| = \text{dist}(\zeta, \sigma(H)).
\]
From (1) we know that given \( \varepsilon > 0 \) there is \( 0 \neq u_\varepsilon \in R(E(\lambda_0 + \varepsilon) - E(\lambda_0 - \varepsilon)) \). Hence
\[
\|(H - \zeta)^{-1}u_\varepsilon\|^2 = \int_{\lambda_0 - \varepsilon}^{\lambda_0 + \varepsilon} |\lambda - \zeta|^{-2} \, d\langle E(\lambda)u_\varepsilon, u_\varepsilon \rangle \\
\quad \geq \int_{\lambda_0 - \varepsilon}^{\lambda_0 + \varepsilon} (|\lambda_0 - \zeta| + \varepsilon)^{-2} \, d\langle E(\lambda)u_\varepsilon, u_\varepsilon \rangle \\
\quad = (|\lambda_0 - \zeta| + \varepsilon)^{-2} \|u_\varepsilon\|^2,
\]
as \( |\lambda - \zeta| \leq |\lambda_0 - \zeta| + |\lambda - \lambda_0| \leq |\lambda_0 - \zeta| + \varepsilon \). Part (3) is trivial.

We now show that the discontinuities of a spectral family correspond to the point spectrum of the associated self-adjoint operator whereas the strong continuity of the \( E(\lambda) \) at \( \lambda_0 \in \sigma(H) \) implies \( \lambda_0 \in \sigma_{\text{cont}}(H) \) (and vice versa).

**Theorem 1.14.** For \( \lambda_0 \in \sigma(H) \) we have that
\[
\lambda_0 \in \sigma_p(H) \iff E(\cdot) \text{ is not strongly continuous at } \lambda_0,
\]
and
\[
\lambda_0 \in \sigma_{\text{cont}}(H) \iff E(\cdot) \text{ is strongly continuous at } \lambda_0.
\]

**Proof.** Obviously, \( E(\lambda) \) is strongly continuous at \( \lambda_0 \) if and only if \( E(\lambda_0 - 0) = E(\lambda_0) \).
For \( \lambda_0 \in \sigma_p(H) \) and \( u_0 \in N(H - \lambda_0) \) with \( \|u_0\| = 1 \),
\[
0 = \|(H - \lambda_0)u_0\|^2 = \int_{-\infty}^{\infty} (\lambda - \lambda_0)^2 \, d\langle E(\lambda)u_0, u_0 \rangle.
\]
Hence \( \langle E(\cdot)u_0, u_0 \rangle \) is constant for \( \lambda < \lambda_0 \) and \( \lambda > \lambda_0 \), i.e. \( \langle E(\lambda)u_0, u_0 \rangle = 0 \) for \( \lambda < \lambda_0 \) and \( \langle E(\lambda)u_0, u_0 \rangle = 1 \) for \( \lambda > \lambda_0 \). Then \( E(\cdot) \) is not strongly continuous at
\( \lambda_0 \). On the contrary, assume that \( E(\cdot) \) is not strongly continuous at \( \lambda_0 \). Then there is \( u \in \mathcal{H} \) with \( \|u\| = 1 \) so that

\[
E(\lambda_0 - 0)u = 0, \quad E(\lambda_0)u = u,
\]
i.e. \( u \in R(E(\lambda_0 - 0))^\perp \cap R(E(\lambda_0)) = R(E(\lambda_0) - E(\lambda_0 - 0)) \), and hence

\[
\| (H - \lambda_0)u \|^2 = \int_{\lambda_0 - \varepsilon}^{\lambda_0} (\lambda - \lambda_0)^2 \langle E(\lambda)u, u \rangle = 0,
\]
i.e. \( \lambda_0 \in \sigma_p(H) \). \( \square \)

The following theorem characterizes the essential and the discrete spectrum of a self-adjoint operator with the aid of its spectral family.

**Theorem 1.15.** A number \( \lambda \in \mathbb{R} \) belongs to \( \sigma_{\text{disc}}(H) \) if and only if the following two properties are satisfied:

1. There is \( \varepsilon > 0 \) such that \( E(\cdot) \) is constant in \( (\lambda - \varepsilon, \lambda) \) and \([\lambda, \lambda + \varepsilon)\).
2. \( 0 < \dim R(E(\lambda) - E(\lambda) - 0)) < \infty \).

Moreover, \( \lambda \in \sigma_{\text{ess}}(H) \) if and only if \( \dim R(E(\lambda + \varepsilon) - E(\lambda - \varepsilon)) = \infty \) for any \( \varepsilon > 0 \).

**Proof.** The statement concerning \( \sigma_{\text{disc}}(H) \) is clear. If \( \lambda \in \sigma_{\text{ess}}(H) \), then \( \lambda \in \sigma(H) \) and this implies that \( E(\lambda - \varepsilon) \neq E(\lambda + \varepsilon) \) for any \( \varepsilon > 0 \). If \( \dim R(E(\lambda + \varepsilon) - E(\lambda - \varepsilon)) \) would be finite for some \( \varepsilon_0 > 0 \), then \( \lambda \in \sigma_{\text{disc}}(H) \). To prove the other direction, we assume for a contradiction that \( \dim R(E(\lambda + \varepsilon) - E(\lambda - \varepsilon)) = \infty \) for any \( \varepsilon > 0 \) and that \( \lambda \in \sigma_{\text{disc}}(H) \). By (1) we can find \( \eta > 0 \) so that \( E(\cdot) \) is constant in the intervals \( (\lambda - \eta, \lambda) \) and \([\lambda, \lambda + \eta)\). Our assumption implies that \( \dim R(E(\lambda) - E(\lambda - 0)) = \infty \) which contradicts the assumption \( \lambda \in \sigma_{\text{disc}}(H) \). \( \square \)

To characterize the essential spectrum of self-adjoint operators, *singular sequences* are useful tools.

**Definition 1.16.** Let \( H : D(H) \to \mathcal{H} \) be self-adjoint and let \( \lambda \in \mathbb{R} \). A sequence \( (u_n)_{n \in \mathbb{N}} \subset D(H) \) is called a *singular sequence for \( H \) and \( \lambda \)* if the following three properties are satisfied:

1. \( \|u_n\| = 1 \) or \( \liminf_{n \to \infty} \|u_n\| > 0 \),
2. \( (u_n)_{n \in \mathbb{N}} \) is a weak null sequence, i.e. \( u_n \xrightarrow{w} 0 \),
3. \( \|(H - \lambda)u_n\| \to 0 \).

Singular sequences are sequences of approximate eigenfunctions. We have the following important theorem.
Theorem 1.17. $\lambda \in \sigma_{\text{ess}}(H) \iff$ There is a singular sequence for $H$ and $\lambda$.

Proof. We write

$$H = \int_{-\infty}^{\infty} \lambda \, dE(\lambda)$$

and assume that $\lambda_0 \in \sigma_{\text{ess}}(H)$. By Theorem 1.15,

$$\dim R(E(\lambda_0 + \varepsilon) - E(\lambda_0 - \varepsilon)) = \infty, \quad \forall \varepsilon > 0.$$  

Let $u_1 \in R(E(\lambda_0 + 1) - E(\lambda_0 - 1))$ with $\|u_1\| = 1$ be given. Then $u_1 \in D(H)$ and

$$\|(H - \lambda_0)u_1\|^2 = \int_{\lambda_0 - 1}^{\lambda_0 + 1} (\lambda - \lambda_0)^2 \, d\langle E(\lambda)u_1, u_1 \rangle \leq 1.$$  

We then choose successively $u_k \in R(E(\lambda_0 + 1/k) - E(\lambda_0 - 1/k))$ with $\|u_k\| = 1$ and $\langle u_k, u_j \rangle = 0$ for all $j = 1, \ldots, k; k - 1$; this is possible as

$$\dim R(E(\lambda_0 + 1/k) - E(\lambda_0 - 1/k)) = \infty, \quad \forall k \in \mathbb{N},$$

and $\dim \text{span}\{u_1, \ldots, u_{k-1}\} < \infty$, i.e.

$$R(E(\lambda_0 + 1/k) - E(\lambda_0 - 1/k)) \cap \text{span}\{u_1, \ldots, u_{k-1}\}^\perp \neq \{0\}.$$  

Analogously, we get that $u_k \in D(H)$ with

$$\|(H - \lambda_0)u_k\| \leq k^{-1}.$$  

Hence $(u_k)_{k \in \mathbb{N}} \subset D(H)$ is a singular sequence for $H$ and $\lambda_0$. Contrariwise, we consider a singular sequence $(u_k)_{k \in \mathbb{N}}$ for $H$ and $\lambda_0$, i.e.

$$\|u_k\| = 1, \quad u_k \xrightarrow{k \to \infty} 0, \quad \|(H - \lambda_0)u_k\| \to 0.$$  

First, $\lambda_0 \in \sigma(H)$ since otherwise there would be $\eta > 0$ with $\|(H - \lambda_0)u\| \geq \eta \|u\|$ for all $u \in D(H)$. If $\lambda_0 \in \sigma_{\text{disc}}(H)$ then $E(\cdot)$ would be constant on the intervals $(\lambda_0 - \varepsilon_0, \lambda_0)$ and $[\lambda_0, \lambda_0 + \varepsilon_0]$ for some $\varepsilon_0 > 0$. Then the sequence $(u_k)_{k \in \mathbb{N}}$ satisfies

$$\|(H - \lambda_0)u_k\|^2 = \left( \int_{-\infty}^{\lambda_0 - \varepsilon_0} + \int_{\lambda_0 + \varepsilon_0}^{\lambda_0 + \varepsilon_0} + \int_{\lambda_0 + \varepsilon_0}^{\infty} \rangle \langle (\lambda - \lambda_0)^2 \, d\langle E(\lambda)u_k, u_k \rangle \right.$$  

$$\geq \varepsilon_0^2 \left( \int_{-\infty}^{\lambda_0 - \varepsilon_0} + \int_{\lambda_0 + \varepsilon_0}^{\infty} \right) \, d\langle E(\lambda)u_k, u_k \rangle$$

$$= \varepsilon_0^2 \int_{-\infty}^{\lambda_0 - \varepsilon_0} \, d\langle E(\lambda)u_k, u_k \rangle - \varepsilon_0^2 \int_{\lambda_0 + \varepsilon_0}^{\lambda_0 + \varepsilon_0} \, d\langle E(\lambda)u_k, u_k \rangle$$

$$= \varepsilon_0^2 \|u_k\|^2 - \varepsilon_0^2 \left( \langle E(\lambda_0 + \varepsilon_0)u_k, u_k \rangle - \langle E(\lambda_0 - \varepsilon_0)u_k, u_k \rangle \right).$$

By our assumption, $\dim R(E(\lambda_0) - E(\lambda_0 - 0)) < \infty$ and hence

$$\dim (R(E(\lambda_0 + \varepsilon_0) - E(\lambda_0 - \varepsilon_0)) < \infty.$$  

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Consequently $E(\lambda_0 + \varepsilon_0) - E(\lambda_0 - \varepsilon_0)$ is compact. As $u_k \xrightarrow{w} 0$ we get that

$$E(\lambda_0 + \varepsilon_0) u_k - E(\lambda_0 - \varepsilon_0) u_k \to 0 \text{ (strongly)}.$$ 

Thus

$$\liminf_{k \to \infty} \|(H - \lambda_0) u_k\|^2 \geq \varepsilon_0^2 \|u_k\|^2,$$

a contradiction. □
Bibliography


