K-theory, Cyclic Cohomology and Pairings for Quantum Heisenberg Manifolds

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Abstract

The $C^*$-algebras called Quantum Heisenberg Manifolds (QHM) were introduced by Rieffel in 1989 as strict deformation quantizations of Heisenberg manifolds. In this article, we compute the pairings of $K$-theory and cyclic cohomology on the QHM. Combining these calculations with other results proved elsewhere, we also determine the periodic cyclic homology and cohomology of these algebras, and obtain explicit bases of the periodic cyclic cohomology of the QHM. We further isolate bases of periodic cyclic homology, expressed as Chern characters of the $K$-theory.

Keywords: quantum Heisenberg manifolds; $K$-theory; cyclic cohomology; Chern-Connes pairings; generalized crossed product; Pimsner algebra; Heisenberg group.

1 Introduction

Quantum Heisenberg Manifolds (QHM) were introduced by Rieffel in [22]. They are a family of $C^*$-algebras $D_{c,\mu,\nu}$ indexed by $c \in \mathbb{Z}$ and $\mu, \nu \in \mathbb{R}$. These algebras $D$ have since been closely studied in series of articles by Abadie (see for instance [1] [2] [5] [3]) and Chakraborty (see [7] and [6]). In the article [4] by Abadie, Eilers and Exel, it was proved that QHM are generalised crossed products. This notion was later unified by Katsura [15, 16] with that of Pimsner algebra [20]. In their article [11], Connes and Dubois-Violette related QHM to the noncommutative 3-spheres they introduced in [10].

The present article is a refinement of the previous studies of QHM, by means of pairings between cyclic cohomology and $K$-theory. These pairings were first defined by Connes in his article [8] of 1985. They have a parity: odd cocycles pair with $K_1$ and even cocycles with $K_0$.

After reviewing QHM and the associated Heisenberg group action, we define a smooth subalgebra $D$ of $D$. Using the general proposition 3.5, we then construct explicit cyclic cocycles $(\varphi_1)_{i=1,2,3}$, $(\varphi_{1,3})_{i=1,2}$ and $\varphi_{1,2,3}$ out of the Lie group action (see proposition 3.8). We rely on previous work by Abadie to obtain projective finitely generated modules $N$, $N^\dagger$ over $D$. We then construct a generating set $(U_1, U_2, U_3)$ of $K_1(D \otimes \mathbb{C})$.

There are two main results. The first one are the theorems 8.3 and 8.4:

- the family $(\tau, \varphi_{1,3}, \varphi_{2,3})$ is a basis of $HP^0(D)$;
- the family $(\varphi_1, \varphi_2, \varphi_{1,2,3})$ is a basis of $HP^1(D)$;
- the family $(\operatorname{Ch}(U_1), \operatorname{Ch}(U_2), \operatorname{Ch}(U_3))$ is a basis of $HP_0(D)$.

Moreover, if $\mu \neq 0 \neq \nu$, then the family $(\operatorname{Ch}(D_{c,\mu,\nu}), \operatorname{Ch}(N_{c,\mu,\nu}^\circ), \operatorname{Ch}(N_{c,\mu,\nu}^\dagger))$ is a basis of $HP_0(D_{c,\mu,\nu})$.

Notice that this theorem relies on $kk$-equivalences that are constructed in [13]. The proof also depends on the two tables (4.1) and (5.3):

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\varphi_{1,3}$</th>
<th>$\varphi_{2,3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$N$</td>
<td>$2\mu$</td>
<td>$-i2\pi$</td>
</tr>
<tr>
<td>$N^\dagger$</td>
<td>$-2\nu$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\varphi_1$</th>
<th>$\varphi_2$</th>
<th>$\varphi_{1,2,3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\sqrt{i2\pi}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$-\sqrt{i2\pi}$</td>
<td>0</td>
</tr>
<tr>
<td>$\sqrt{i2\pi} 2c\nu$</td>
<td>$-\sqrt{i2\pi} 2c\mu$</td>
<td>$(i2\pi)^{1/2}c/3$</td>
</tr>
</tbody>
</table>
that give the pairings of cyclic cohomology with $K$-theory in the even and odd cases. The computation of these tables is the second main result.

This paper is organised as follows. In section 2, we review the properties of the QHM. Section 3 is devoted to the construction of cyclic cocycles. The pairings in the even cases are computed in section 4. Sections 5 and 6 are concerned with the calculation in the odd cases. Section 7 focuses on the unfolding the consequences of the previous computations. The computation of the periodic homology and cohomology is completed in section 8. We conclude in section 9 by some remarks, and finally include a construction of the Toeplitz algebra of the Pimsner algebra.

## 2 Quick Review of QHM

We will define the QHM as generalised crossed product of $A = C(T^2)$ by a certain Hilbert bimodule $M_{\mu,\nu}^c$. A general reference for Hilbert modules is [17]. As several definitions of “Hilbert bimodules” exist in the literature, we specify our notion:

**Definition 2.1** (Hilbert bimodule). A $A$-$A$-Hilbert bimodule $AE_A$ is a vector space $E$ such that:

- $E_A$ is a right Hilbert module with $A$-valued scalar product $\langle \cdot, \cdot \rangle_A$;
- $AE$ is a left Hilbert module with scalar product $A\langle \cdot, \cdot \rangle$;
- the scalar products are compatible in the sense that $\xi\langle \zeta, \eta \rangle_A = A\langle \xi, \zeta \rangle \eta$.

We follow the definitions 2.1 and 2.4 of [4]:

**Definition 2.2** (covariant representation). Let $AE_A$ be a $A$-$A$ Hilbert bimodule. A covariant representation of $E$ on a $C^*$-algebra $B$ is a pair $(\pi, \tau)$ made up of:

- a *-homomorphism of algebras $\pi : A \to B$;
- a linear map $\tau : E \to B$ satisfying

  - (i) $\tau(\xi)^*\tau(\zeta) = \pi(\langle \xi| \zeta \rangle_A)$
  - (ii) $\tau(\xi)\pi(a) = \tau(\xi a)$
  - (iii) $\pi(a)\tau(\xi) = \tau(a\xi)$
  - (iv) $\tau(\xi)\tau(\zeta)^* = \pi(A\langle \xi| \zeta \rangle)$.

The associated $C^*$-algebra is:

**Definition 2.3** (generalised crossed product). Let $AE_A$ be a $A$-$A$ Hilbert bimodule. The generalised crossed product $A \rtimes_{E} \mathbb{Z}$ of $A$ by $E$ is the universal $C^*$-algebra generated by the covariant representations of $AE_A$.

We now define the Hilbert bimodule $M_{\mu,\nu}^c$ that we will use.

**Definition 2.4** (Hilbert bimodule over $A$). Given two real numbers $\mu, \nu$ and an integer $c > 0$, we define a Hilbert bimodule $M_{\mu,\nu}^c$ over $A = C(T^2)$ as the set of continuous functions $\xi : \mathbb{R} \times S^1 \to \mathbb{C}$ which satisfy:

$$\xi(x + 1, y) = e(-c(y - \nu))\xi(x, y) \quad \xi(x, y + 1) = \xi(x, y),$$

where $e(x) = e^{2\pi ix}$. We give it a bimodule structure by (direct) pointwise multiplication on the left and right multiplication defined by $(\xi \cdot a)(x, y) = \xi(x, y)\sigma(a)(x, y)$ where $\sigma$ is the automorphism of $A$ defined by $\sigma(a)(x, y) = a(x - 2\mu, y - 2\nu)$. Finally, the scalar products are:

$$\langle \xi_1, \xi_2 \rangle_A(x, y) = \overline{\xi_1(x + 2\mu, y + 2\nu)}\xi_2(x + 2\mu, y + 2\nu)$$

$$A\langle \xi_1, \xi_2 \rangle(x, y) = \xi_1(x, y)\overline{\xi_2(x, y)},$$
For convenience, we will use the shorthand notation $M$ instead of $M\mu,\nu$. We can easily verify that this indeed defines a Hilbert bimodule. We are now ready to define the QHM:

**Definition 2.5** (quantum Heisenberg manifolds). Given an integer $c > 0$ and $\mu, \nu \in \mathbb{R}$, the quantum Heisenberg manifold $D\mu,\nu$ is the generalised crossed product $A \rtimes M\mu,\nu$.

In the following, we will write $D$ instead of $D\mu,\nu$ whenever possible. We will identify the elements $\xi \in M$ with their images in $D$. From the definition of $D$, it appears that for any $\xi, \zeta \in M$

$$A(\xi, \zeta) = \xi \zeta^* \quad \langle \xi, \zeta \rangle_A = \xi^* \zeta.$$

To give a more concrete definition (see [4]), $D$ is a completion of the algebra:

$$D_0 = \{ F \in C_c(\mathbb{Z} \to C_b(\mathbb{R} \times S^1)) | F(p, x + 1, y) = e(-cp(y - p\nu))F(p, x, y) \}$$

equipped with multiplication:

$$(F_1 \cdot F_2)(p, x, y) = \sum_{q \in \mathbb{Z}} F_1(q, x, y)F_2(p - q, x - q2\mu, y - q2\nu) \quad (2.4)$$

and involution:

$$F^*(p, x, y) = \overline{F}(-p, x - 2p\mu, y - 2p\nu).$$

As every generalised crossed product, $D$ is endowed with a gradation by $\mathbb{Z}$: $a \in A \subseteq D$ has degree 0 and $\xi \in M \subseteq D$ has degree 1.

**Proposition 2.6.** For all $M\mu,\nu$, we can find $\xi_1, \xi_2 \in M\mu,\nu$ such that:

$$\sum_{i=1}^{2} \xi_i^* \xi_i = \sum_{i=1}^{2} \langle \xi_i, \xi_i \rangle_A = 1 \quad \sum_{i=1}^{2} \xi_i \xi_i^* = \sum_{i=1}^{2} A(\xi_i, \xi_i) = 1. \quad (2.5)$$

*Proof.* Let $U_1, U_2$ be the respective images of $V_1 = \lfloor -1/3, 1/3 \rfloor$ and $V_2 = \lfloor 1/6, 5/6 \rfloor$ by the quotient map $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \simeq S^1$. $U_1, U_2$ is an open cover of $S^1$. We call $\chi_1, \chi_2$ a subordinated smooth partition of the unity of $S^1$. Setting $\chi_i = \frac{\chi_i}{\sqrt{\chi_1^2 + \chi_2^2}}$, we get $\chi_1$ and $\chi_2$ such that $\chi_1^2 + \chi_2^2 = 1$ and $\text{Supp} \chi_i \subseteq U_i$.

To define $\xi_i$ on the cylinder $\mathbb{R} \times S^1$, we first set $\xi_1(x, y) = \chi_1(x)$ on $[-1/2, 1/2] \times S^1$. This function can then be extended to the whole $\mathbb{R} \times S^1$ by enforcing the equations \((2.1)\). Notice that the extension is possible because $\xi_1$ vanishes on the boundaries of $[-1/2, 1/2] \times S^1$.

The same process can be applied on $[0, 1] \times S^1$ to define $\xi_2$. In the end, we get $\xi_i$ such that

$$\langle \xi_i, \xi_i \rangle_A(x, y) = \chi_i(x + 2\mu, y + 2\nu)\chi_i^*(x + 2\mu, y + 2\nu) = \chi_i^2(x + 2\mu, y + 2\nu).$$

The first equation of \((2.5)\) hence comes from the property of the $\chi_i$. The same kind of computation provides the second equation. \(\square\)
Definition 2.7 (Heisenberg group $H_1$). The Heisenberg group $H_1$ is the subgroup of $GL_3(\mathbb{R})$ of the matrices

$$
\begin{pmatrix}
1 & s & t \\
0 & 1 & r \\
0 & 0 & 1 \\
\end{pmatrix}, \quad r, s, t \in \mathbb{R}
$$

(2.6)

The following definition has its origin in [22], proposition 5.6.

Definition 2.8 (action of the Heisenberg group). There is a pointwise continuous action of $H_1$ on $D$ defined on $D_0$ by:

$$
\alpha_{(r,s,t)}(F)(p,x,y) = e(\gamma p(t + cs(x - r)))F(p,x - r, y - s).
$$

Using the parameters $(r, s, t)$, the infinitesimal generators of this action are:

$$
\partial_1 F(p,x,y) = -\frac{\partial F}{\partial x}(p,x,y) \quad \partial_3 (F)(p,x,y) = i2\pi pF(p,x,y)
$$

$$
\partial_2 (F)(p,x,y) = -\frac{\partial F}{\partial y}(p,x,y) + i2\pi pcxF(p,x,y).
$$

and they fulfill the commutation relations:

$$
[\partial_1, \partial_2] = -c\partial_3 \quad [\partial_1, \partial_3] = 0 \quad [\partial_2, \partial_3] = 0.
$$

(2.7)

Definition 2.9 (trace on QHM). A trace $\tau$ is given by $\tau(F) = \int_0^1 \int_{S^1} F(x,y,0)dydx.$ It is invariant under the action of $H_1$.

Finally, the $K$-theory of the QHM was computed in [2] (theorem 3.4):

Theorem 2.10 (Abadie, 1995).

$$
K_0(D) = \mathbb{Z}^3 \oplus \mathbb{Z}/c\mathbb{Z} \quad K_1(D) = \mathbb{Z}^3.
$$

3 Cyclic Cocycles and Lie Groups Actions

The definition of cyclic cohomology was given in [9] III.1 p.182:

Definition 3.1 (Cyclic Cohomology). Given an algebra $\mathcal{A}$, the cyclic cohomology $HC^n(\mathcal{A})$ is the cohomology of the complex $(C^n_\lambda, b)$ where $C^n_\lambda$ is the space $(n + 1)$-linear forms $\phi$ on $\mathcal{A}$ such that:

$$
\phi(a^0, a^1, \cdots, a^n) = (-1)^n \phi(a^n, a^0, \cdots, a^n)
$$

and the coboundary map $b$ is given by:

$$
b\phi(a^0, \cdots, a^n, a^{n+1}) = \sum_{j=0}^n (-1)^j \phi(a^0, \cdots, a^j a^{j+1}, \cdots, a^{n+1}) + (-1)^{n+1} \phi(a^{n+1} a^0, \cdots, a^n).
$$

A cyclic cocycle is a closed cochain in the above sense.
Definition 3.2 (Cycle). A cycle of dimension \( n \) is a triple \((\Omega, d, \int)\) where \( \Omega = \bigoplus_{j=0}^{n} \Omega^j \) is a graded algebra over \( \mathbb{C} \), \( d \) is a graded derivation of degree 1 such that \( d^2 = 0 \) and \( \int : \Omega^n \to \mathbb{C} \) is a closed graded trace on \( \Omega \).

A cycle over an algebra \( \mathcal{A} \) is a cycle \((\Omega, d, \int)\) together with an homomorphism \( \rho : \mathcal{A} \to \Omega^0 \).

We can restate the proposition 4, III.1 in [9]:

**Proposition 3.3.** All cyclic cocycle arise as trace of a cycle and vice versa.

In the following definition and proposition, we adapt the construction of [9], III.6, example 12 c) p.254:

**Definition 3.4** (Differential Algebra Associated to a Lie Group Action). Let \( A \) be a Banach algebra equipped with a pointwise continuous action \( \alpha \) of the Lie group \( G \). We let \( A = \{ x \in A : g \mapsto \alpha_g(x) \in C^\infty(G \to A) \} \), and \( \mathcal{G} \) be the Lie algebra associated to \( G \).

The differential algebra \( \Omega_G \) associated with the action \( \alpha \) is the graded differential algebra defined by the alternating \( A \)-valued multilinear forms on \( G \) equipped with the differential \( d \):

\[
d\omega(X_1, \ldots, X_{n+1}) = \sum_{i=1}^{n+1} (-1)^i X_i \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{n+1})
+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{n+1}).
\]

The following proposition is a restatement of the property described in [9], p.255. We adapt it to our simple case and state it:

**Proposition 3.5.** If \( \tau \) is a \( G \)-invariant trace over \( A \) and \( \xi_1 \wedge \cdots \wedge \xi_k \in \Lambda^k \mathcal{G} \) satisfies

\[
\sum_{i<j} (-1)^{i+j} [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \hat{\xi}_j \wedge \cdots \wedge \xi_k = 0,
\]

then

\[
(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \mapsto \sum_{\sigma \in \Sigma_n} \epsilon(\sigma) \tau(a_0 \xi_{\sigma(1)}(a_1) \cdots \xi_{\sigma(n)}(a_n))
\]

is a cyclic cocycle on \( \mathcal{A} \).

**Definition 3.6** (smooth QHM). We define the smooth QHM \( \mathcal{D} \) by \( \mathcal{D} = \{ F \in D : g \mapsto \alpha_g(F) \in C^\infty(H_1 \to D) \} \).

As \( \alpha \) is pointwise continuous, the following property is well known (see [23], p.138):

**Lemma 3.7.** \( \mathcal{D} \) is a dense subalgebra that is stable under holomorphic functional calculus. In particular

\[
K_0(\mathcal{D}) = K_0(D) = \mathbb{Z}^3 \oplus \mathbb{Z}/c\mathbb{Z} \quad K_1(\mathcal{D}) = K_1(D) = \mathbb{Z}^3.
\]

**Proposition 3.8.** The 7 following multilinear forms are cyclic cocycles on \( \mathcal{D} \):

- Degree 0: trace \( \tau \).
• Degree 1: $\varphi_i$ for $i = 1, 2, 3$ where $\varphi_i(a_0, a_1) = \tau(a_0 \partial_i(a_1))$.

• Degree 2: $\varphi_{1,3}$ and $\varphi_{2,3}$ where

$$\varphi_{1,3}(a_0, a_1, a_2) = \tau\left(a_0 (\partial_1(a_1) \partial_2(a_2) - \partial_2(a_1) \partial_1(a_2))\right).$$

• Degree 3: $\varphi_{1,2,3}$ given by:

$$\varphi_{1,2,3}(a_0, a_1, a_2, a_3) = \sum_{\sigma \in \Sigma_2} \varepsilon(\sigma) \tau\left(a_0 \partial_{\sigma(1)}a_1 \partial_{\sigma(2)}a_2 \partial_{\sigma(3)}a_3\right).$$

Proof. This is a straightforward application of proposition 3.5.

• Degree 0: there is nothing to check since $\tau$ is a trace.

• Degree 1: any derivation generates a cyclic cocycle.

• Degree 2: the commutation relations (2.7) show that $\xi_1 \wedge \xi_3$ and $\xi_2 \wedge \xi_3$ satisfy the condition (3.1).

• Degree 3: start with $\xi_1 \wedge \xi_2 \wedge \xi_3$. The condition can therefore be written:

$$(-1)^3[\xi_1, \xi_2] \wedge \xi_3 + (-1)^4[\xi_1, \xi_3] \wedge \xi_2 + (-1)^5[\xi_2, \xi_3] \wedge \xi_1 = 0.$$

The commutation relations (2.7) ensure that this expression vanishes.

Remark 3.9. One could expect a third 2-cyclic cocycle $\varphi_{1,2}$ given by $\varphi_{1,2}(a_0, a_1, a_2) = \tau\left(a_0 (\partial_1(a_1) \partial_2(a_2) - \partial_2(a_1) \partial_1(a_2))\right)$. $\varphi_{1,2}$ is indeed a Hochschild cocycle, as an easy computation shows. However, proposition 9.1 implies that $\varphi_{1,2}$ is not a cyclic cocycle.

Proof. Indeed, if $\varphi_{1,2}$ were a cyclic cocycle, then $\varphi_{1,2}(1, a_1, a_2) = 0$. Yet,

$$\varphi_{1,2}(1, a_1, a_2) = \tau(\partial_1(a_1) \partial_2(a_2) - \partial_2(a_1) \partial_1(a_2)) = \tau(\partial_1(a_1) \partial_2(a_2) - \partial_2(a_1) \partial_1(a_2)) - \tau(\partial_1 \partial_2(a_2) - \partial_2 \partial_1(a_2)) = c\tau(a_1 \partial_1(a_2)) = c\varphi_3(a_1, a_2),$$

using (2.7). But the proposition 9.1 proves that $\varphi_3$ pairs nontrivially with a Hochschild cocycle and therefore is nonzero.

4 Modules and even Pairings

Bear in mind (see [3] part II, theorem 9) that if $a_i \otimes b_i \in A \otimes M_n(\mathbb{C})$ and $\phi \in HC^*$, the cup product $\phi \# \text{Tr}$ is defined by:

$$(\phi \# \text{Tr})(a^0 \otimes b^0, \ldots, a^n \otimes b^n) = \phi(a^0, \ldots, a^n) \text{Tr}(b^0 \cdots b^n).$$

We follow the definition and normalisations of [3] (III.3 proposition 2):
Definition 4.1 (even Chern-Connes Pairings). The following formula defines a bilinear pairing between $K_0(\mathcal{A})$ and $HC^{2m}(\mathcal{A})$:

$$\langle [e], [\phi] \rangle = \frac{1}{m!} (\phi \# \text{Tr})(e, \ldots, e)$$

where $[e] \in K_0$ and $\phi \in ZC^{2m}$.

Furthermore, there is a periodicity map $S: HC^n \to HC^{n+2}$ which enables us to define the groups $HP^*(\mathcal{A})$ – see [23], 10.1, definition 10.5 p.445:

Definition 4.2 (periodic cyclic cohomology). The periodic cyclic cohomology are the two groups obtained as inductive limits:

$$HP^0(\mathcal{A}) = \lim_{\rightarrow} HC^{2k}(\mathcal{A}) \quad HP^1(\mathcal{A}) = \lim_{\rightarrow} HC^{2k+1}(\mathcal{A}).$$

The above pairings are in fact defined on $HP^0(\mathcal{A})$ because they satisfy $\langle [e], [S\phi] \rangle = \langle [e], [\phi] \rangle$.

Theorem 4.3. If $\mu \neq 0 \neq \nu$, we can define projective finitely generated modules $N$ and $N^\dagger$ over $D$, and the values of the pairings are given by the table:

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\varphi_{1,3}$</th>
<th>$\varphi_{2,3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\mathcal{A}]$</td>
<td>2$\mu$</td>
<td>$-i2\pi$</td>
</tr>
<tr>
<td>$[\mathcal{A}^\dagger]$</td>
<td>$-2\nu$</td>
<td>0</td>
</tr>
</tbody>
</table>

(4.1)

Notice that:

- the first column of this table was computed by B. Abadie in [3];
- if $\mu = 0$ or $\nu = 0$, then the projective finitely generated modules $\mathcal{A}$ and $\mathcal{A}^\dagger$ are not both defined. In this case, we have to use the isomorphism between $D^c_{\mu+1,\nu+1}$ and $D^c_{\mu,\nu}$ that was proved by Abadie in [1]. This phenomenon of “disappearing module” may be surprising, but it also happens with “Schwartz modules” in the case of noncommutative tori – see [5] part II, definition above lemma 54.

The projective finitely generated modules over $D^c_{\mu,\nu}$ were studied by Abadie in [1] and [2]. We present the results of [1] pp. 2–3:

Theorem 4.4 (Abadie, 1992). For all $c \in \mathbb{N}^*$, $\mu, \nu \in \mathbb{R}$ such that $\mu^2 + \nu^2 \neq 0$, there is a projective and finitely generated right module $N^c_{\mu,\nu}$ over $D^c_{\mu,\nu}$. $N^c_{\mu,\nu}$ is obtained by completing $C_c(\mathbb{R} \times S^1)$ with respect to the $D^c_{\mu,\nu}$-valued scalar product:

$$\langle f, g \rangle_{D^c_{\mu,\nu}}(p, x, y) = \sum_{n \in \mathbb{Z}} e(cn(p - \nu y))f(x + n, y)g(x - 2p\mu + n, y - 2p\nu).$$

The right action of $D^c_{\mu,\nu}$ is given by

$$\langle f \cdot F \rangle(x, y) = \sum_{q} f(x - 2q\mu, y - 2q\nu)F(-q, x - 2q\mu, y - 2q\nu)$$

Moreover $\text{Tr} \left( \text{Id}_{N^c_{\mu,\nu}} \right) = 2\mu$.

We will write $N$ instead of $N^c_{\mu,\nu}$ whenever possible.

Remark 4.5. This module $N$ is in fact the dual to the module $X$ given in [1], p.2.
4.1 Connexions and Pairings for \( \mathcal{N} \)

The definition of a (noncommutative) connexion is given in [9], III.3. definition 5:

**Definition 4.6 (Connexion).** Let \( \mathcal{A} \xrightarrow{\rho} \Omega \) be a cycle over \( \mathcal{A} \), and \( \mathcal{E} \) a finite projective module over \( \mathcal{A} \), a connexion \( \nabla \) on \( \mathcal{E} \) is a linear map \( \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega \) such that:

\[
\forall \xi \in \mathcal{E}, \forall a \in \mathcal{A}, \quad \nabla(\xi a) = (\nabla \xi)a + \xi \otimes d\rho(a)
\]

Notice that in the above definition, we do not actually need the trace \( f \) of the cycle. We could equally consider a “graded differential algebra over \( \mathcal{A} \)”.

The following proposition enables us to compute the even pairings using the connexions – see [9], III.3 proposition 8:

**Proposition 4.7.** Let \( \mathcal{E}_{\mathcal{A}} \) be a finitely generated projective module over \( \mathcal{A} \). Assume that we have a cycle \( (\Omega, d, \int) \) over \( \mathcal{A} \).

1. \( \tilde{\mathcal{E}} = \mathcal{E} \otimes_{\mathcal{A}} \Omega \) is a finitely generated projective \( \Omega \)-module.
2. All connexion \( \nabla \) can be uniquely extended to a linear application from \( \tilde{\mathcal{E}} \) into itself that satisfies :

\[
\forall \xi \in \mathcal{E}, \omega \in \Omega, \quad \nabla(\xi \otimes \omega) = (\nabla \xi)\omega + \xi \otimes d\omega
\]

3. One has

\[
\langle [\mathcal{E}], [\tau] \rangle = \frac{1}{m!} \int \theta^m
\]

where

- \( n \) is the (even) dimension of the cycle \( \Omega \) and \( n = 2m \);
- \( [\mathcal{E}] \in K_0(\mathcal{A}) \) is the class of \( \mathcal{E} \);
- \( \tau \) is the character of \( \Omega \);
- \( f \) is the trace of the cycle;
- \( \theta \) is the endormorphism of \( \tilde{\mathcal{E}} \) defined by \( \theta = \nabla^2 \).

**Definition 4.8 (Covariant Action).** Let \( E_A \) be a module over a \( C^* \)-algebra \( A \). A covariant action of a Lie group \( G \) on \( E_A \) is a pair \((\alpha, \beta)\) where \( \alpha \) and \( \beta \) are actions of \( G \) on \( A \) and \( E \) respectively, which satisfy

\[
\forall \xi \in E, \forall a \in A \quad \beta_g(\xi a) = \beta_g(\xi)\alpha_g(a).
\]

**Proposition 4.9.** If \((\alpha, \beta)\) is a covariant action on \( E_A \), then

- we can define a \( \mathcal{A} \)-module \( \mathcal{E}_{\mathcal{A}} \) as the set of “\( G \)-regular elements”;
- the module \( \mathcal{E} \) is equipped with a connexion \( \nabla : \mathcal{E} \to \mathcal{E} \otimes \mathcal{G}^* \) over the differential algebra \( \Omega_G \).

**Proof.** We define \( \mathcal{E} \) in the following way:

\[
\mathcal{E} = \{ \xi \in E : g \mapsto \beta_g(\xi) \text{ is in } C^\infty(G \to E) \}.
\]

This is a module over \( \mathcal{A} \): if \( \xi \in \mathcal{E} \), then \( \xi a \in \mathcal{E} \) because \( g \mapsto \beta_g(\xi a) = \beta_g(\xi)\alpha_g(a) \).
Define $\nabla$ by

$$\nabla_X \xi = \lim_{t \to 0} \frac{\beta_{t^X}(\xi) - \xi}{t}.$$ 

This is clearly a map $\mathcal{E} \to \mathcal{E} \otimes \mathcal{G}^\ast$.

Let us check that $\nabla$ is a connexion:

$$\nabla_X (\xi a) = \lim_{t \to 0} \frac{\beta_{t^X}(\xi a) - \xi a}{t} = \lim_{t \to 0} \frac{\beta_{t^X}(\xi)\alpha_{t^X}(a) - \xi a}{t} =

= \lim_{t \to 0} \frac{1}{t} (\beta_{t^X}(\xi)\alpha_{t^X}(a) - \xi \alpha_{t^X}(a) + \xi \alpha_{t^X}(a) - \xi a) = (\nabla_X \xi) a + \xi \partial_X (a).

$$

**Proposition 4.10.** If $\mu \neq 0$, there is a covariant action $\beta$ of $H_1$ on $N_D$ given by

$$\beta_{(r,s,t)}(f)(x,y) = e^{ix \frac{\pi}{\mu}(t + scx/2)} f(x - r, y - s).$$

**Proof.** On the one hand,

$$\beta_{(r,s,t)}(f \cdot F)(x,y) =

= e^{ix \frac{\pi}{\mu}(t + scx/2)} \sum_q f(x - r - 2q\mu, y - s - 2q\nu) F(-q, x - r - 2q\mu, y - s - 2q\nu).$$

On the other hand,

$$\left(\beta_{(r,s,t)}(f) \cdot \alpha_{(r,s,t)}(F)\right)(x,y) =

= \sum_q e^{i(x-2q\mu) \frac{\pi}{\mu}(t + sc(x-2q\mu)/2)} f(x - r - 2q\mu, y - s - 2q\nu) \times e^{(q(t + cs(x - 2q\mu) - r + q\mu))} F(-q, x - r - 2q\mu, y - s - 2q\nu).$$

We can restrict this to study the phase factor

$$e^{i(x-2q\mu) \frac{\pi}{\mu}(t + sc(x-2q\mu)/2)} e^{i2\pi q(t + cs(x-q\mu))}$$

while ignoring the $e^i$:

$$(x - 2q\mu) \frac{\pi}{\mu}(t + sc(x - 2q\mu)/2) + 2\pi q(t + cs(x - q\mu)) =

= \frac{\pi}{\mu} t \left(\frac{\pi}{\mu} - 2\pi q + 2\pi q\right) + s \left(\frac{\pi}{\mu} - 2\pi q\right) c(x - 2q\mu)/2 + 2\pi q c(x - q\mu)

= \frac{\pi}{\mu} t + sc \left((x - 2q\mu)^2/2 + 2\mu c(x - q\mu)\right) = x \frac{\pi}{\mu} (t + scx/2)

Integrating this property into the expression of $\beta_{(r,s,t)}(f) \cdot \alpha_{(r,s,t)}(F)$:

$$\left(\beta_{(r,s,t)}(f) \cdot \alpha_{(r,s,t)}(F)\right)(x,y) =

= \sum_q e^{ix \frac{\pi}{\mu}(t + scx/2)} f(x - r - 2q\mu, y - s - 2q\nu) F(-q, x - r - 2q\mu, y - s - 2q\nu)

= \beta_{(r,s,t)}(f \cdot F)(x,y).$$

$\square$
Notation 4.11. We denote by \( \mathcal{N} \) the \( \mathcal{D} \)-module of the elements of \( N \) that are regular under the action \( \beta \) of \( H_1 \). It is a finitely generated projective module over \( \mathcal{D} \).

Using the covariant action on \( N \), it is easy to construct connexions over \( \mathcal{N} \):

**Proposition 4.12.** The connexions over \( \mathcal{N} \) associated to \( \varphi_{1,3} \) and \( \varphi_{2,3} \) respectively, are:

\[
(\nabla f)(x, y) = \frac{\partial f}{\partial x} (x, y) dx - \frac{i \pi}{\mu} xf(x, y) dp
\]

and

\[
(\nabla f)(x, y) = \left( \frac{\partial f}{\partial y} - \frac{i \pi c}{2\mu} x^2 f \right) dy - \frac{i \pi}{\mu} xf(x, y) dp.
\]

**Proof.** This is an obvious application of propositions 4.9 and 4.10. \( \square \)

**Proposition 4.13.** If \( \mu \neq 0 \), the pairings over \( \mathcal{N} \) are given by (4.1).

**Proof.** First consider the case of \( \varphi_{1,3} \). Computing \( \nabla^2 f \) yields:

\[
\nabla(\nabla f) = \nabla \left( \frac{\partial f}{\partial x} dx - \frac{i \pi}{\mu} xf dp \right) = \left( \frac{\partial^2 f}{\partial x^2} dx - \frac{i \pi}{\mu} \frac{\partial f}{\partial x} dp \right) dx - \frac{i \pi}{\mu} \left( \left( f + x \frac{\partial f}{\partial x} \right) dx - \frac{i \pi}{\mu} xf dp \right) dp
\]

thus \( \nabla^2 = -\frac{i \pi}{\mu} \text{Id}_{\mathcal{N}} \otimes dx \wedge dp \). The trace of \( \text{Id}_{\mathcal{N}} \) was computed by Abadie (theorem 4.4). The pairing is given by the table (4.1): \( \tau (p) = 2\mu \). Hence

\[
\langle [\mathcal{N}], \varphi_{1,3} \rangle = -i2\pi.
\]

In the case of \( \varphi_{2,3} \), the two terms in the connexion commute, so \( \nabla^2 = 0 \) and

\[
\langle [\mathcal{N}], \varphi_{2,3} \rangle = 0.
\]

\( \square \)

### 4.2 Connexions and Pairings for \( \mathcal{N}^\dagger \)

We define a second module \( \mathcal{N}^\dagger \) over \( D \) through an isomorphism between two \( D \):

**Proposition 4.14.** There is an isomorphism \( \Phi : D_{\mu,\nu}^c \to D_{\nu,\mu}^c \), induced by the Hilbert bimodule representation:

\[
\pi(a)(x, y) = a(-y, -x) \quad T(\xi)(x, y) = e^{i2\pi c(y+\mu)(x+\nu)} \xi(-y, -x)
\]

Moreover, this isomorphism intertwines the actions of \( H_1 \):

\[
\Phi(\alpha_{r,s,t}(F)) = \alpha'_{-s,-r,-t+c\ar}(\Phi(F)) \quad (4.2)
\]

where \( \alpha \) and \( \alpha' \) are the actions of \( H_1 \) over \( D_{\mu,\nu}^c \) and \( D_{\nu,\mu}^c \), respectively.
The intertwining of the actions proves that $\mathcal{D}_{\mu,\nu}$ and $\mathcal{P}_{\nu,\mu}$ are sent onto one another.

**Proof.** The existence of such isomorphism was proved in [5] theorem 2.2. However, we here give an explicit isomorphism and specify the intertwining relation.

Following the definition 2.5, the algebra $D_{\nu,\mu,0}^c$ of continous functions $\mathbb{R} \times \mathbb{R} \to \mathbb{C}$ which satisfy

$$F(p, x+1, y) = e(-cp(y-p\mu))F(p, x, y) \quad F(p, x, y+1) = F(p, x, y).$$

is dense in $D_{\nu,\mu}^c$. The definition 2.5 implies that it suffices to check that there is a representation of the Hilbert bimodule $M_{\mu,\nu}^c$ in $D_{\nu,\mu}^c$ to get the morphism of algebra $\Phi$.

Let us check that $T(\xi)$ has the degree $-1$ in $D_{\nu,\mu,0}^c$:

$$T(\xi)(x+1, y) = e^{2\pi c(y+\mu)(x+1)} \xi(-y-x-1) = e^{2\pi c(y+\mu)} T(\xi)(x, y)$$

and

$$T(\xi)(x, y+1) = e^{2\pi c(y+\mu)(x+\nu)} \xi(-y-1, -x) = e^{2\pi c(y+\mu)} e^{2\pi c(x+\mu)} e^{2\pi c(-x-\nu)} \xi(-y, -x) = T(\xi)(x, y).$$

To prove that $\pi$ and $T$ induce a bimodule representation, it suffices to prove the points (i) and (iv) of definition 2.2. The others are consequences of these two (see definition 2.1).

Regarding point (i):

$$T(\xi)^* T(\xi)(x, y, 0) = e^{-2\pi c(y+\mu)(x+\nu)} \xi(-y+2\mu, -x+2\nu)e^{2\pi c(y+\mu)(x+\nu)} \xi(-y+2\mu, -x+2\nu) = \xi(-y+2\mu, -x+2\nu) \xi(-y+2\mu, -x+2\nu) = \pi(\langle \xi, \xi \rangle)(x, y).$$

As for point (iv):

$$T(\xi) T(\xi)^*(x, y, 0) = e^{2\pi c(y+\mu)(x+\nu)} e^{-2\pi c(y+\mu)(x+\nu)} e^{2\pi c(y+\mu)(x+\nu)} \xi(-y, -x) = \xi(-y, -x) \xi(-y, -x) = \pi(\langle \xi, \xi \rangle)(x, y).$$

Hence, there is a homomorphism from $D_{\mu,\nu}^c$ into $D_{\nu,\mu}^c$.

$\pi(A)$ and $T(E)$ generate the algebra, hence it suffices to check the intertwining relation 4.2 on these two sets. Denoting by $\alpha'$ the action of $H_1$ on $D_{\nu,\mu}^c$, we get:

$$\pi(\alpha_{r,s, t}(a))(x, y) = a(-r-y, -x-s) = \alpha'(s,-r,-t+cr) \pi(a)(x, y)$$

as well as:

$$T(\alpha_{r,s, t}(\xi))(x, y) = T\left( e\left(-t+cs(x-r-\mu)\right) \xi(x-r, y-s) \right)$$

$$= e^{2\pi c(y+\mu)(x+\nu)} e\left(-t+cs(-y-r-\mu)\right) \xi(-y-r, -x-s)$$

$$= e\left(-t+c(y+\mu)(x+s+\nu)+cr\right) \xi(-y-r, -x-s)$$

$$= e\left(-(t-csr) - cr(x+s+\nu)\right) e(c(y+r+\mu)(x+s+\nu)) \xi(-y-r, -x-s)$$

$$= (\alpha'(s,-r,-t+cr) T(\xi))(x, y).$$

$\square$
We simply have to derive the intertwining relation to obtain:

\[ \text{Proof.} \]

Over \( D \), the cyclic cocycles\footnote{Lemma 4.17.} \( \tau_{\partial} \) linking the derivations \( \partial \) of \( \langle \cdot \rangle_{\mu} \) induced from \( \Phi \) are:

\[ \Phi^* \varphi_{1,3} = \varphi_{2,3} \quad \Phi^* \varphi'_{2,3} = \varphi_{1,3} \]

\[ \text{Proof.} \] We simply have to derive the intertwining relation to obtain:

\[ \begin{align*}
-\Phi \circ \partial_1 &= \partial_2' \circ \Phi \\
-\Phi \circ \partial_2 &= \partial_1' \circ \Phi \\
-\Phi \circ \partial_3 &= \partial_3' \circ \Phi
\end{align*} \]

linking the derivations \( \partial'_i \) of \( \mathcal{D}_{\mu,\nu} \) and \( \partial_i \) of \( \mathcal{N}_{\nu,\mu} \). The traces \( \tau'_{\mu,\nu} \) and \( \tau_{\nu,\mu} \) are invariant in the sense that \( \tau_{\mu,\nu} (\Phi(F)) = \tau'_{\nu,\mu}(F) \). Consequently, the induced cocycles are:

\[ \Phi^* \varphi_{i,3}(a_0, a_1, a_2) = \tau \left( \Phi(a_0) \left[ \partial'_i(\Phi(a_1))\partial'_3(\Phi(a_2)) - \partial'_3(\Phi(a_1))\partial'_i(\Phi(a_2)) \right] \right) = \tau \left( \Phi(a_0) \partial_j(a_1)\partial_j(a_2) - \partial_j(a_1)\partial_j(a_2) \right) = \varphi_{j,3}(a_0, a_1, a_2) \]

where \( (i, j) \) is a permutation of \( (1, 2) \).

\[ \square \]

\[ \text{Proposition 4.18.} \quad \text{If } \nu \neq 0 \neq \mu, \text{ the pairings of } \mathcal{N}^+ \text{ are given by the table } \mathbb{N}_4.1. \]

\[ \text{Proof.} \] Considering remark \(^4.16\) the definition of \( \Phi^* \varphi_{1,3} \) and the previous lemma, we clearly have

\[ \langle \mathcal{N}^+_{\mu,\nu}, \varphi_{1,3} \rangle = \langle \mathcal{N}^+_{\mu,\nu}, \Phi^* \varphi_{1,3} \rangle = \langle \mathcal{N}^+_{\nu,\mu}, \varphi_{1,3} \rangle. \]

Since \(^4.1\) gives us the values of the pairings \( \langle \mathcal{N}^+_{\nu,\mu}, \varphi_{1,3} \rangle \) and \( \langle \mathcal{N}^+_{\nu,\mu}, \varphi'_{1,3} \rangle \), we get

\[ \langle \mathcal{N}^+_{\mu,\nu}, \varphi_{1,3} \rangle = 0 \quad \langle \mathcal{N}^+_{\mu,\nu}, \varphi'_{1,3} \rangle = -i2\pi. \]

\[ \square \]

\section{Units and First Odd Pairings}

The inclusion \( A \hookrightarrow \mathcal{D}_{\mu,\nu}^c \) induces 2 elements of \( K_1(\mathcal{D}_{\mu,\nu}^c) \):

\[ U_1(p, x, y) = \delta_0, p \epsilon(x) \quad U_2(p, x, y) = \delta_0, p \epsilon(y) \]

We want to construct a third unitary. As a first step, notice that the definition \(^2.4\) entails:

\[ \forall \xi \in M, \forall a \in A, \quad \xi a = \sigma(a) \xi. \quad (5.1) \]

\[ \text{Corollary 5.1.} \quad \text{For all } \xi, \zeta \in \mathcal{M}_{\mu,\nu}^c \subseteq \mathcal{D}_{\mu,\nu}^c, \sigma(\xi^* \zeta) = \zeta \xi^*. \]
Remark 5.2. The automorphism $\sigma$ of $A$ can be extended to an automorphism of $D$, also noted $\sigma$. It suffices to set $\sigma = \sigma(2\mu, 2\nu, 0)$.

Proposition 5.3. Let

$$M_+ = \begin{pmatrix} \xi_1 & 0 \\ -\xi_2 & 0 \end{pmatrix} \quad M_- = \begin{pmatrix} 0 & \sigma(\xi_2)^* \\ 0 & \sigma(\xi_1)^* \end{pmatrix}.$$ 

We get the relations:

$$M_+ M_+^* = P_+ = \begin{pmatrix} \xi_1 \xi_1^* -\xi_1 \xi_2^* \\ -\xi_2 \xi_1^* & \xi_2 \xi_2^* \end{pmatrix} \quad M_- M_-^* = P_- = \begin{pmatrix} 0 & \sigma(\xi_2)^* \\ \sigma(\xi_1)^* & 0 \end{pmatrix},$$

$$M_+ M_-^* = P_- = \begin{pmatrix} 0 & \sigma(\xi_2^2 \xi_2) \\ \sigma(\xi_1 \xi_2) & 0 \end{pmatrix} \quad M_- M_+^* = Q_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$P_\pm = P_\pm^2 = P_\pm^* \quad P_+ + P_- = I_2 \quad Q_\pm = Q_\pm^2 = Q_\pm^* \quad Q_+ + Q_- = I_2 \quad P_\pm M_\pm = M_\pm \quad M_\pm Q_\pm = M_\pm \quad M_\pm M_\pm^* = 0 \quad M_\pm^* M_\pm = 0.$$

Thus $U_3 = M_+ + M_-$ is a unitary.

We will compute the index of this element in proposition 7.3 and make further remarks on this element in section A.

Proof. The first four relations are proved by direct calculation. The relations between the $Q_\pm$ are obvious. The $P_\pm$ are clearly self-adjoint. A direct computation using $\xi_1 \xi_1^* + \xi_2 \xi_2^* = \sum_{i=1}^2 A(\xi_i, \xi_i) = 1$ shows that $P_2^2 = P_2$.

Let us now show that $P_+ + P_- = I_2$, which proves the relations between $P_\pm$:

$$\begin{pmatrix} \xi_1 \xi_1^* -\xi_1 \xi_2^* \\ -\xi_2 \xi_1^* & \xi_2 \xi_2^* \end{pmatrix} + \begin{pmatrix} \sigma(\xi_2^2 \xi_2) \\ \sigma(\xi_1 \xi_2) \end{pmatrix} = \begin{pmatrix} \xi_1 \xi_1^* + \xi_2 \xi_2^* \\ -\xi_2 \xi_1^* & \xi_2 \xi_2^* \end{pmatrix} = I_2,$$

through a systematic use of corollary 5.1. Finally, developing

$$(M_\pm Q_\pm - M_\pm)^*(M_\pm Q_\pm - M_\pm) = (Q_\pm M_\pm^* - M_\pm^*)(M_\pm Q_\pm - M_\pm) = 0$$

ensures that $M_\pm Q_\pm = M_\pm$. Now $P_\pm M_\pm = M_\pm M_\pm^* M_\pm = M_\pm Q_\pm = M_\pm$, and

$$M_\pm M_\pm^* = M_\pm Q_\pm Q_\pm M_\pm^* = 0 \quad M_\pm^* M_\pm = M_\pm^* P_\pm P_\pm M_\pm = 0.$$

The definition of odd pairings can be found in III.3 proposition 3.
Definition 5.4 (Chern-Connes pairings). The following formula defines a bilinear pairing between $K_1(A)$ and $HC^n(A)$:

$$\langle [u], [\phi] \rangle = \frac{2^{-n}}{\sqrt{2i}} \Gamma \left( \frac{n}{2} + 1 \right)^{-1} (\phi \# \text{Tr})(u^* - 1, u - 1, u^* - 1, \ldots, u - 1) \quad (5.2)$$

where $n = 2m + 1$, $[U] \in K_1$ and $\phi \in ZC^{2m+1}$. These pairings satisfy $\langle [u], [S\phi] \rangle = \langle [u], [\phi] \rangle$.

Theorem 5.5. Using the previous unitaries $U_i$, the values of the pairings are given by the table:

<table>
<thead>
<tr>
<th></th>
<th>$\varphi_1$</th>
<th>$\varphi_2$</th>
<th>$\varphi_3$</th>
<th>$\varphi_{1,2,3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[U_1]$</td>
<td>$-\sqrt{i2\pi}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$[U_2]$</td>
<td>0</td>
<td>$-\sqrt{i2\pi}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$[U_3]$</td>
<td>$\sqrt{i2\pi} 2\nu$</td>
<td>$-\sqrt{i2\pi} 2\mu$</td>
<td>0</td>
<td>$(i2\pi)^{3/2}c/3$</td>
</tr>
</tbody>
</table>

(5.3)

The proof of these results will occupy the next two sections.

Proposition 5.6. The pairings with $U_1$ and $U_2$ are given by the first two lines of (5.3).

Proof. For $i = 1, 2$ and $j = 1, 2, 3$, let us evaluate:

$$\langle [U_i], [\varphi_j] \rangle = \frac{1}{\sqrt{2i\pi}} \tau((U_i^* - 1)\partial_j(U_i - 1)) = \frac{1}{\sqrt{2i\pi}} \tau(U_i^* \partial_j(U_i));$$

using $\tau(\partial_j(X)) = 0$ and $\partial_j(1) = 0$. The explicit expressions of $\partial_j$ in degree 0 and $U_i$ yields:

$$\langle [U_i], [\varphi_j] \rangle = -\delta_{i,j} \sqrt{2i\pi}.$$

The pairings with $\varphi_{1,2,3}$ are given by:

$$\langle [U_i], \varphi_{1,2,3} \rangle = \sum_{\sigma \in \Sigma_3} \varepsilon(\sigma) \tau((U_i^* - 1)\partial_{\sigma(1)}(U_i - 1)\partial_{\sigma(2)}(U_i^* - 1)\partial_{\sigma(3)}(U_i - 1)).$$

Each term of the above sum contains a vanishing derivation, hence $\langle [U_i], \varphi_{1,2,3} \rangle = 0$.

To evaluate the other odd pairings, we need to estimate the commutation relations between $\alpha$ and the derivations $\partial_i$:

Notation 5.7. We use the notation $\partial_{(u,v,w)} = u\partial_1 + v\partial_2 + w\partial_3$.

Proposition 5.8. The equality

$$\alpha_{r,s,t}(\partial_{(u,v,w)}(F)) = \partial_{(u',v',w')}(\alpha_{r,s,t}(F))$$

is fulfilled if and only if

$$u = u' \quad v = v' \quad w = w' + c(v'r - su'$$

(5.4)

Proof. This is a straightforward computation.

Proposition 5.9. The pairings of $U_3$ with the degree 1 cocycles are given by (5.3).
Explicit computations yields:

\[\sqrt{2i\pi}(U_3, \varphi) = \tau(M_+^* \partial M_+ + M_-^* \partial M_-)\]

because \(\tau(\partial(M_+ + M_-)) = 0\). The trace vanishes on nonzero degree elements, hence we keep only the degree 0 terms of the expression:

Explicit computations yields:

\[
M_+^* \partial M_+ = \begin{pmatrix} \xi_1^* \partial \xi_1 + \xi_2^* \partial \xi_2 & 0 \\ 0 & 0 \end{pmatrix},
M_-^* \partial M_- = \begin{pmatrix} 0 & 0 \\ \sigma(\xi_2) \partial \sigma(\xi_2) + \sigma(\xi_1) \partial \sigma(\xi_1) \end{pmatrix}.
\]

The commutation relations \(\partial \sigma(F) = \sigma(\partial(F) + k \partial_3(F))\) holds for all \(\partial_i, i \in \{1, 2, 3\}\), but with different constants \(k_i\). Proposition 5.8 ensures

\[
k_1 = -2\pi\nu, \quad k_2 = 2\pi\mu, \quad k_3 = 0.
\]

Upon integrating the commutations relations in the trace of \(M_+^* \partial M_+\):

\[
\sigma(\xi_1) \sigma(\partial_1^* + k \partial_3(\xi_1^*)) + \sigma(\xi_2^*) \sigma(\partial_2^* + k \partial_3(\xi_2^*)) = \sigma(\xi_1 \partial_1(\xi_1^*) + \xi_2 \partial_2(\xi_2^*) + k(\xi_1 \partial_3(\xi_1^*) + \xi_2 \partial_3(\xi_2^*) + \xi_3)).
\]

Yet \(\partial_3(\xi_i) = -i2\pi\xi_i^* + \xi_1\xi_i^* + \xi_2\xi_i^* = 1\), so:

\[
\sqrt{2i\pi}(U_3, \varphi) = \tau(\xi_1^* \partial \xi_1 + \xi_2^* \partial \xi_2) + \tau(\sigma(\xi_1^* \partial_1 + \xi_2^* \partial_2)) - i2\pi k
\]

\[
= \tau(\xi_1^* \partial \xi_1 + \xi_2^* \partial \xi_2 + \sigma(\xi_1 \partial \xi_1^* + \xi_2 \partial \xi_2^*)) - i2\pi k
\]

\[
= \tau(\xi_1^* \partial \xi_1 + \xi_2^* \partial \xi_2 + \partial(\xi_1^* \xi_1^*) + \partial(\xi_2^* \xi_2^*)) - i2\pi k
\]

\[
= \tau(\partial(\xi_1^* \xi_1 + \xi_2^* \xi_2)) - i2\pi k = \tau(\partial(1)) - i2\pi k,
\]

where we applied corollary 5.1. Taking into account the different values of \(k_i\):

\[
\langle U_3, \varphi_1 \rangle = \sqrt{2i\pi} 2\pi\nu, \quad \langle U_3, \varphi_2 \rangle = -\sqrt{2i\pi} 2\nu, \quad \langle U_3, \varphi_3 \rangle = 0.
\]

\[
\square
\]

6 Top Degree Pairing

**Proposition 6.1.** The pairing \([U_3, \varphi_{1,2,3}]\) is given by the table 5.3.

The proof of this proposition will fill the remainder of this section. Setting \(U = U_3 = M_+ + M_-\), we want to evaluate:

\[
[U_3, \varphi_{1,2,3}] = \frac{1}{8\sqrt{2\pi}} \frac{4}{3\sqrt{\pi}} \sum_{\sigma \in \Sigma_3} \varepsilon(\sigma) T\sigma(1)\sigma(2)\sigma(3)
\]

where

\[
T_{ijk} = \tau((U^* - 1) \partial_i(\partial_j(U^* - 1) - \partial_k(U - 1))
\]

\[
= \tau((\partial_i(U) \partial_j(U) \partial_k(U))).
\]

All the following terms have odd degree – and therefore vanishing trace:

\[
\tau(\partial_i(U) \partial_j(U) \partial_k(U)) = \tau(\partial_i(M_+ + M_-) \partial_j(M_+ + M_-) \partial_k(M_+ + M_-)) = 0.
\]
Lemma 6.2. The following relations hold:
\[
\partial_i(M_\pm)M_\mp = -M_\pm \partial_i(M_\mp) \quad \partial_i(M_\mp^*)P_\mp = -M_\pm \partial_i(P_\mp) \quad (6.2)
\]
\[
\tau(P_\pm \partial_i(M_\pm)\partial_j(M_\mp^*)) = \tau(P_\pm \partial_i(P_\pm)\partial_j(P_\pm)) \quad (6.3)
\]
\[
\tau(P_\pm \partial_i(M_\pm)\partial_j(M_\mp^*)) = \tau(\partial_i(M_\pm)\partial_j(M_\mp^*) + P_\pm \partial_j(P_\pm)\partial_i(P_\pm)) \quad (6.4)
\]

Proof. The first series can be proved by integration by parts:
\[
\partial_i(M_\pm)M_\mp = \partial_i(M_\pm M_\mp) - M_\pm \partial_i(M_\mp) = -M_\pm \partial_i(M_\mp)
\]
because (lemma 6.3) \(M_\pm M_\mp = 0\). We also have \(M_\pm^* P_\mp = M_\pm^* P_\mp P_\mp = 0\), which enables to prove the second equality of (6.2) using the same method. Now
\[
\tau(P_\pm \partial_i(M_\pm)\partial_j(M_\mp^*)) = \tau(\partial_j(M_\mp^*)P_\pm \partial_i(M_\pm)) = \\
= -\tau(M_\pm^* \partial_j(P_\mp)\partial_i(M_\pm)) = \tau(M_\pm^* \partial_j(P_\pm)\partial_i(M_\pm)) = \\
= \tau(P_\pm \partial_j(P_\pm)\partial_i(P_\pm) + M_\pm^* P_\pm \partial_j(P_\pm)\partial_i(P_\pm)) = \\
= \tau(P_\pm \partial_j(P_\pm)\partial_i(P_\pm)),
\]
using (6.2), \(\partial_i(P_\pm) = -\partial_i(P_\mp)\) and \(P_\pm \partial_i(P_\pm)P_\mp = 0\). This last equality is true because \(P_\pm\) is an idempotent and \(\partial_i\) a derivation. Regarding the last equation:
\[
\tau(P_\pm \partial_i(M_\pm)\partial_j(M_\mp^*)) = \tau(\partial_i(P_\pm)M_\mp - \partial_i(P_\pm)M_\pm)\partial_j(M_\mp^*) = \\
= \tau(\partial_i(M_\pm)\partial_j(M_\mp^*) - \partial_i(P_\pm)M_\pm \partial_j(M_\mp^*)M_\mp^* \partial_j(P_\pm)) = \\
= \tau(\partial_i(M_\pm)\partial_j(M_\mp^*) - P_\pm \partial_i(P_\pm)P_\pm M_\pm \partial_j(M_\mp^*)M_\mp^* \partial_j(P_\pm)) = \\
= \tau(\partial_i(M_\pm)\partial_j(M_\mp^*) + P_\pm \partial_j(P_\pm)\partial_i(P_\pm)),
\]
possible because \(M_\pm^* = M_\pm^* P_\pm\).

We will also need the following lemma:

Lemma 6.3. If \((i, j) = (1, 2)\) or \((2, 1)\), we have both
\[
\tau(\partial_i(U)\partial_j(U^*)) = \tau(\partial_i(M_+\partial_j(M_+^*) + \partial_i(M_-\partial_j(M_-^*)))
\]
and
\[
\tau(\partial_1(U)\partial_2(U^*) - \partial_2(U)\partial_1(U^*)) = 0.
\]

Proof. The first equality is obvious by keeping only the degree 0 terms in
\[
\tau(\partial_i(U)\partial_j(U^*)) = \tau(\partial_i(M_+ \partial_j(M_+^*) + \partial_i(M_- \partial_j(M_-^*))).
\]
The second one is obtained by integration by parts:
\[
\tau(\partial_i(U)\partial_j(U^*) - \partial_2(U)\partial_1(U^*)) = \\
= \tau(\partial_1(U\partial_2(U^*)) - U\partial_1\partial_2(U^*) - \partial_2(U\partial_1(U^*)) + U\partial_2\partial_1(U^*)) = \\
= c\tau(U\partial_1(U^*)) = 0,
\]
using \([\partial_1, \partial_2] = -c\partial_3\) and \(<\varphi_3, U> = 0\).
6.0.1 Terms $T_{132}$ and $T_{231}$

Using the identity $\partial_3(M_\pm) = \mp i2\pi M_\pm$, we can evaluate $T_{231}$:

$$T_{231} = -i2\pi \tau((M_+^* + M_-^*)\partial_2(\mathcal{U})(M_+^* - M_-^*)\partial_1(\mathcal{U}))$$

$$= -i2\pi \tau\left(M_+^* \partial_2(\mathcal{U})M_+^*\partial_1(\mathcal{U}) - M_-^* \partial_2(\mathcal{U})M_-^*\partial_1(\mathcal{U})\right) + M_-^* \partial_2(\mathcal{U})M_-^*\partial_1(\mathcal{U}) - M_-^* \partial_2(\mathcal{U})M_-^*\partial_1(\mathcal{U})\right).$$

Same thing for $T_{132}$:

$$T_{132} = -i2\pi \tau((M_+^* + M_-^*)\partial_1(\mathcal{U})(M_+^* - M_-^*)\partial_3(\mathcal{U}))$$

$$= -i2\pi \tau\left(M_+^* \partial_1(\mathcal{U})M_+^*\partial_3(\mathcal{U}) - M_-^* \partial_1(\mathcal{U})M_-^*\partial_3(\mathcal{U})\right) + M_-^* \partial_1(\mathcal{U})M_-^*\partial_3(\mathcal{U}) - M_-^* \partial_1(\mathcal{U})M_-^*\partial_3(\mathcal{U})\right).$$

Taking into account the $\varepsilon(\sigma)$ of (6.1) and the fact that $\tau$ is a trace,

$$T_{231} - T_{132} = i4\pi \tau\left(M_+^* \partial_2(\mathcal{U})M_+^*\partial_1(\mathcal{U}) - M_-^* \partial_2(\mathcal{U})M_-^*\partial_1(\mathcal{U})\right).$$

As $\mathcal{U} = M_+ + M_-$, we can put both terms in the general form:

$$\tau(M_+^* \partial_i(M_+ + M_-)M_-^*\partial_j(M_+ + M_-)).$$

Keeping only elements of total degree 0:

$$\tau(M_+^* \partial_i(M_+ + M_-)M_-^*\partial_j(M_+ + M_-)).$$

The first term integrates by parts:

$$(\partial_3(M_+^*M_+) - \partial_4(M_-^*)M_+^*)M_-^*\partial_j(M_-),$$

which vanishes because $\partial_4(M_+^*M_+) = \partial_4(Q_+) = 0$ and $M_+ M_- = 0$. For the second term,

$$\tau\left(M_+^* (\partial_i(M_+^*M_+) - M_-^*\partial_i(M_-^*))\partial_j(M_+)\right) = \tau(M_+^* (P_+\partial_i(M_+) - M_-^*\partial_i(M_+)) =$$

$$= -\tau(P_- \partial_i(M_+)\partial_j(M_+)) = -\tau(P_- \partial_i(P_+)\partial_j(P_+))$$

applying (6.2), trace property and then (6.3).

Therefore, the contribution of $T_{132}$ and $T_{231}$ is

$$T_{231} - T_{132} = -i4\pi \tau(P_+ (\partial_2P_+\partial_3P_+ - \partial_1P_+\partial_3P_+)).$$

6.0.2 Terms $T_{123}$ and $T_{213}$

As $\partial_3(M_\pm) = \pm i2\pi M_\pm$, the terms $T_{123}$ and $T_{213}$ can be written:

$$T_{123} = i2\pi \tau((M_+^* + M_-^*)\partial_1(\mathcal{U})\partial_3(\mathcal{U}^*)(M_+ - M_-))$$

$$= i2\pi \tau((P_+ - P_-)\partial_1(\mathcal{U})\partial_3(\mathcal{U}^*)) = i2\pi \tau((2P_+ - I_2)\partial_1(\mathcal{U})\partial_3(\mathcal{U}^*))$$

$$= i2\pi \tau((2P_+ - I_2)\partial_1(\mathcal{U})\partial_3(\mathcal{U}^*))$$
Taking the difference $T_{123} - T_{213}$, then using lemma 6.3, the expression becomes:

$$T_{123} - T_{213} = i4\pi T\left(P_\tau (\partial_1(U)\partial_2(U^*) - \partial_2(U)\partial_1(U^*))\right).$$

Let us study the term

$$\tau(P_+\partial_i(M_+ - M_-)\partial_j(M^*_+ + M^*_-)) = \tau(P_+ (\partial_i(M_+\partial_j(M^*_+) + \partial_i(M_-)\partial_j(M^*_-))))$$

$$= \tau(\partial_i(M_+)\partial_j(M_+) + P_+\partial_j(P_+\partial_i(P_+) + P_-\partial_j(P_-))\partial_i(P_+)) =$$

$$= \tau(\partial_i(M_+)\partial_j(M^*_+) + \partial_j(P_+)\partial_i(P_+),$$

using the equations (6.4) and (6.3), and then $\partial_i(P_-)\partial_j(P_-) = \partial_i(P_+)\partial_j(P_+)$. Next,

$$\tau(\partial_i(P_+)\partial_2(P_+) - \partial_2(P_+)\partial_1(P_+)) = 0$$

because $\tau$ is a trace.

Finally, the contribution of the terms $T_{123}$ and $T_{213}$ is

$$T_{123} - T_{213} = i4\pi T\left(\partial_i(M_+)\partial_2(M^*_+) - \partial_2(M_+)\partial_1(M^*_+)\right).$$

### 6.0.3 Termes $T_{312}$ and $T_{321}$

**Lemma 6.4.** The analogs of (6.3) hold.

$$Q_\pm \partial_1(M^*_\pm) = -\partial_1(Q_\mp)M^*_\pm = 0 \hspace{1cm} Q_\pm \partial_1(M^*_\pm) = \partial_1(M^*_\pm) \hspace{1cm} (6.5)$$

$$Q_\pm \partial_1(M^*_\pm)\partial_j(M_\pm) = 0 \hspace{1cm} (6.6)$$

$$Q_\pm \partial_i(M^*_\pm)\partial_j(M_\pm) = \partial_i(M^*_\pm)\partial_j(M_\pm) \hspace{1cm} (6.7)$$

**Proof.** For the first series of relations,

$$Q_\pm \partial_1(M^*_\pm) = \partial_1(Q_\mp)M^*_\pm - \partial_i(Q_\mp)M^*_\pm = 0$$

because $Q_\pm M^*_\pm = Q_\pm Q_\mp M^*_\pm = 0$ and $\partial(Q_\mp) = 0$. Likewise,

$$Q_\pm \partial_i(M^*_\pm) = \partial_i(Q_\pm M^*_\pm) - \partial_i(Q_\pm)M^*_\pm = \partial_i(M^*_\pm),$$

thanks to $Q_\pm M^*_\pm = M^*_\pm$. For the second series,

$$\tau(Q_\mp \partial_1(M^*_\pm)\partial_j(M_\pm)) = -\tau(\partial_j(Q_\mp)M^*_\pm\partial_i(M_\pm)) = 0,$$

using the previous relations. The third relation is obvious starting from (6.5) \[ \square \]

The terms $T_{312}$ and $T_{321}$ can be written:

$$T_{312} = i2\pi T((M^*_+ + M^*_-)(M_+ - M_-)\partial_3(U^*)\partial_j(U)) =$$

$$i2\pi T((Q_+ - Q_-)\partial_j(U^*)\partial_j(U)) = i2\pi T((2Q_+ - 1)\partial_j(U^*)\partial_j(U)).$$

Taking the difference $T_{312} - T_{321}$, then using lemma 6.3, we get:

$$T_{312} - T_{321} = i4\pi T\left(Q_+ (\partial_1(U^*)\partial_2(U) - \partial_2(U^*)\partial_1(U))\right).$$

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Let us study the following term
\[ \tau(Q_+ \partial_i (M^*_+ + M^*_-) \partial_j (M_+ + M_-)) = \tau(Q_+ (\partial_i (M^*_+) \partial_j (M_+) + \partial_i (M^*_-) \partial_j (M_-)) = \tau(\partial_i (M^*_+) \partial_j (M_+)). \]
The difference can thus be written:
\[ T_{312} - T_{321} = i4\pi \tau(\partial_i (M^*_+) \partial_j (M_+) - \partial_i (M^*_+) \partial_j (M_+)). \]

6.0.4 Synthesis and Final Computation

Forming the synthesis of the studied terms:
\[
6\sqrt{2i\pi} \langle [U_3], [\varphi_{1.2}] \rangle = (T_{231} - T_{132} + T_{123} - T_{312} - T_{321})
= i4\pi \tau(P_+ (\partial_1 P_+ \partial_2 P_+ - \partial_2 P_+ \partial_1 P_+))
= i4\pi \tau(\partial_1 (M^*_+) \partial_2 (M^*_+) - \partial_2 (M^*_+) \partial_1 (M^*_+))
= i4\pi \tau(P_+ (\partial_1 P_+ \partial_2 P_+ - \partial_2 P_+ \partial_1 P_+))
= i4\pi \langle [P_+], [\varphi_{1.2}] \rangle
\]
This expression is a coupling on the algebra \( A = C(T^2) \). We can compute it using a connexion.

First identify the module of \( C(T^2) \) which corresponds to the projector \( P_+ \).

**Lemma 6.5.** The module \( P_+ A^2 \) is isomorphic to \( M^\sigma_{0,0} \).

Notice that we only need an identification as module, and not as bimodule.

**Notation 6.6.** We denote the elements of \( M^\sigma_{0,0} \) by \( \zeta^* \). More generally, we include \( M^\sigma_{0,0} \) into \( D^\sigma_{0,0} \) in the following computation.

**Proof.** To identify the module \( M \) associated to \( P_+ \), the simplest thing is to interpret \( M^\sigma_+ \) and \( M_+ \) respectively as maps \( P_+ A^2 \to M \) and \( M \to P_+ A^2 \). Formally, we introduce the maps \( \Phi : P_+ A^2 \to M^\sigma_{0,0} \) and \( \Psi : M^\sigma_{0,0} \to P_+ A^2 \):
\[
\Phi \left( \begin{array}{c} a_1 \\ a_2 \end{array} \right) = \left( \begin{array}{c} \xi^*_1 \\ -\xi^*_2 \end{array} \right) \begin{array}{c} a_1 \\ a_2 \end{array} \\
\Psi(\zeta^*) = \left( \begin{array}{c} \xi_1 \\ -\xi_2 \end{array} \right) \zeta^*.
\]

Using the properties of \( M_+ \) and \( M^\sigma_+ \), we see that the maps \( \Phi \) and \( \Psi \) are inverse to one another, and that they preserve the scalar products. \( \Box \)

The following result is well known:

**Proposition 6.7.** The pairing of \( M^\sigma \) with \( \varphi_{1.2} \) is given by:
\[
\langle [M^\sigma_{0,0}], [\varphi_{1.2}] \rangle = i2\pi c
\]
This proposition enables us to complete the computation of \( \langle [U_3], [\varphi_{1.2,3}] \rangle : \)
\[
\langle [U_3], [\varphi_{1.2,3}] \rangle = \frac{i4\pi}{6\sqrt{2i\pi}} i2\pi c = c \frac{(2\pi)^{3/2}}{3}
\]
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7 Consequences of the Pairings

An immediate consequence of (3.2) and table (5.3) is that \((U_1, U_2, U_3)\) form a basis of \(K_1(D) \otimes \mathbb{C}\). It is therefore natural to wonder if these elements generate \(K_1(D)\). The answer is no.

To prove the above, first notice that since the QHM are Pimsner algebras associated to the Hilbert bimodule \(M_c\), their \(K\)-theory fit in the 6-terms exact sequence (see [20]):

\[
\begin{array}{c}
K_0(A) \xrightarrow{\text{Id} - [M^*]} K_0(A) \xrightarrow{\iota} K_0(D) \\
\downarrow \quad \downarrow \\
K_1(D) \xleftarrow{\iota} K_1(A) \xleftarrow{\text{Id} - [M^*]} K_1(A)
\end{array}
\]  \hspace{1cm} (7.1)

To establish this exact sequence, Pimsner’s approach is in fact to start with a short exact sequence

\[
0 \to J \to T \to D \to 0 \tag{7.2}
\]

and to prove that both \(J\) and \(T\) are \(KK\)-equivalent to \(A\). The algebra \(T\) was defined in [20]. In our special case, one can prove (see section A) that \(T\) can be identified with the \(\ast\)-subalgebra of \(T \otimes D\) generated by \(1 \otimes a\) and \(S \otimes \xi\) for \(a \in A\) and \(\xi \in M\). Notice that \(T\) is a nuclear \(C^\ast\)-algebra, thus we don’t need to specify which \(C^\ast\)-norm we are taking in the tensor product. We identify \(T\) with the unital \(C^\ast\)-algebra generated by the projector \(P\) and the isometry \(S\) with the relations

\[
SS^* = 1 - P \\
S^*S = 1.
\]

To prove that \((U_1, U_2, U_3)\) does not generate \(K_1(D)\), we study their images by the index map \(\partial: K_1(D) \to K_0(A)\).

**Proposition 7.1.** The index map of the \(U_i\) is given by:

\[
\partial(U_1) = 0 \\
\partial(U_2) = 0 \\
\partial(U_3) = [Q_-] \otimes [P_+]
\]

with the notation of proposition 5.3.

**Proof.** Since \(U_1\) and \(U_2\) come from the inclusion \(A \hookrightarrow D\), the first two equalities are clear.

Regarding the last equality, we are really going to compute the index map in the 6-term exact sequence associated to (7.2) and then translate the result in terms of \(K\)-theory of \(A\).

It is readily checked that

\[
U_3 = \begin{pmatrix} S \otimes M_+ + S^* \otimes M_- & P \otimes P_+ \\ P \otimes Q_- & S^* \otimes M_+^* + S \otimes M_-^* \end{pmatrix}
\]

is a unitary lift of \(U_3\) in \(T\). It is then easy to complete the index computation:

\[
U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^* = \begin{pmatrix} S \otimes M_+ + S^* \otimes M_- & 0 \\ P \otimes Q_- & 0 \end{pmatrix} = \begin{pmatrix} 1 \otimes 1 & 0 \\ 0 & P \otimes Q_- \end{pmatrix}
\]

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Hence, the index of \( U_3 \) in \( J \) is \( P \otimes Q_- - P \otimes P_+ \), which is the image of \( Q_- \otimes P_+ \) in \( J \). The \( KK \)-equivalence between \( A \) and \( J \) then entails \( \partial(U_3) = Q_- \otimes P_+ \).

In [21], Rieffel has given an explicit description of the finitely generated projective modules over \( C(T^2) \). He proves that \( K_0(A) \) can be identified with the pairs \((d, t)\) where \( d \) (dimension) and \( t \) (twist) are in \( \mathbb{Z} \). We can choose the definition of the twist such that \( \left[ M^c \right] \simeq (1, c) \), and thus \( [P_+] = (1, -c) \).

From this identification, together with the proposition 3.10 of [21], we see that

\[
\text{Id} - [M] : (d, t) \mapsto (d, t) - (d, t + cd) = (0, -cd)
\]

Thus the kernel of \( \text{Id} - [M] \) is generated by \((0, 1)\). As \( \partial(U_3) = [Q_-] \otimes [P_+] \simeq (0, c) \), \( U_3 \) is not a generator of \( K_1(D) \).

Another consequence of proposition [7.1] and the previous computation is

**Corollary 7.2.** For any \( U \in K_1(D) \),

\[
\langle [U], \varphi_{1,2,3} \rangle = \frac{-\sqrt{32\pi}}{3} \langle \partial[U], \varphi_{1,2} \rangle.
\] (7.3)

**Proof.** Since the pairing is bilinear, we only have to check this on a basis of \( K_1(D) \otimes \mathbb{C} \). The equality is true for \( U = U_1 \) and \( U = U_2 \), since \( \partial(U_i) = 0 \).

For \( U_3 \), notice that \( \langle [Q_-], \varphi_{1,2,3} \rangle = \langle [1], \varphi_{1,2,3} \rangle = 0 \). Hence

\[
\langle [U_3], \varphi_{1,2,3} \rangle = \frac{-i4\pi}{6\sqrt{12\pi}} (-[P_+], \varphi_{1,2}) = \frac{-\sqrt{12\pi}}{3} \langle \partial(U_3), \varphi_{1,2} \rangle.
\]

\( \square \)

The formula (7.3) enables us to prove further properties of the pairing with \( \varphi_{1,2,3} \):

**Corollary 7.3.** There is a \( K \)-homology element \( K' \in K^1(D) \) such that for any \( [U] \in K_1(D) \),

\[
\langle [U], \varphi_{1,2,3} \rangle = \frac{(i2\pi)^{3/2}}{3} \langle [U], K' \rangle.
\]

In particular, this pairing takes only integer values, as one can check by direct examination of (7.3).

**Proof.** We know that in \( A = C(T^2) \), one can find a \( K \)-homology element \( K \) such that

\[
\langle [E], \varphi_{1,2} \rangle = -i2\pi \langle [E], K \rangle
\]

for any \([E] \in K_0(A)\). It is known that in the 6-terms exact sequence (7.1), one can see the boundary maps as multiplication by some element \( \delta \in KK^1(D, A) \). Multiplying \( K \in K^0(A) = KK_0(A, \mathbb{C}) \) by \( \delta \), we get \( K' \in K^1(D) \) such that

\[
i2\pi \langle [U], K' \rangle = i2\pi \langle \partial[U], K \rangle = -\langle \partial[U], \varphi_{1,2} \rangle = \frac{3}{\sqrt{12\pi}} \langle [U], \varphi_{1,2,3} \rangle.
\]

\( \square \)
8 Dimension of $HP^*(\mathcal{D})$

To complete the description of $HP^*(\mathcal{D})$, it only remains to be proven that there are no extra cyclic cocycles. We therefore compute the dimension of $HP^0(\mathcal{D})$ and $HP^1(\mathcal{D})$. We stick to computation of pairings and periodic theory. We also use results from [13]: in this article, it is proved that the smooth QHM $\mathcal{D}$ fit in a short exact sequence of locally convex algebras. In this section, we denote by $\mathcal{A}$ the algebra $C^\infty(S^1)$ with its usual Fréchet algebra structure. We start with the following definition (see [12], 2.2 and 2.3):

**Definition 8.1 (Smooth Toeplitz algebra).** The smooth compact operators $K$ are the $N \times N$ matrices $(a_{i,j})$ with rapidly decreasing complex entries. The topology on $K$ is given by the norms:

$$p_n((a_{i,j})) = \sum_{i,j} |1 + i|^n |1 + j|^n |a_{i,j}|.$$ 

The smooth Toeplitz algebra $\mathcal{F}$ is topologically the direct sum $\mathcal{F} = K \oplus C^\infty(S^1)$, where $C^\infty(S^1)$ is equipped with its usual Fréchet structure. The multiplication in $\mathcal{F}$ is described via an action of $\mathcal{F}$ on $s(N)$ (rapidly decreasing sequences). $K$ acts in the natural way, and $C^\infty(S^1)$ by truncated convolution. The function $\sum a_k z^k$ acts on $(\xi_i) \in s(N)$ by:

$$(a_\xi)_i = \sum_{k+j=i} a_k \xi_j.$$ 

The algebra $\mathcal{F}$ fits in the linearly split exact sequence:

$$0 \to \mathcal{J} \to \mathcal{I} \xrightarrow{\pi} \mathcal{D} \to 0,$$

where $\mathcal{I}$ is the subalgebra generated in the projective tensor product $\mathcal{F} \hat{\otimes} \mathcal{D}$ by $1 \otimes a, S \otimes \xi$ and $S^* \otimes \xi^*$ for $a \in \mathcal{A}$ and $\xi \in \mathcal{M}$. The map $\pi$ is defined on the generators by:

$$\pi(1 \otimes a) = a \quad \pi(S \otimes \xi) = \xi \quad \pi(S^* \otimes \xi^*) = \xi^*.$$ 

From this short exact sequence, the article [13] ensures us that there is a 6-terms exact sequence in periodic cyclic cohomology:

$$
\begin{array}{c}
HP^0(\mathcal{J}) & \xleftarrow{\pi^*} & HP^0(\mathcal{I}) & \xleftarrow{\pi^*} & HP^0(\mathcal{D}) \\
HP^1(\mathcal{D}) & \xrightarrow{\pi^*} & HP^1(\mathcal{I}) & \xrightarrow{\pi^*} & HP^1(\mathcal{J})
\end{array}
$$

(8.1)

It was also proved in [14] that there are $kk$-equivalences $\mathcal{I} \simeq \mathcal{A}$ and $\mathcal{J} \simeq \mathcal{A}$. Consequently

$$
HP^0(\mathcal{J}) = HP^0(\mathcal{I}) = HP^0(\mathcal{A}) = \mathbb{C}^2 \quad (8.2)
$$

$$
HP^1(\mathcal{J}) = HP^1(\mathcal{I}) = HP^1(\mathcal{A}) = \mathbb{C}^2. \quad (8.3)
$$

Moreover, the inclusion $\mathcal{A} \hookrightarrow \mathcal{I}$ induces an isomorphism at the level of $K$-theory. Hence the unitaries $V_i = 1 \otimes U_i \in \mathcal{I}$ form a generating set of $K_1(\mathcal{I})$, and if $E$ is a projector in $M_n(\mathcal{A})$ with dimension 1 and twisting 1, then $1 \otimes 1$ and $1 \otimes E$ are a generating set of $K_0(\mathcal{I})$. These generating sets separate the periodic cyclic cocycles.

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Let us now evaluate the image of $\pi^*$ by computing pairings on $T_M$. In the odd case,
\[
(V_i, \pi^* \varphi_j) = \langle \pi^*(V_i), \varphi_j \rangle = \langle U_i, \varphi_j \rangle,
\]
which together with (8.2) proves that $(\pi^* \varphi_j)_{j = 1, 2}$ is a basis of $HP^1(\mathcal{D}_\mathcal{M})$.

In the even case, our analyse of the map $\text{Id} - [M^\mathcal{C}]$ ensures that in $K_0(\mathcal{D})$ $\iota(E) \otimes \iota(1)$ is a torsion element. Here we used $\iota : \mathcal{A} \hookrightarrow \mathcal{D}$ to distinguish $E \in K_0(\mathcal{A})$ from $\iota(E) \in K_0(\mathcal{D})$. The definition of $\pi$ guarantees $\pi(1 \otimes E) = \iota(E)$ and $\pi(1 \otimes 1) = \iota(1)$. The bilinearity of the pairing therefore enforces:
\[
(1 \otimes E, \pi^* \psi) = \langle \iota(E), \psi \rangle = \langle \iota(1), \psi \rangle = \langle 1 \otimes 1, \pi^* \psi \rangle.
\]
Since $1 \otimes 1$ and $1 \otimes E$ separate $HP^*(\mathcal{D})$, we see that $\pi^*(HP^0(\mathcal{D}))$ has dimension at most 1. It is easy to use the trace to check that the image is non zero.

Putting the above results in the exact sequence (8.1), we get:

**Proposition 8.2.** The periodic cyclic cohomology of $\mathcal{D}$ is given by:
\[
HP^0(\mathcal{D}) = \mathbb{C}^3 \quad HP^1(\mathcal{D}) = \mathbb{C}^3.
\]

Yet, proposition 3.8 provided us with 7 cyclic cocycles. The tables (4.1) and (5.3) enable us to exhibit linearly independent families of cyclic cocycles. Notice that the pairing $\langle [K], [\varphi] \rangle$ only depends on the class of $\varphi$ in periodic cyclic cohomology and thus the families are independent in $HP^*(\mathcal{D})$. Hence we get:

**Theorem 8.3.** Taking the notations of proposition 3.8,
- the family $(\tau, \varphi_{1,3}, \varphi_{2,3})$ is a basis of $HP^0(\mathcal{D})$;
- the family $(\varphi_1, \varphi_{2,3})$ is a basis of $HP^1(\mathcal{D})$.

It follows from [14] that the (tensorised) Chern character is an isomorphism for the QHM. As we know the $K$-theory of the QHM, we get:
\[
HP^0(\mathcal{D}) = K_0(\mathcal{D}) \otimes \mathbb{Z} \mathbb{C} = \mathbb{C}^3 \quad HP^1(\mathcal{D}) = K_1(\mathcal{D}) \otimes \mathbb{Z} \mathbb{C} = \mathbb{C}^3.
\]
Since the Chern-Connes pairings factorise through the Chern-Character from $K$-theory to cyclic homology (see [18], section 8.3), we can describe the periodic cyclic homology. Notice that in the following we require $\mu \neq 0 \neq \nu$ so that the modules $\mathcal{N}$ and $\mathcal{N}_\dual$ over $\mathcal{D}_{\mu, \nu}$ exist (see notation 4.11 and definition 4.13).

**Theorem 8.4.**
- The family $(\text{Ch}(U_1), \text{Ch}(U_2), \text{Ch}(U_3))$ is a basis of $HP^0(\mathcal{D})$.
- If $\mu \neq 0 \neq \nu$, then the family $(\text{Ch}(\mathcal{D}_{\mu, \nu}), \text{Ch}(\mathcal{N}_{\mu, \nu}^\dual), \text{Ch}(\mathcal{N}_{\mu, \nu}^\dual))$ is a basis of $HP^1(\mathcal{D}_{\mu, \nu})$.

9 Final Remarks

Notice that if we have an element $K$ of $K$-homology, the tables (5.3) and (4.1) are precisely what we need to determine the Chern character of $K$ by simply computing its pairings with $K$-theory.
The previous discussion on linear independence in $HP^*(\mathcal{D})$ or $HC^*(\mathcal{D})$ is relevant because the cyclic cocycle $\varphi_3$ is nonzero in $HC^1(\mathcal{D})$ and $(\varphi_1, \varphi_2, \varphi_3)$ is linearly independent in $HC^1(\mathcal{D})$, yet $[\varphi_3] = 0$ in $HP^1(\mathcal{D})$.

More generally, we have the following proposition which ensures the linear independence of the cyclic cocycles of proposition 3.8.

**Proposition 9.1.** The following Hochschild cycles are “dual” to the cocycles of proposition 3.8 in the sense that

$$
\langle c_k, \varphi_i \rangle = \delta_{k,i} k_i, \quad \langle c_{k,j}, \varphi_{i,j} \rangle = \delta_{i,k} \delta_{j,l} k_{i,j}, \quad \langle c_{1,2,3}, \varphi_{1,2,3} \rangle \neq 0,
$$

for some nonzero constants $k_i \in \mathbb{C}$, where the Hochschild cycles are:

- **Degree 1:** $c_1 = U_1^* \otimes U_1$, $c_2 = U_2^* \otimes U_2$ and $c_3 = \xi_1^* \otimes \xi_1 + \xi_2^* \otimes \xi_2$.

- **Degree 2:** “skewsymmetrisation” $c_{j,3}$ of $\sum_p \xi_p^* U_j^* \otimes U_j \otimes \xi_p$:

$$
c_{j,3} = \sum_{p=1}^2 \xi_p^* U_j^* \otimes U_j \otimes \xi_p - U_j^* \xi_p^* \otimes \xi_p \otimes U_j
$$

- **Degree 3:** setting $U_{p,j}$ equal to $U_j$ for $j = 1, 2$ and equal to $\xi_p$ for $j = 3$,

$$
c_{1,2,3} = \sum_{p=1}^2 \sum_{\sigma \in \Sigma_3} \varepsilon(\sigma) U_{p,\sigma(3)}^* U_{p,\sigma(2)}^* U_{p,\sigma(1)}^* \otimes U_{p,\sigma(1)} \otimes U_{p,\sigma(2)} \otimes U_{p,\sigma(3)}.
$$

**Remark 9.2.** The Hochschild cycle $c_{1,2,3}$ is the analog of a fundamental form for $\mathcal{D}$.

**Remark 9.3.** Cyclic cocycles are Hochschild cocycles and thus the pairings $\langle c_1, \varphi_K \rangle$ between Hochschild cycles and cocycles exist. If we consider only Hochschild cocycles, $(\varphi_1, \varphi_2, \varphi_3, \varphi_1, \varphi_2)$ are dual in the above sense to $(c_1, c_2, c_3, c_1)$ where $c_{1,2} = U_1^* U_2^* \otimes U_1 - U_2^* U_1^* \otimes U_2$.

**Remark 9.4.** The formulas of these cycles are very similar to shuffle products of $c_1$. However, the shuffle product doesn’t apply directly, since $D$ is not commutative.

**Proof.** It is obvious that $c_1$ and $c_2$ are Hochschild cycles. It is a straightforward consequence of (2.5) that $c_3$ is closed.

To prove that the $c_{1,3}$ and $c_{1,2,3}$ are Hochschild cycles, we essentially adapt the proof of [23], lemma 12.15. We can apply the same arguments because

$$
e^{-i4\pi\mu}U_1^* \xi = \xi U_1 \quad \quad \quad e^{-i4\pi\nu}U_2^* \xi = \xi U_2.
$$

To compensate the fact that $\xi_p$ is not unitary, we sum over $p$ and use (2.5). The proof is then a set of lengthy but otherwise straightforward computations.

We only prove that $c_{1,2,3}$ is a Hochschild cycle and that it pairs non trivially with $\varphi_{1,2,3}$, the other calculations are easier cases of the same thing. We want to evaluate

$$
\langle \phi_{1,2,3}, \sum_i v_i \rangle = 
\sum_{\sigma \in \Sigma_3} \sum_{\sigma' \in \Sigma_3} \sum_i \varepsilon(\sigma) \varepsilon(\sigma') (U_{i,\sigma(3)}^* U_{i,\sigma(2)}^* U_{i,\sigma(1)}^* \partial_{\sigma(1)} U_{i,\sigma(1)} \partial_{\sigma(2)} U_{i,\sigma(2)} \partial_{\sigma(3)} U_{i,\sigma(3)}).
$$
Fix $\sigma$ and $\sigma'$ and denote by $T_{\sigma,\sigma'}$ the associated term in the above sum. Notice that $\partial_3(U_{p,j}) = \delta_{3,j}2\pi U_{p,j}$. We can find $j_3$ such that $\sigma(j_3) = 3$. If $\sigma(j_3) = 3 \neq \sigma'(j_3)$, then $T_{\sigma,\sigma'} = 0$. As $\partial_3(U_{p,j}) = -\delta_{2,j}i2\pi U_{p,j}$, if $\sigma(j_2) = 2 \neq \sigma'(j_2)$ then $T_{\sigma,\sigma'} = 0$. Thus $\sigma \neq \sigma'$ implies $T_{\sigma,\sigma'} = 0$. Finally, $\partial_1(U_{p,j}) = -\delta_{1,j}i2\pi U_{p,j}$ and

$$
\left\langle \phi_{1,2,3}, \sum_i v_i \right\rangle = (i2\pi)^3 \sum_i \sum_{\sigma \in \Sigma_3} \tau \left( U^{*}_{i,\sigma(3)} U^{*}_{i,\sigma(2)} U^{*}_{i,\sigma(1)} U_{i,\sigma(1)} U_{i,\sigma(2)} U_{i,\sigma(3)} \right) = (i2\pi)^3 \sum_i \sum_{\sigma \in \Sigma_3} \tau(\xi_i^* \xi_i) = (i2\pi)^3 6 \neq 0,
$$

by using the trace property of $\tau$.

About the construction of $U_3$: the proposition 5.3 and the lemma 6.5 essentially show that $U_3$ realises an isomorphism $E^\sigma \oplus E^{-1} \simeq 2E^0$, where $E^t$ is the line bundle over $T^2$ with twisting $t$. Another unitary could be constructed, that would realise the isomorphism $E^t \oplus E^{-1} \simeq E^{t-1} \oplus E^{-(c-1)}$. This unitary together with $U_1, U_2$ would probably be a generator of $K_1(D)$. However, the necessary computations are much more involved.

To conclude, on top of its possible implications on noncommutative 3-spheres, our study may foster intuition on Pimsner algebras. In the case of QHM, the “transfer formula” (7.3) shows how we can “transfer” a pairing from the Pimsner algebra to the basis algebra, thereby generalising a property of Nest [19]. It would be interesting to investigate if this is a general phenomenon for Pimsner algebras.

### A From Pimsner Algebras to Toeplitz Algebras

This section is concerned with the identification of the Toeplitz algebras for general Pimsner algebras in the sense of [16]. We will rely on the gauge-invariant uniqueness theorem of [16]. In the following, we denote by $\mathcal{L}(E_B)$ (resp. $\mathcal{K}(E_B)$) the adjointable (resp. compact) operators of the Hilbert module $E_B$ (see [17]) and by $\Theta_{\xi,\eta}$ the adjointable operator defined by $\Theta_{\xi,\eta}(\zeta) = \xi(\eta)\zeta$. We also let $A$ and $B$ be two $C^*$-algebras.

**Definition A.1** ($C^*$-correspondence). A $C^*$-correspondence $\Lambda E_B$ from $A$ to $B$ is given by

1. a right Hilbert module $E_B$, with scalar product $\langle \cdot | \cdot \rangle_B$;
2. a $*$-homomorphism $\phi: A \to \mathcal{L}(E_B)$.

**Definition A.2** (representation of $C^*$-correspondence). A representation of the $C^*$-correspondence $\Lambda E_A$ on a $C^*$-algebra $B$ is a pair $(\pi, T)$ consisting of

- a $*$-homomorphism of algebras $\pi: A \to B$;
- a linear map $T: E \to B$ which satisfies
  1. $T(\xi)^* T(\zeta) = \pi(\langle \xi | \zeta \rangle_A)$,
  2. $T(\xi) \pi(a) = T(\xi a)$,
  3. $\pi(a) T(\xi) = T(\phi(a)\xi)$.

Notice that (ii) is in fact a consequence of (i) (see [16], definition 2.1).
Notation A.3. Given a representation $(\pi, T)$, we denote by $C^*(\pi, T)$ the $C^*$-algebra in $B$ by $\pi(A) \cup T(E)$.

Definition A.4 (Toeplitz algebra). The Toeplitz algebra $T^A_E$ associated to a $C^*$-correspondence $AE_A$ is the universal $C^*$-algebra generated by the representations of $AE_A$.

We also need the following definition (compare [16], definition 2.3):

Definition A.5. If $AE_A$ is a $C^*$-correspondence and $(\pi, T)$ a representation of $E$ on $B$, there is a $*$-homomorphism $\pi^{(1)} : \mathcal{K}(E) \to B$ defined by:

$$\pi^{(1)}(\Theta_{\xi, \zeta}) = T(\xi)T(\zeta)^*.$$ 

Proposition A.6. Let $AE_A$ be a $C^*$-correspondence. One can define a representation $(\pi, \tau)$ of $E$ on $T^C \otimes O_E$ by:

$$\pi(a) = 1 \otimes a \quad \tau(\xi) = S \otimes S\xi.$$

This representation induces an isomorphism $\Phi : T_E \to C^*(\pi, \tau)$.

Since the Toeplitz algebra $T_C$ is nuclear, we do not need to specify which $C^*$-norm we use to define $T^C \otimes O_E$.

Proof. Check first that $(\pi, \tau)$ is indeed a representation:

(i) $\tau(\xi)^*\tau(\zeta) = (S^* \otimes S^*_\xi)(S \otimes S\xi) = 1 \otimes \langle \xi, \zeta \rangle = \pi(\langle \xi, \zeta \rangle);$ 

(ii) this point is a consequence of (i); 

(iii) $\pi(a)\tau(\xi) = (1 \otimes a)(S \otimes S\xi) = S \otimes aS\xi = S \otimes S\phi(a)\xi = \tau(\phi(a)\xi).$

Therefore, there is a homomorphism $\Phi : T_E \to T^C \otimes O_E$ defined on generators by:

$$\Psi(a) = 1 \otimes a \quad \Psi(T^C) = S \otimes S\xi.$$

To prove that this is an isomorphism, we use gauge-invariant uniqueness theorem for Toeplitz algebra (theorem 6.2, [16]).

For this representation $(\pi, T)$, 

$$\tau(\xi)^*\tau(\zeta) = SS^* \otimes S\xi S^*_\xi = (1 - P) \otimes S\xi S^*_\xi$$

hence $\text{Im} \pi^{(1)} \subseteq (1 - P) \otimes O_E^{(0)}$. Since the image of $A$ by $\pi$ is $1 \otimes A$, $I'_{(\pi, \tau)} = \{ a \in A | \pi(a) \in B_1 = \pi^{(1)}(\mathcal{K}(E)) \} = 0$.

One can define an action $\tilde{\beta} : S^1 \curvearrowright T_C \otimes O_E$ by

$$\tilde{\beta}_t(S^n \otimes F) = e^{2\pi nt}S^n \otimes F,$$

where $F \in O_E$ and $n \in \mathbb{N}$.

We hence get a gauge action $\beta$ on $C^*(\pi, \tau)$ by restriction of $\tilde{\beta}$:

$$\beta_t(\pi(a)) = \beta_t(1 \otimes a) = \pi(a) \quad \beta_t(\tau(\xi)) = \beta_t(S \otimes S\xi) = e^{2\pi t}S \otimes S\xi = e^{2\pi t}\tau(\xi).$$

Consequently, $\Phi : T_E \to C^*(\pi, \tau)$ is an isomorphism. □

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References


