Final Report

Geometry of Gauged Vortices

Group C – JHTP Mathematical Physics, HIM 2012

Nuno M. Romão (University of Göttingen)

1 The group

We applied to the JHTP as a group of ten researchers — Dennis Eriksson, Eduardo González, Lotte Hollands, Ignasi Mundet, Timothy Nguyen, Andreas Ott, Nuno Romão, Martin Speight, Christian Wegner and Fabian Ziltener — with the general aim of studying various aspects of the vortex equations in fibre bundles over Riemann surfaces. Gauge theory has had a fruitful interaction with many branches of mathematics, and this was well reflected in the broad spectrum of our backgrounds. To a large extent, we accredit many of the successes described in this report to the interdisciplinarity advocated in our original proposal.

2 Research projects

Our projects revolved around three main themes: **(i)** geometry and topology of vortex moduli spaces, relating to the Kähler metrics supported by vortex moduli, as well as the geometric and topological structures emerging from their quantisation; **(ii)** gauged Gromov–Witten invariants, which yield a generalisation of the theory of pseudoholomorphic curves in symplectic manifolds to the setting of holomorphic Hamiltonian actions; **(iii)** vortices and higher-dimensional field theories.

(i) Geometry and topology of vortex moduli spaces

Let G be a compact Lie group, for which a holomorphic Hamiltonian action on a Kähler manifold (X, J, ω_X) with moment map μ is given. The vortex equations are the PDEs

$$\bar{\partial}_A^{j_\Sigma, J} u = 0, \qquad F_A + \mu(u)\,\omega_\Sigma = 0$$

for pairs (A, u), where A is a connection on a principal G-bundle $P \to \Sigma$ over a surface $(\Sigma, j_\Sigma, \omega_\Sigma)$ on which another Kähler structure is given, and $u : P \to X$ is a G-equivariant map. Each u determines a G-equivariant 2-homology class $[u] \in H_2^G(X; \mathbb{Z})$, and fixing such a class $\mathbf{k}$ one obtains a moduli space $\mathcal{M}_{\mathbf{k}}$ of vortex solutions modulo gauge equivalence. The natural flat L^2-metric on the space of all pairs (A, u) descends to a nontrivial Kähler metric g_{L^2} on $\mathcal{M}_{\mathbf{k}}$, defining what we call the L^2-geometry of the moduli. In the project [25], the metrics g_{L^2} were investigated for $\Sigma = \mathbb{C}$ and $\Sigma = \mathbb{P}^1$ with the standard action of $G = \mathrm{U}(1)$ on $\mathbb{P}^1$. Combining a localisation formula with a heuristic description of these metrics as a limit of the L^2-geometry associated with the linear action of $G = \mathrm{U}(1)^2$ on $\mathbb{C}^2$, Nuno Romão and Martin Speight obtained formulas for the symplectic volume of the moduli space in the case $\Sigma = \mathbb{P}^1$ and $\mathbf{k} = (1, 1)$, which turns out to be finite. Numerical work on the PDEs has not only yielded further support for this prediction, which was expected in analogy with results on moduli of ungauged sigma-models, but it also uncovered that the vortex metrics g_{L^2} are complete, in striking contrast with the ungauged models.

Both of these properties of the L^2-geometry (completeness and finite volume) are necessary to validate a very compelling semiclassical picture for the canonical quantisation of the A-twisted supersymmetric sigma-models associated to the vortex equations above. These lead to topological quantum field theories whose correlators localise to integrals over the moduli spaces $\mathcal{M}_{\mathbf{k}}$. In the project [8], the quantum ground states of these gauged A-models are studied for compact Σ via $N = (2, 2)$ supersymmetric quantum mechanics on $\mathcal{M}_{\mathbf{k}}$; X is taken to be a symplectic compact toric manifold with torus G. In this setting, it is most natural to allow the waveforms to take values in local systems over the moduli spaces — equivalently, one lifts the supersymmetric quantum mechanics to the universal covers $\widetilde{\mathcal{M}_{\mathbf{k}}}$ with L^2-geometry induced by pull-back. The easier situation where $X = \mathbb{C}$ with usual circle action can be understood rather directly from results on the topology of $\widetilde{\mathrm{Sym}}^k\Sigma$ obtained in [5]: if Σ has genus $g > 1$, and for all $k \geq 1$, one must deal with a torus $\mathrm{Hom}(H_1(\pi_1(\mathcal{M}_k), \mathrm{U}(1)) \cong \mathrm{U}(1)^{2g}$ parametrising the possible choices of rank-one local systems over $\mathcal{M}_k = \mathrm{Sym}^k\Sigma$, interpreted as dual (electric) Aharonov–Bohm charges in Σ itself, and we showed that the L^2-cohomology $H^j_{(2)}(\widetilde{\mathrm{Sym}}^k\Sigma, \mathbb{C})$ is infinitely generated precisely in the middle degree $j = k$. To obtain sensible information about the action of $\pi_1(\mathcal{M}_k) \cong H_1(\Sigma, \mathbb{Z})$, one needs to extract L^2-Betti numbers, which is not hard in this co-compact example. The result shows that vortices in line bundles (in their ground states) are fermionic particles; moreover, Bökstedt and Romão were able to obtain a very intuitive interpretation of this result based on a 'lemniscate principle' [7] for pair-of-pants decompositions of Σ. As an application, a complete discussion of the spectrum of quantum particles has already been obtained by Romão and Wegner [8] for the rank-one ground states of the simplest nonlinear model $X = \mathbb{P}^1$: alongside the fermionic states associated to the two fixed points of the circle action, we detected the existence of a bosonic particle arising from the fusion of two fermions of each type. Moreover, the fundamental groups of the moduli spaces were computed in [6] using generalised braids that we call 'divisor links' on Σ. This calculation relied on the construction of invariants for divisor links, and it shows that the centres of $\pi_1(\mathcal{M}_{\mathbf{k}})$ are nontrivial abelian groups with possibly nonzero rank; their torsion is determined arithmetically from the classes in $H_2^G(X, \mathbb{Z})$. Conceptually, one of the most startling outcomes of this project is the realisation that abelian gauge theories may give rise to nonabelionic particles at the quantum level. The presence of such 'nonabelions' potentiates applications of abelian gauged nonlinear sigma-models in areas of recent interest in condensed matter physics such as topological quantum computation.

Another viewpoint on quantising vortices consists of regarding the moduli spaces $\mathcal{M}_{\mathbf{k}}$ themselves as phase spaces, and then proceed to the geometric quantisation of the symplectic structures ω_{L^2} in the complex polarisation induced by j_Σ. The project [11] by Dennis Eriksson and Nuno Romão studies this problem for the case of vortices in line bundles using Quillen determinants. The quantum Hilbert spaces were calculated from a choice of prequantisation of the Kähler structure $(\Sigma, \frac{\tau}{2}\omega_\Sigma, j_\Sigma)$ on the surface, where $\tau > 4\pi k/\mathrm{Vol}(\Sigma)$ reflects the choice of moment map μ. There is a convenient way of incorporating metaplectic corrections and model the different polarisations on the family $\mathrm{Pic}^k\Sigma$. It was found that there is a special case (corresponding to a metric of constant scalar curvature on Σ) where the choice of quantisation of Σ is immaterial, and one can naturally ask whether this situation also leads to special properties of the L^2-geometry. One of the main results in [11] was the proof that, whenever $g > 1$, there does not exist a projectively flat connection over $\mathrm{Pic}^k\Sigma$ relating Hilbert spaces for different holomorphic structures. There were also developments beyond previous attempts to understand localisation formulas for the prequantum connection, refining the already well-established formulas for its curvature ω_{L^2}.

(ii) Gauged Gromov–Witten invariants

A new idea in gauged Gromov–Witten theory is to incorporate the complex structure j_Σ as part of the moduli problem; a natural question to ask then is what happens when the complex curve Σ degenerates and becomes singular. The simplest example is of course the degeneration of a rational curve when a node is created. Here j_Σ remains constant up to diffeomorphism as long as Σ remains smooth, but nonetheless this example can be quite useful to explore the new phenomena related to the appearance of nodes — in fact, the need to allow for nodes in the compactification is the only feature that makes the inclusion of j_Σ in the moduli problem a highly nontrivial task. This was the starting point of a project by Ignasi Mundet; the forthcoming paper [21] studies the behaviour of the Hitchin–Kobayashi correspondence for $\Sigma \cong \mathbb{P}^1$ and X any Kähler manifold, for a family of area forms $\{\omega_{\Sigma,t}\}_t$ induced by restricting the Fubini–Study symplectic form on $\mathbb{P}^2$ to the quadrics $xy = tz^2$ (parametrised by $\mathbb{P}^1$) and letting $t \to 0$. Applying the Hitchin–Kobayashi correspondence to a family of pairs $\{(A_t, u_t) : \bar{\partial}^{j,J}_{A_t} u_t = 0\}_t$ on a fixed bundle over $\mathbb{P}^1$ and depending on t (note that the latter is an essential ingredient in the Hitchin–Kobayashi correspondence for general holomorphic pairs), one obtains a one-parameter family of vortices. Their possible limits as

$t \to 0$ are studied in algebraic terms from the point of view of a compactness theorem established by the author and Tian.

Andreas Ott and Fabian Ziltener made progress on a long-term project [24] that aims at generalising the existing definitions of gauged Gromov–Witten invariants (which so far have required rather restrictive assumptions) to monotone Hamiltonian manifolds X, including important examples such as Grassmannians and toric varieties. This new approach is based on a nonlocal version of the vortex equations (of integro-differential type) which relies on a certain holonomy perturbation $\bar{\partial}_{A,\Theta}^{j_\Sigma,J}$ of the holomorphic structure under a classifying map Θ for the gauge group action on the space of pairs (A, u). A novelty is that J now gets replaced by an equivariant family of not necessarily G-invariant almost complex structures on X, with an additional nonlocal dependence on the pair (A, u) itself. In this way, transversality for the sphere bubbles in the compactification of the moduli space can be achieved as long as X is assumed to be monotone. An axiomatic characterisation of such holonomy perturbations Θ was obtained. An important step in the whole programme is a certain a priori estimate that requires a careful study of the first and second derivatives of Θ. Ott and Ziltener found a way of constructing suitable maps Θ explicitly by taking the holonomy of the connection A along certain paths on Σ and then averaging over a compact family. This provides a conceptually clear framework for the analysis required in the generalisation of gauged Gromov–Witten invariants.

Another extension of the moduli space of vortices was the subject of a project [12] by Eduardo González, where parabolic structures are added at finitely many smooth points of Σ; the main motivation is to relate the invariants to those for gauged maps on stacky curves (orbifolds). The first step is to understand gauged maps whose principal component is a Galois quotient from a smooth curve by a finite group. The relevant semistability condition for these objects is a version of Mundet semistability supplemented by weights at the parabolic points. Progress has now been made on understanding the algebraic structure of the corresponding invariants. The usual Gromov–Witten invariants yield a cohomological field theory in the sense of Kontsevich–Manin, since the moduli space is a stack over the Deligne–Mumford stack $\mathcal{M}_{n,g}$ of stable curves; similarly, gauged Gromov–Witten invariants form a cohomological field theory trace, since the moduli spaces are now defined over the Fulton–McPherson space $\mathcal{M}_{n,g}(\Sigma)$ of stable curves with a principal component Σ by results of Woodward, and this can be extended to the parabolic case. A similar problem is treated with a slightly different perspective in ongoing work [14] on vortices over cylindrical ends; here, the main goal was to establish the existence of a virtual fundamental class for the moduli, which is rendered difficult by the presence of symmetries.

One area where gauged Gromov–Witten theory has been finding applications is mirror symmetry. Vortices (or gauged maps) interpolate equivariant cohomology with the quantum cohomology of a GIT quotient via the so-called Kirwan map. The effect of varying this quotient on the invariants was studied in the paper [15], which was revised by the first author during his stay at HIM. In another project [13] of Eduardo González with Hiroshi Iritani, the Kirwan map was identified with objects called 'Seidel elements' in the setup of toric mirror symmetry.

(iii) Vortices and higher-dimensional field theories

The vortex equations appear in higher-dimensional field theory when one introduces certain nonlocal sources (or defects). Particularly interesting are four-dimensional supersymmetric gauge theories, in which the vortex equations govern two-dimensional surface defects. One of the interesting questions is to classify such surface defects and figure out their intrinsic two-dimensional description, for example in terms of gauged sigma-models. In collaboration work [9] where Lotte Hollands took part, it was found that surface defects in an important class of four-dimensional field theories, indexed by a Lie algebra, are labeled by an irreducible representation of this Lie algebra. In two extreme cases, it was possible to obtain the corresponding two-dimensional description in terms of a gauged linear sigma-model, and to associate a new invariant to it: a two-dimensional superconformal index. Perhaps most interesting is that this invariant can be phrased as a certain elliptic difference operator, and that the kinematics of the surface defects gives rise to an algebra of such operators.

At the quantum level, vortices in supersymmetric theories provide examples of what are called BPS states: representations of a supersymmetry algebra which saturate a mass bound. There has been much interest in developing tools to understand phenomena we already encounter in the vortex setting, like wall-crossing, in higher-dimensional theories. One insightful way to study wall-crossing of BPS states in certain supersymmetric four-dimensional gauge theories is through so-called 'spectral networks', which Greg Moore told us about in his inspiring Felix Klein Lectures (hosted by HCM at the time of the JHTP). A spectral network consists of a collection of trajectories on a Riemann surface S that obey various rules; one set of examples is given by foliations of S that are generated by meromorphic quadratic differentials. Together with Andrew Neitzke, Lotte Hollands studied a more mathematical application of such spectral networks [17]. Given a spectral network $\mathcal{W}$, one can define a one-to-one mapping from nonabelian flat connections on S to abelian flat connections on a ramified cover $\tilde{S} \to S$ (a spectral curve). Particularly interesting in that this naturally defines a Darboux coordinate system on the moduli space of nonabelian flat connections on S for every choice of $\mathcal{W}$. Known coordinate systems follow from particular choices; the paper [17] discusses how to obtain the Fenchel–Nielsen coordinates, as well as the Fock–Goncharov coordinates.

3 Activities

A first meeting of the group members was organised at the occasion of the

- **Master Class:** "Vortex Equations and Hamiltonian GW-Invariants", 17th–20th January 2012 (by Ignasi Mundet), 16 lectures [22] at the Centre for Quantum Geometry of Moduli Spaces (QGM), Aarhus University, Denmark

for which we received very generous support from QGM.

In Bonn, we structured our research activities around three events that were open to everyone, and which we took care to advertise at the local mathematical institutions (HCM, HIM and MPIM):

- **Trimester Seminar:** This was run weekly in coordination with the other three groups at the JHTP; we contributed with four talks on vortices at pedagogical level by members of the group;
- **Vortex Seminar:** Meeting every week, this was intended mainly as a forum to discuss informally recent research by our visitors. A total of twelve talks were presented, in addition to a **minicourse** on "Vortices and Quantum Kirwan Maps" by Fabian Ziltener, which spanned two days;
- **Workshop** "Geometry of the Vortex Equations", 27th–30th November 2012**:** This was the main event in the whole programme, and consisted of three days dedicated to each of the central topics in our proposal, plus an extra day with a more interdisciplinary flavour. Nineteen talks were presented altogether, and they are all documented (through slides and notes) on the HIM webpage; a total of 45 participants registered. The social apex of this meeting was a memorable **recital** by and for vortex researchers, featuring vocal and piano performances by virtuosi among our group.

It should be mentioned that some of the researchers we brought to HIM from abroad made themselves available to give seminars about their work at other locations in Germany; among the institutions that profited from us in this way were the MPIM in Bonn and the Universities of Augsburg, Bielefeld, Duisburg-Essen, Heidelberg and Münster, as well as group D in our JHTP.

To ensure continuity and give visibility to the research we completed for the JHTP, we set up two further workshops at the Simons Center for Geometry and Physics, Stony Brook University, USA:

- **Workshop 1:** "Equivariant Gromov–Witten Theory and Applications" 12th–16th May 2014 (organisers: Eduardo González and Chris Woodward), in topics related to theme **(ii)** and mirror symmetry;
- **Workshop 2:** "Gauged Sigma-Models in Two Dimensions" 3rd–7th November 2014 (organisers: Sergei Gukov, Nuno Romão and Samson Shatashvili), in topics related to themes **(i)** and **(iii)**.

4 Our visitors

At the time of the JHTP, we were able to invite a total of 22 external visitors to our group, either as speakers in the weekly Vortex Seminar or participants of our workshop. In a limited amount of space, we can only mention a few highlights of such visits and of their impact on further research.

Nick Manton was present at the opening of the JHTP, and he later reported on experiments [10] aimed at understanding the nonlocal geometric flows for metrics on surfaces defined by the L^2-geometry of 1-vortices with varying size. Tudor Dimofte spent three weeks at HIM within September, and he gave an account of a project [4] on holomorphic blocks in 3-dimensional gauge theories; his fascinating talk illustrated far-reaching applications of the 'vortex counting' introduced to us by Lotte Hollands in this initial period.

In October, we had a rather short visit by Amihay Hanany, who presented his viewpoint [16] on the B-twisted gauged sigma-models associated to linear nonabelian vortex equations; supersymmetric QFTs of this type are alternative to those mentioned in **(i)**. Markus Szymik gave an account of his attempt [28] at constructing co-homotopy invariants via the vortex equations, in analogy with ideas that have led to successful applications for the Seiberg–Witten equations. João Baptista payed us a visit to provide further insight on his study of the L^2-geometry of ungauged nonlinear sigma-models using linear vortex metrics, and to describe his new results [3] on vortices in simply connected manifolds and in abelian varieties. Tim Perutz also spent a week in Bonn to discuss with Tim Nguyen ramifications of a project that stems from Nguyen's PhD thesis [23]. Developing the ideas discussed during his visit, Perutz is now working on a project with graduate supervisee Andrew Lee to devise an analogue of Heegaard–Floer theory for 3-manifolds. This theory will follow the same basic pattern as Heegaard–Floer theory, but will be based not on symmetric products of a Riemann surface Σ (alias vortices on Σ with $G = \mathrm{U}(1)$), but rather on stable (i.e. Bradlow) pairs on Σ (alias vortices with $G = \mathrm{SU}(2)$); the visit to HIM was useful in getting this project off the ground.

As a warm-up for the workshop, we had a row of lectures on gauged Gromov–Witten theory in the algebraic setting by Eduardo González as well as Bumsig Kim, who visited HIM with his graduate student Jeongseok Oh. Kim explained his theory of quasimaps in joint work with Ciocan-Fontanine, which provides a powerful framework to discuss vortex moduli in higher rank for affine targets; while in Bonn, he got inspiration for an extension of this setup [18] that should lead to interesting applications. The workshop itself was meant to be a period of congregation of vortex researchers from all over the world, some of them visiting our group for the second time. The three keynote speakers (Steve Bradlow, Chris Woodward and Sergei Gukov) gave extremely inspiring talks that covered broad ground in each of the three themes in our proposal. Around the time of his visit to take part in the workshop, João Baptista profited from various discussions to develop his new idea of looking at vortices of a given charge on a smooth surface as vortices of a lower charge in a singular background metric [2], whereas Óscar García-Prada made progress on a new project about gravitating vortices [1]. Sushmita Venugopalan and Chris Woodward were able to complete their work on classification of affine vortices [29] at the time of their visit; after the workshop, she provided us with details about her progress on vortices in noncompact surfaces, also benefiting from discussions with Fabian Ziltener.

5 Publications and preprints

The papers listed here are either by group members and connected to the projects described in section 2, or they result from research done by our visitors which has intertwined with our activities at HIM. Author names are written in **boldface** for group members, and underlined for visitors.

[1] L. Álvarez-Cónsul, M. García Fernandez, Ó. García-Prada: *Gravitating vortices* (in preparation)
[2] J.M. Baptista: *Vortices as degenerate metrics*, Lett. Math. Phys. **104** (2014) 731–747
[3] J.M. Baptista: *Moduli spaces of abelian vortices on Kähler manifolds*; `arXiv:1211.0012`

[4] C. Beem, T. Dimofte, S. Pasquetti: *Holomorphic blocks in three dimensions*; `arXiv:1211.1986`
[5] M. Bökstedt, **N.M. Romão**: *On the curvature of vortex moduli spaces*, Math. Z. **277** (2014) 549–573
[6] M. Bökstedt, **N.M. Romão**: *Divisor links and fundamental groups of toric vortex moduli* (in preparation)
[7] M. Bökstedt, **N.M. Romão**: *Pairs-of-pants, lemniscates and L^2-invariants* (in preparation)
[8] M. Bökstedt, **N.M. Romão**, **C. Wegner**: *L^2-invariants and supersymmetric quantum mechanics on vortex moduli spaces* (in preparation)
[9] M. Bullimore, M. Fluder, **L. Hollands**, P. Richmond: *The superconformal index and an elliptic algebra of surface defects*; `arXiv:1401.3379`
[10] D. Dorigoni, M. Dunajski, N.S. Manton: *Vortex motion on surfaces of small curvature*, Ann. Phys. **339** (2013) 570–587
[11] **D. Eriksson**, **N.M. Romão**: *Kähler quantisation of vortex moduli* (in preparation)
[12] **E. González**: *Parabolic gauged maps* (in preparation)
[13] **E. González**, H. Iritani: *Seidel elements and potential functions of holomorphic disc counting*; `arXiv:1301.5454`
[14] **E. González**, **A. Ott**, C. Woodward, **F. Ziltener**: *Symplectic vortices with fixed holonomy at infinity* (in preparation)
[15] **E. González**, C. Woodward: *A wall-crossing formula for Gromov-Witten invariants under variation of GIT quotient*; `arXiv:1208.1727`
[16] A. Hanany, R.-K. Seong: *Hilbert series and moduli spaces of k $\mathrm{U}(N)$ vortices*; `arXiv:1403.4950` (submitted to JHEP)
[17] **L. Hollands**, A. Neitzke: *Spectral networks and Fenchel–Nielsen coordinates*; `arXiv:1312.2979`
[18] B. Kim, J. Oh: *Quasimaps for fibrations* (in preparation)
[19] A. Lee, T. Perutz: *Floer theory in spaces of rank 2 stable pairs* (work in progress)
[20] N.S. Manton: *Vortex solutions of the Popov equations*, J. Phys. A: Math. Theor. **46** (2013) 145402
[21] **I. Mundet i Riera**: *Hitchin–Kobayashi correspondence for twisted holomorphic maps on nearly singular conics* (in preparation)
[22] **I. Mundet i Riera**: *Vortex Equations and Hamiltonian GW-Invariants* (in preparation; to be published as a book in the QGM Master Class Series, European Mathematical Society)
[23] **T. Nguyen**: *Lagrangian correspondences and Donaldson's TQFT construction of the Seiberg-Witten invariants of 3-manifolds*, Alg. Geom. Top. **14** (2014) 863–923
[24] **A. Ott**, **F. Ziltener**: *Gauged Gromov–Witten invariants for monotone symplectic manifolds* (in preparation)
[25] **N.M. Romão**, **J.M. Speight**: *L^2-geometry in gauged nonlinear sigma-models* (in preparation)
[26] **N.M. Romão**, **C. Wegner**: *L^2-Betti numbers and particle counting in a gauged nonlinear sigma-model* (in preparation)
[27] **J.M. Speight**: *Solitons on tori and soliton crystals*, Commun. Math. Phys. (to appear); DOI 10.1007/s00220-014-2104-z
[28] M. Szymik: *The stable homotopy theory of vortices on Riemann surfaces*; `arXiv:1310.7737`
[29] S. Venugopalan, C. Woodward: *Classification of affine vortices*; `arXiv:1301.7052`