EXERCISES AND SOLUTIONS FOR A COURSE ON LIE GROUPS

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Abstract. These are the solutions prepared by the author for the exercises given to the students in the Mathematics 261A course on Lie groups and Lie algebras in Berkeley taught by Vera Serganova in the Fall 1996 semester.

1. *Let G be a Lie group with identity element e, $L_g(x) = gx$, and $R_g(x) = xg$. Show that the map* Ad *from G to $\mathrm{Gl}(\mathrm{T}_eG)$, given by $\mathrm{Ad}(g) = D(L_gR_{g^{-1}})$ is a homomorphism of Lie groups.*

By definition, $L_gR_{g^{-1}}(x) = gxg^{-1}$ for all $g, x \in G$. Since G is a Lie group, the mapping $\phi\colon G \times G \to G$, $\phi(g, x) = gxg^{-1}$ is a smooth mapping as a composition of the smooth mappings

$$(g, x) \mapsto (g, x, g) \mapsto (g, x, g^{-1}) \mapsto (gx, g^{-1}) \mapsto gxg^{-1}.$$

This implies immediately that Ad is a smooth mapping. Furthermore,

$$\mathrm{Ad}(e) = D(\mathrm{id}_G) = \mathrm{id}_{\mathrm{T}_eG}$$

and

$$\begin{aligned}\mathrm{Ad}(gh) &= D(L_{gh}R_{h^{-1}g^{-1}}) = D(L_gL_hR_{g^{-1}}R_{h^{-1}})\\ &= D(L_gR_{g^{-1}}L_hR_{h^{-1}}) = D(L_gR_{g^{-1}}) \circ D(L_hR_{h^{-1}})\\ &= \mathrm{Ad}(g) \circ \mathrm{Ad}(h),\end{aligned}$$

i.e. Ad is also a group homomorphism. Thus Ad is a homomorphism of Lie Groups.

2. *Prove that any connected Lie group is generated by any open neighborhood of the identity.*

We will show more: If G is a connected topological group and U is a neighborhood of the identity element e of G, then U generates G.

Let $\tilde{U} = U \cap U^{-1}$, then $\tilde{U} \subset U$ is a symmetric neighborhood of e with $\tilde{U}^{-1} = \tilde{U}$. Since it suffices to show that $\tilde{U}$ generates G, we can assume without loss of generality that $U = U^{-1}$. Let $U^n = \{u_1 \cdots u_n \mid u_k \in U\}$. Then the group generated by U is $V = \bigcup_{n=1}^{\infty} U^n$ because U is symmetric. If $x \in V$, say $x = u_1 \cdots u_n$, then $U \cdot x \subset U^{n+1} \subset V$ is a neighborhood of x, so that V is open. Hence all cosets gV with $g \in G$ are open as well, so that $V = G \setminus \bigcup_{g \notin V} gV$ is closed. Furthermore, $V \supset U \neq \emptyset$, so that we must have $V = G$ by connectedness.

3. *Show that the group $\mathrm{Gl}_n(\mathbb{H})$ is connected and simply connected for any $n \in \mathbb{N}$.*

For $n = 1$, we have $\mathrm{Gl}_1(\mathbb{H}) \cong \mathbb{H}^* \cong \mathbb{R}^4 \setminus \{0\}$ as a manifold. The latter space is homotopy equivalent to the 3-sphere S^3 and thus connected and simply connected. We will show the assertion for arbitrary n by induction. Assume that we already established that $\mathrm{Gl}_{n-1}(\mathbb{H})$ is connected and simply connected.

First we want to show that $\mathrm{Gl}_n(\mathbb{H})$ operates transitively on $\mathbb{H}^n \setminus \{0\}$. Let $E_k = (0, \dots, 0, 1, 0, \dots, 0)$ be the k-th vector of the standard basis of $\mathbb{H}^n$ and $X \in \mathbb{H}^n \setminus \{0\}$, then we need an element of $\mathrm{Gl}_n(\mathbb{H})$ mapping E_1 to X. Since matrices permuting the coordinates are clearly in $\mathrm{Gl}_n(\mathbb{H})$ (in fact, $\mathrm{Gl}_n(\mathbb{R}) \subset \mathrm{Gl}_n(\mathbb{H})$), we may assume $X_1 \neq 0$. Since $a \cdot \mathrm{id}_{\mathbb{H}^n} \in \mathrm{Gl}_n(\mathbb{H})$ for all $a \in \mathbb{H}^* = \mathbb{H}\setminus\{0\}$, we may further assume that $X_1 = 1$. Then it is trivial to see that the matrix $(X, E_2, \dots, E_n)$ has the inverse $(\hat{X}, E_2, \dots, E_n)$ with $\hat{X}_1 = 1$ and $\hat{X}_k = -X_k$ for $k = 2, \dots, n$. Since it maps E_1 to X, the operation of $\mathrm{Gl}_n(\mathbb{H})$ on $\mathbb{H}^n \setminus \{0\}$ is transitive.

The next step is to compute the stabilizer of E_1. It consists of all matrices of the form

$$M(b, A) = \begin{pmatrix} 1 & b \\ 0 & A \end{pmatrix}, \qquad b \in \mathbb{H}^{n-1}, A \in \mathbb{H}^{n-1 \times n-1}$$

that are invertible. Since the inverse of such a matrix, if it exists, lies again in the stabilizer, it is of the above form. This easily yields that A must be invertible. On the other hand, if A is invertible, then

$$\begin{pmatrix} 1 & 0 \\ 0 & A^{-1} \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & A \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & A \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & A^{-1} \end{pmatrix}$$

is invertible by the preceding paragraph. Hence $M(b, A) \in \mathrm{Gl}_n(\mathbb{H})$ if and only if $A \in \mathrm{Gl}_{n-1}(\mathbb{H})$. Thus the stabilizer of E_1 is homeomorphic to $\mathbb{H}^{n-1} \times \mathrm{Gl}_{n-1}(\mathbb{H})$.

The mapping $\mathrm{Gl}_n(\mathbb{H}) \to \mathbb{H}^n \setminus \{0\}$, $g \mapsto gE_1$ is a locally trivial bundle since the fibers are precisely the cosets of the stabilizer of E_1. Furthermore, $X = \mathrm{Gl}_n(\mathbb{H})$, $Y = \mathbb{H}^n \setminus \{0\}$ and the fibers $Z = \mathbb{H}^{n-1} \times \mathrm{Gl}_{n-1}(\mathbb{H})$ are all smooth manifolds. Obviously Y is connected and simply connected, and so is Z by assumption on $\mathrm{Gl}_{n-1}(\mathbb{H})$. Hence X must be connected as well, and this allows us to apply the exact homotopy sequence to finish the argument:

$$\pi_1(Z) \longrightarrow \pi_1(X) \longrightarrow \pi_1(Y) \longrightarrow \pi_0(Z) \longrightarrow \{0\}$$

is an exact sequence, and filling in the information $\pi_1(Z) = \pi_1(Y) = \{0\}$, we obtain $\pi_1(X) = \{0\}$ as desired.

4. *Show that the group* $\mathrm{Sl}_2(\mathbb{C})$ *is the universal covering of* $\mathrm{SO}_3(\mathbb{C})$.

First, we have to show that $\mathrm{Sl}_2(\mathbb{C})$ and $\mathrm{SO}_3(\mathbb{C})$ are connected complex Lie groups of complex dimension 3 and that $\mathrm{Sl}_2(\mathbb{C})$ is simply connected. All ocuring manifolds in this exercise are complex manifolds. Hence if we give dimensions, they are to be thought of as complex dimensions, i.e. twice the real dimension.

By definition,

$$\mathrm{Sl}_2(\mathbb{C}) = \{M \in \mathrm{Gl}_2(\mathbb{C}) \mid \det(M) = 1\}.$$

Since $\det\colon \mathrm{Gl}_2(\mathbb{C}) \to \mathbb{C}^*$ is a surjective Lie group homomorphism it has constant rank 1 everywhere, so that its kernel $\mathrm{Sl}_2(\mathbb{C})$ is indeed a Lie group of dimension 3. Let $x = (1, 0) \in \mathbb{C}^2 \setminus \{0\}$. It is easy to see that $\mathrm{Sl}_2(\mathbb{C}) \cdot x = \mathbb{C}^2 \setminus \{0\}$, i.e. the operation of $\mathrm{Sl}_2(\mathbb{C})$ on $\mathbb{C}^2 \setminus \{0\}$ is transitive. Furthermore, the stabilizer of x is the group of all matrices of the form

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \qquad c \in \mathbb{C}$$

and hence isomorphic to $\mathbb{C}$. Both $\mathbb{C}$ and $\mathbb{C}^2 \setminus \{0\}$ are connected and simply connected. The same argument as in exercise 3 yields that $\mathrm{Sl}_2(\mathbb{C})$ is connected and simply connected.

In order to obtain the needed results about $\mathrm{SO}_3(\mathbb{C})$, we will prove instead that $\mathrm{SO}_n(\mathbb{C})$ is a $\frac{1}{2}n(n-1)$-dimensional connected manifold, using induction on n. Since $\mathrm{SO}_1(\mathbb{C}) = \{1\}$, the assertion is trivial for $n = 1$. Assume it to be known for $n-1$, $n > 1$. Let $x = (1, 0, \ldots, 0) \in \mathbb{C}^n$, let $Q(v, w) = v^t w$ be the standard bilinear form, and assume that Q is $\mathrm{SO}_n(\mathbb{C})$-invariant.

Then for arbitrary $y \in \mathbb{C}^n$ with $Q(y, y) = 1$ there is an $s \in \mathrm{SO}_3(\mathbb{C})$ with $sx = y$: Simply extend $\{y\}$ to an "orthonormal" basis $(y_1, \ldots, y_n)$ (in the sense $y_j^t y_k = \delta_{jk}$) and then adjust y_2 so that $\det(y_1, \ldots, y_n) = 1$. Hence $\mathrm{SO}_3(\mathbb{C})$ operates transitively on the set

$$M = \{y \in \mathbb{C}^n \mid Q(y, y) = 1\}.$$

The equation $Q(y, y) - 1 = 0$ is easily seen to be an irreducible polynomial equation. Since the origin is the only point where the rank of DQ is not maximal, M has no singularities and hence is a connected complex manifold by results of algebraic geometry. Clearly, it is 1-codimensional and hence has dimension $n-1$.

The stabilizer of $x \in M$ consists of all those matrices of the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A} \end{pmatrix}, \qquad \tilde{A} \in \mathrm{SO}_{n-1}(\mathbb{C})$$

because the columns and rows of A must be "orthonormal". Hence it is isomorphic to $\mathrm{SO}_{n-1}(\mathbb{C})$. Thus locally $\mathrm{SO}_n(\mathbb{C})$ looks like $M \times \mathrm{SO}_{n-1}(\mathbb{C})$ and hence is a complex manifold of dimension

$$\dim \mathrm{SO}_n(\mathbb{C}) = (n-1) + \dim \mathrm{SO}_{n-1}(\mathbb{C}) = \frac{1}{2}n(n-1)$$

as desired. Furthermore, since the stabilizer $\mathrm{SO}_{n-1}(\mathbb{C})$ and the manifold M are connected, $\mathrm{SO}_n(\mathbb{C})$ is connected as well. Consequently, our claim follows by induction.

Furthermore, $\mathrm{SO}_n(\mathbb{C})$ is the connected component of the identity in $\mathrm{O}_n(\mathbb{C})$ because a matrix $A \in \mathrm{O}_n(\mathbb{C})$ satisfies $\det(A)^2 = 1$ and hence has determinant ± 1.

Let $V_0 = \mathbb{M}_2(\mathbb{C})$ be the space of 2×2-matrices over $\mathbb{C}$. Then $\mathrm{Sl}_2(\mathbb{C})$ operates on V_0 by conjugation, i.e. we have a mapping $\rho\colon \mathrm{Sl}_2(\mathbb{C}) \to \mathrm{Gl}(V_0)$ given by $\rho(s)M = sMs^{-1}$ for $s \in \mathrm{Sl}_2(\mathbb{C})$, $M \in \mathbb{M}_2(\mathbb{C})$. It is trivial to verify that this is a Lie group representation.

Furthermore, if $\mathrm{Tr}(M)$ denotes the trace of the matrix M, then

$$\mathrm{Tr}(\rho(s)M) = \mathrm{Tr}(sMs^{-1}) = \mathrm{Tr}(Ms^{-1}s) = \mathrm{Tr}(M)$$

because of the rule $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. This implies that the subspace V of traceless matrices is invariant for $\rho(\mathrm{Sl}_2(\mathbb{C}))$ and hence yields a subrepresentation $\sigma\colon \mathrm{Sl}_2(\mathbb{C}) \to \mathrm{Gl}(V)$.

We claim that the bilinear form $Q\colon V^2 \to \mathbb{C}$, $Q(X,Y) = \mathrm{Tr}(XY)$ is non-degenerate and $\sigma(\mathrm{Sl}_2(\mathbb{C}))$-invariant. If $X \in V$, then also $X^* \in V$, and XX^* is a positive operator on V. If $X \neq 0$, then $XX^* \neq 0$ and the latter matrix must have a positive trace. Hence Q is non-degenerate. Furthermore,

$$Q(\rho(s)X, \rho(s)Y) = \mathrm{Tr}(sXYs^{-1}) = \mathrm{Tr}(XY) = Q(X,Y),$$

i.e. Q is $\rho(s)$-invariant, for all $s \in \mathrm{Sl}_2(\mathbb{C})$. Hence the image of ρ is contained in $\mathrm{O}_3(\mathbb{C})$. Since $\mathrm{Sl}_2(\mathbb{C})$ is connected and ρ is continuous, $\mathrm{Im}(\rho)$ is contained in the connected component of the identity in $\mathrm{O}_3(\mathbb{C})$, which is equal to $\mathrm{SO}_3(\mathbb{C})$.

The kernel of ρ is characterized by $sM = Ms$ for all $M \in V$ and hence for all $M \in \mathbb{M}_2(\mathbb{C})$. This implies that $s \in \mathbb{C} \cdot \mathrm{id}_{\mathbb{C}^2}$ and hence

$$\mathrm{Ker}(\rho) = \{\pm 1\}.$$

Since ρ is a Lie group homomorphism, its derivative has constant rank

$$\dim \mathrm{Im}(\rho) = \dim \mathrm{Sl}_2(\mathbb{C}) - \dim \mathrm{Ker}(\rho) = 3.$$

Since $\dim \mathrm{Ker}(\rho) = 0$, the derivative of ρ has maximal rank everywhere, i.e. ρ is an immersion. Since $\mathrm{Ker}(\rho)$ is finite, it is trivial to see that ρ is proper, so that $\rho\colon \mathrm{Sl}_2(\mathbb{C}) \to \mathrm{Im}(\rho)$ is a smooth covering map. $\mathrm{Im}(\rho)$ is a 3-dimensional subgroup of the 3-dimensional Lie group $\mathrm{SO}_3(\mathbb{C})$ and thus contains a neighborhood of the identity in $\mathrm{SO}_3(\mathbb{C})$. By exercise 2 and because $\mathrm{SO}_3(\mathbb{C})$ is connected, this neighborhood generates $\mathrm{SO}_3(\mathbb{C})$, so that indeed $\mathrm{Im}(\rho) = \mathrm{SO}_3(\mathbb{C})$.

Since $\mathrm{Sl}_2(\mathbb{C})$ is connected and simply connected, it is the universal covering of $\mathrm{SO}_3(\mathbb{C})$. Thus the fundamental group of $\mathrm{SO}_3(\mathbb{C})$ is isomorphic to the kernel of ρ, which is isomorphic to the group $\mathbb{Z}_2$ with two elements.

5. *Let $\phi\colon G \to H$ be a homomorphism of Lie groups. Prove that $D\phi|_e\colon \mathrm{T}_eG \to \mathrm{T}_eH$ is a homomorphism of Lie algebras.*

Let $\mathfrak{g} = \mathrm{T}_eG$ and $\mathfrak{h} = \mathrm{T}_eH$. Clearly, $\phi_* = D\phi|_e\colon \mathfrak{g} \to \mathfrak{h}$ is a linear map. For $x \in G$, define $\Psi_x\colon G \to G$ by $\Psi_x(y) = xyx^{-1}$, and similarly for $x \in H$. Then $\mathrm{Ad}(x) \in \mathrm{Gl}(\mathfrak{g})$ is the differential of Ψ_x at the identity.

Since ϕ is a Lie group homomorhpism, $\phi(\Psi_x(y)) = \phi(x)\phi(y)\phi(x)^{-1} = \Psi_{\phi(x)}(\phi(y))$, i.e. $\phi \circ \Psi_x = \Psi_{\phi(x)} \circ \phi$. Differentiating this at the identity yields

$$\phi_* \circ \mathrm{Ad}(x) = (D\phi|_e) \circ (D\Psi_x|_e)(x) = (D\Psi_{\phi(x)}|_e) \circ (D\phi|_e) = \mathrm{Ad}(\phi(x)) \circ \phi_*$$

for all $x \in G$. For fixed $Y \in \mathfrak{g}$, this equation becomes

$$\phi_* \circ \mathrm{Ad}(x)(Y) = \mathrm{Ad}(\phi(x))(\phi_*(Y)). \tag{1}$$

Let $\mathrm{Ad}\colon G \to \mathrm{Gl}(\mathfrak{g})$ be the mapping $x \mapsto \mathrm{Ad}(x)$ and let $\mathrm{ad} = D\,\mathrm{Ad}\,|_e\colon \mathrm{T}_eG \to \mathrm{T}_e\mathrm{Gl}(\mathfrak{g}) = \mathrm{End}(\mathfrak{g})$. Then $[X,Y] = \mathrm{ad}\,X(Y)$, where we view $\mathrm{ad}\,X \in \mathrm{T}_e\mathrm{Gl}(\mathfrak{g})$ as an endomorphism of $\mathfrak{g}$.

Differentiating (1) at e with respect to x and using the linearity of various maps and the chain rule, we obtain

$$\begin{aligned}
\phi_*([X,Y]) &= \phi_*(\mathrm{ad}\,X(Y)) = \frac{\partial}{\partial X}(\phi_* \circ \mathrm{Ad}(x))|_{x=e}(Y) \\
&= \frac{\partial}{\partial X}(\mathrm{Ad}(\phi(x)))|_{x=e}(\phi_*(Y)) = (\mathrm{ad} \circ \phi|_e) \circ \frac{\partial \phi}{\partial X}(e)(\phi_*(Y)) \\
&= \mathrm{ad}(\phi_*(X))(\phi_*(Y)) = [\phi_*(X), \phi_*(Y)],
\end{aligned}$$

where $\partial/\partial X$ is the derivative in the direction of X, i.e. $D\sqcup|_e(X)$. Hence ϕ_* is a homomorphism of Lie groups as desired.

6. *Let $\phi\colon \mathbb{R} \to G$ be a homomorphism of Lie groups. Show that $\phi(t) = \exp Xt$ for some $X \in \mathrm{T}_eG$.*

Let $X = \dot\phi(0)$. We want to show $\phi(t) = \exp(tX)$. For this purpose, let v_X be the left invariant vector field on G with $v_X(e) = X$. Then $\phi(s+t) = \phi(s)\phi(t)$ implies

$$\dot\phi(s) = \left.\frac{d}{dt}\phi(s+t)\right|_{t=0} = (L_{\phi(s)})_*\dot\phi(0) = v_X(\phi(s))$$

due to the left invariance of v_X. Hence ϕ is the unique solution of the differential equation $\dot\phi = v_X \circ \phi$, $\phi(0) = e$. By definition, $\phi(1) = \exp(X)$. Since the same argument applies to the Lie group homomorphism $\phi_\lambda\colon t \mapsto \phi(\lambda t)$ for some $\lambda \in \mathbb{R} \setminus \{0\}$, and since $\dot\phi_\lambda(0) = \lambda X$, we obtain $\phi(\lambda) = \exp(\lambda X)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. Furthermore, $\phi(0) = e = \exp(0)$, which proves our assertion.

7. *Let $\mathfrak{g}$ be a Lie algebra. A linear map $d\colon \mathfrak{g} \to \mathfrak{g}$ is called a derivation of $\mathfrak{g}$ if*

$$d([X,Y]) = [d(X), Y] + [X, d(Y)]$$

for any $X, Y \in \mathfrak{g}$.

a): *Show that the set* $\mathrm{Der}(\mathfrak{g})$ *of all derivations of* $\mathfrak{g}$ *is a Lie algebra under the bracket*

$$[d_1, d_2] = d_1 d_2 - d_2 d_1.$$

b): *Show that* $\operatorname{ad} X \in \mathrm{Der}(\mathfrak{g})$ *for any* $X \in \mathfrak{g}$ *and* $\operatorname{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ *is a homomorphism of Lie algebras.*

Let $d_1, d_2 \in \mathrm{Der}(\mathfrak{g})$, $X, Y \in \mathfrak{g}$. Then

$$\begin{aligned}
&[d_1, d_2]([X,Y]) = d_1 d_2([X,Y]) - d_2 d_1([X,Y]) \\
= \;& d_1([d_2(X), Y]) + d_1([X, d_2(Y)]) - d_2([d_1(X), Y]) - d_2([X, d_1(Y)]) \\
= \;& [d_1 d_2(X), Y] + [d_2(X), d_1(Y)] + [d_1(X), d_2(Y)] + [X, d_1 d_2(Y)] \\
& \quad - [d_2 d_1(X), Y] - [d_1(X), d_2(Y)] - [d_2(X), d_1(Y)] - [X, d_2 d_1(Y)] \\
= \;& [[d_1, d_2](X), Y] + [X, [d_1, d_2](Y)].
\end{aligned}$$

Thus $[d_1, d_2] \in \mathrm{Der}(\mathfrak{g})$. Clearly, the bracket is linear and skew-symmetric and the Jacobi identity is true as well: If an associative algebra A is furnished with the bracket $[x, y] = xy - yx$, then the Jacobi identity is always true, as shown by the trivial computation

$$\begin{aligned}
[a,[b,c]] + [b,[c,a]] + [c,[a,b]] &= abc - acb - bca + cba + bca - bac - cab \\
&\quad + acb + cab - cba - abc + bac = 0.
\end{aligned}$$

The mapping $\operatorname{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is defined by $\operatorname{ad} X(Y) = [X, Y]$. The assertion that $\operatorname{ad}(X) \in \mathrm{Der}(\mathfrak{g})$ for all $X \in \mathfrak{g}$ is equivalent to the Jacobi identity:

$$\begin{aligned}
\operatorname{ad} X([Y,Z]) &= [X,[Y,Z]] = -[Y,[Z,X]] - [Z,[X,Y]] \\
&= [Y,[X,Z]] + [[X,Y],Z] = [Y, \operatorname{ad} X(Z)] + [\operatorname{ad} X(Y), Z].
\end{aligned}$$

Furthermore, if $X, Y, Z \in \mathfrak{g}$, then

$$\begin{aligned}
\operatorname{ad}[X,Y](Z) &= [[X,Y],Z] = -[[Y,Z],X] - [[Z,X],Y] \\
&= [X,[Y,Z]] - [Y,[X,Z]] = \operatorname{ad} X(\operatorname{ad} Y(Z)) - \operatorname{ad} Y(\operatorname{ad} X(Z)) \\
&= [\operatorname{ad} X, \operatorname{ad} Y](Z),
\end{aligned}$$

i.e. ad is a homomorphism of Lie algebras.

8. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra and* $\mathrm{Aut}(\mathfrak{g})$ *be the group of all automorphisms of* $\mathfrak{g}$*. Show that* $\mathrm{Aut}(\mathfrak{g})$ *is a Lie group and the Lie algebra of* $\mathrm{Aut}(\mathfrak{g})$ *is* $\mathrm{Der}(\mathfrak{g})$*.*

Let V be the linear space of skew-symmetric bilinear mappings $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then $[,] \in V$. If $L\colon \mathfrak{g} \to \mathfrak{g}$ is linear and invertible, then L operates on the space V by $L_*(Q)(X,Y) = L \circ Q(L^{-1}X, L^{-1}Y)$ for $X, Y \in \mathfrak{g}$, $Q \in V$. We have

$$\begin{aligned}
&L_* \circ M_*(Q)(X,Y) = L \circ (M_* Q)(L^{-1}X, L^{-1}Y) \\
= \;& L \circ M \circ Q(M^{-1}L^{-1}X, M^{-1}L^{-1}Y) = (L \circ M)_*(Q)(X,Y)
\end{aligned}$$

for all $L, M \in \mathrm{Gl}(\mathfrak{g})$, and $\mathrm{id}_{\mathfrak{g}*} = \mathrm{id}_V$. Hence $\mathrm{Gl}(\mathfrak{g})$ operates on V. The group operation is given by polynomials and hence smooth. The mapping $L \in \mathrm{Gl}(\mathfrak{g})$ is an automorphism of the Lie algebra iff $L^{-1} \in \mathrm{Aut}(\mathfrak{g})$ iff $[L^{-1}X, L^{-1}Y] = L^{-1}[X,Y]$ for all $X, Y \in \mathfrak{g}$ iff $L_*[,] = [,]$. Hence $\mathrm{Aut}(\mathfrak{g})$ is precisely the stabilizer of the element $[,] \in V$ corresponding to the operation of $\mathrm{Gl}(\mathfrak{g})$ on V. Hence our assertion follows from the following more general result:

Theorem 1. *Let the Lie group* G *act on the smooth manifold* M *and let* $m \in M$*. Then the stabilizer* $G_m = \{g \in G \mid g \cdot m = m\}$ *of* m *is a closed Lie subgroup of* G*.*

Clearly, G_m is a subgroup of G. Define a smooth mapping $\rho\colon G \to M$ by $\rho(g) = g \cdot m$. Then $G_m = \rho^{-1}(m)$. Since ρ is continuous and $\{m\}$ is closed, so is G_m. If $g \in G$, then $\phi_g\colon M \to M$, $x \mapsto g \cdot x$, is a diffeomorphism. Clearly,

$$\rho \circ L_g(h) = \rho(gh) = \phi_g \circ \rho(h),$$

so that

$$\begin{aligned}
\operatorname{rk} D\rho|_g &= \operatorname{rk}(D\rho|_g) \circ (DL_g|_e) = \operatorname{rk} D(\rho \circ L_g)|_e = \operatorname{rk} D(\phi_g \circ \rho)|_e \\
&= \operatorname{rk}(D\phi_g|_m) \circ (D\rho|_e) = \operatorname{rk} D\rho|_e.
\end{aligned}$$

This ensures that G_m is an embedded submanifold of G by standard arguments of calculus on manifolds.

It still has to be shown that the multiplication in G_m is smooth, considered as a mapping from $G_m \times G_m$ to G_m. Since G_m is an embedded submanifold, every $g \in G_m$ has a neighborhood U in G such that there is a smooth rectraction $p\colon U \to G_m \cap U$. Hence if $i\colon G_m \to G$ is the canonical injection and $\mu\colon G \times G \to G$ is the multiplication in G, then $\mu_m\colon G_m \times G_m \to G_m$ is smooth near $(g_1, g_2) \in G_m$ with $g_1 \cdot g_2 = g$ because locally we have $\mu_m = p \circ \mu \circ (i \times i)$.

It remains to identify the tangent space of $\mathrm{Aut}(\mathfrak{g})$. For this purpose, let $\phi\colon \mathbb{R} \to \mathrm{Aut}(\mathfrak{g})$ represent a tangent vector at 0. Then

$$[\phi(t)(X), \phi(t)(Y)] = \phi(t)([X,Y])$$

for all $t \in \mathbb{R}$, $X, Y \in \mathfrak{g}$. Differentiating this equality at $t = 0$ yields

$$[\dot\phi(0)(X), Y] + [X, \dot\phi(0)(Y)] = \dot\phi(0)([X,Y]),$$

i.e. $\dot\phi(0) \in \mathrm{Der}(\mathfrak{g})$. Conversely, if $Z \in \mathrm{Der}(\mathfrak{g})$, then

$$\begin{aligned}\frac{d}{dt}[\exp(Zt)X, \exp(Zt)Y] &= [Z\exp(Zt)X, \exp(Zt)Y] \\ &\quad + [\exp(Zt)X, Z\exp(Zt)Y] = Z([\exp(Zt)X, \exp(Zt)Y]).\end{aligned}$$

Hence the function $\psi(t) = [\exp(Zt)X, \exp(Zt)Y]$ solves the differential equation $\dot\psi(t) = Z\psi(t)$ with $\psi(0) = [X,Y]$. The same differential equation is solved by the function $\Psi(t) = \exp(Zt)([X,Y])$. Hence $\psi = \Psi$, so that indeed $\exp(Zt) \in \mathrm{Aut}(\mathfrak{g})$ for all $t \in \mathbb{R}$ and hence $Z \in \mathrm{T}_e(\mathrm{Aut}(\mathfrak{g}))$ as desired.

9. *Let G be a (connected) Lie group with Lie algebra $\mathfrak{g}$, $H \subset G$ be an immersed connected subgroup. Prove that H is normal if and only if its Lie algebra $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.*

If H is normal, we have $ghg^{-1} \in H$ for all $g \in G$, $h \in H$. Differentiating this once yields that if $X \in \mathfrak{h}$, then $\mathrm{Ad}(g)X \in \mathfrak{h}$ as well. Differentiating again, we obtain $\mathrm{ad}(Y)X \in \mathfrak{h}$ for all $Y \in \mathfrak{g}$, $X \in \mathfrak{h}$, because $\mathfrak{h}$ is a linear subspace of $\mathfrak{g}$. Consequently, $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$, i.e. $\mathfrak{h}$ is an ideal.

Assume conversely that $\mathfrak{h}$ is an ideal. The subgroup H is generated by any neighborhood of the identity because it is connected. If we take this neighborhood small enough, it is in the image of $\exp\colon \mathfrak{h} \to H$. Hence H is generated by elements of the form $\exp(X)$ with $X \in \mathfrak{h}$. Let U be a neighborhood of 0 in $\mathfrak{g}$ such that the Campbell-Hausdorff series for $\log(\exp(X)\exp(Y)\exp(Z))$ converges for $X, Y, Z \in U$. Then $\log(\exp(X)\exp(Y)\exp(-X)) = X + Y - X + [\ldots, Y] + [\ldots, [\ldots, Y]] + \cdots$ for some $\ldots \in \mathfrak{g}$ by the Campbell-Hausdorff formula. Hence if $Y \in \mathfrak{h}$ and $X \in \mathfrak{g}$, then $\log(\exp(X)\exp(Y)\exp(-X)) \in \mathfrak{h}$, because $\mathfrak{h}$ is an ideal. This implies $g(\exp(U) \cap H)g^{-1} \subset H$ for all $g \in \exp(U)$. Since $\exp(U) \cap H$ generates H we obtain $gHg^{-1} \subset H$ for $g \in \exp(U)$. Since $\exp(U)$ is a neighborhood of the identity, it generates the connected component of the identity G_0 of G. Thus we obtain $gHg^{-1} \subset H$ for all $g \in G_0$, i.e. H is normal in G_0. (By connectedness, $H \subset G_0$.) Hence $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ if and only if H is normal in G_0.

However, if G is not connected, then H need not be normal in G. Here is a nice example. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ and consider the automorphism group of the domain $\mathbb{D}^2$. Since $\mathbb{D}^2$ is bounded, $\mathrm{Aut}(\mathbb{D}^2)$ is a Lie group, and the connected component of the identity is $\mathrm{Aut}(\mathbb{D}^2)_0 = \mathrm{Aut}(\mathbb{D}) \times \mathrm{Aut}(\mathbb{D})$. Thus $\mathrm{Aut}(\mathbb{D}) \times \{\mathrm{id}_{\mathbb{D}}\}$ is a normal subgroup of $\mathrm{Aut}(\mathbb{D}^2)_0$. However, the flip $\phi\colon (x,y) \mapsto (y,x)$ clearly is an element of $\mathrm{Aut}(\mathbb{D}^2)$ with $\phi(\mathrm{Aut}(\mathbb{D}) \times \{\mathrm{id}_{\mathbb{D}}\})\phi^{-1} = \{\mathrm{id}_{\mathbb{D}}\} \times \mathrm{Aut}(\mathbb{D})$. Thus $\mathrm{Aut}(\mathbb{D}) \times \{\mathrm{id}_{\mathbb{D}}\}$ is not normal in $\mathrm{Aut}(\mathbb{D}^2)$, although it is normal in $\mathrm{Aut}(\mathbb{D}^2)_0$.

10. *Show that the universal covering of* $\mathrm{SO}_4(\mathbb{C})$ *is isomorphic to* $\mathrm{Sl}_2(\mathbb{C}) \times \mathrm{Sl}_2(\mathbb{C})$.

We have already seen in a previous exercise that $\mathrm{Sl}_2(\mathbb{C})$ is a connected and simply connected 3-dimensional complex Lie group. Hence $\tilde G = \mathrm{Sl}_2(\mathbb{C}) \times \mathrm{Sl}_2(\mathbb{C})$ is a connected and simply connected 6-dimensional complex Lie group. Furthermore, we have seen that $\mathrm{SO}_n(\mathbb{C})$ is a connected complex Lie group of dimension $\frac{1}{2}n(n-1)$. Thus $G = \mathrm{SO}_4(\mathbb{C})$ is a 6-dimensional connected complex Lie group. We have to construct a covering map $\rho\colon \tilde G \to G$.

Let $V = \mathbb{M}_2(\mathbb{C})$ be the space of 2×2-matrices over $\mathbb{C}$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad A^\natural = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The adjunct $A^\natural$ of A has the property that

$$AA^\natural = A^\natural A = \det(A) \cdot \mathrm{id}_{\mathbb{C}^2}. \tag{2}$$

Hence $A^\natural = \det(A)A^{-1}$ if A is invertible. Furthermore, the mapping $A \mapsto A^\natural$ is linear. Since invertible matrices span V, we have $(A^\natural)^\natural = A$ and $(AB)^\natural = B^\natural A^\natural$, because this holds for invertible matrices. Moreover, the definition of $A^\natural$ above implies $\mathrm{Tr}(A^\natural) = \mathrm{Tr}(A)$. Hence $Q(A,B) = \frac{1}{2}\mathrm{Tr}(AB^\natural)$ is a symmetric bilinear form on V:

$$2Q(B,A) = \mathrm{Tr}(BA^\natural) = \mathrm{Tr}((BA^\natural)^\natural) = \mathrm{Tr}(A^{\natural\natural}B^\natural) = \mathrm{Tr}(AB^\natural) = 2Q(A,B).$$

Also, since $\mathrm{Tr}(AA^*) = 0$ implies $A = 0$, for every $A \in \mathbb{M}_2 \setminus \{0\}$ there is $B \in \mathbb{M}_2$ such that $Q(A, B) \neq 0$, i.e., the form Q is non-degenerate. Hence we can view $\mathrm{SO}_4(\mathbb{C})$ as the subgroup of $\mathrm{Gl}(V)$ of linear operators with determinant 1 leaving Q invariant. Moreover,

$$Q(A, A) = \frac{1}{2}\,\mathrm{Tr}(AA^\natural) = \frac{1}{2}\,\mathrm{Tr}(\det(A)\mathrm{id}_{\mathbb{C}^2}) = \det(A).$$

Define $\rho\colon \mathrm{Sl}_2(\mathbb{C}) \times \mathrm{Sl}_2(\mathbb{C}) \to \mathrm{Gl}(V)$ by $\rho(g, h)A = gAh^{-1}$ for $A \in V$ and $g, h \in \mathrm{Sl}_2(\mathbb{C})$. It is trivial to check that this is indeed a group representation. Since $h^{-1} = h^\natural$ for $h \in \mathrm{Sl}_2(\mathbb{C})$, the mapping ρ is analytic. Furthermore, if $g, h \in \mathrm{Sl}_2(\mathbb{C})$, then

$$\begin{aligned} Q(\rho(g,h)A, \rho(g,h)A) &= \det(\rho(g,h)A) = \det(g)\det(A)\det(h)^{-1} \\ &= \det(A) = Q(A, A). \end{aligned}$$

By the polarization identity, Q is $\rho(g, h)$-invariant. Consequently, $\rho(\tilde{G}) \subset \mathrm{O}_4(\mathbb{C})$. Since $\tilde{G}$ is connected, we even have $\rho(\tilde{G}) \subset \mathrm{SO}_4(\mathbb{C}) = G$. If $\rho(g, h) = \mathrm{id}_V$, then especially $\rho(g, h)\mathrm{id}_{\mathbb{C}^2} = \mathrm{id}_{\mathbb{C}^2}$, i.e. $gh^{-1} = \mathrm{id}_{\mathbb{C}^2}$ and hence $g = h$. Then $gAg^{-1} = A$ for all $A \in \mathbb{M}_2(\mathbb{C})$ implies $g \in Z(\mathbb{M}_2(\mathbb{C})) = \mathbb{C} \cdot \mathrm{id}_{\mathbb{C}^2}$. Hence the kernel of ρ consists of the elements $(\mathrm{id}_{\mathbb{C}^2}, \mathrm{id}_{\mathbb{C}^2})$ and $(-\mathrm{id}_{\mathbb{C}^2}, -\mathrm{id}_{\mathbb{C}^2})$ only. Since the rank of ρ is constant, it must be 6 because otherwise the kernel would have non-zero dimension. Thus $\rho(\tilde{G})$ contains a neighborhood of the identity in G and hence all of G, because G is connected and thus generated by any neighborhood of the identity. Consequently, ρ is a covering map with kernel isomorphic to $\mathbb{Z}_2$.

11. *Let G be a Lie group and let $\rho\colon G \to \mathrm{Gl}(V)$ be a representation of G on a vector space V. A bilinear form B on V is G-invariant if*

$$B(\rho(g)v, \rho(g)w) = B(v, w) \qquad \text{for all } g \in G,\ v, w \in V.$$

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let $d\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ be the corresponding representation of $\mathfrak{g}$. Show that B is G-invariant if and only if

$$B(d\rho(X)v, w) + B(v, d\rho(X)w) = 0 \qquad \text{for all } X \in \mathfrak{g},\ v, w \in V.$$

Let $X \in \mathfrak{g}$. Then $\rho(\exp X) = \exp(d\rho(X))$, where the first exponential is the one in $\mathfrak{g}$ and the latter is the exponential in $\mathrm{Gl}(V)$. Hence we have

$$\begin{aligned} \frac{d}{dt}B(\rho(\exp(tX))v, \rho(\exp(tX))w) &= \frac{d}{dt}B(\exp(d\rho(tX))v, \exp(d\rho(tX))w) \\ &= B(\exp(d\rho(tX))d\rho(X)v, \exp(d\rho(tX))w) \\ &+ B(\exp(d\rho(tX))v, \exp(d\rho(tX))d\rho(X)w) \end{aligned}$$

If B is G-invariant, then B is $\rho(\exp(tX))$-invariant for all $X \in \mathfrak{g}$ and hence

$$0 = \frac{d}{dt}B(\rho(\exp(tX))v, \rho(\exp(tX))w) = B(d\rho(X)v, w) + B(v, d\rho(X)w),$$

applying the $\rho(\exp(tX))$-invariance twice. Conversely, if the latter equation is satisfied for all $v, w \in V$, then

$$\begin{aligned} 0 &= B(d\rho(X)\exp(d\rho(tX))v, \exp(d\rho(tX))w) \\ &\quad + B(\exp(d\rho(tX))v, d\rho(X)\exp(d\rho(tX))w) \\ &= \frac{d}{dt}B(\rho(\exp(tX))v, \rho(\exp(tX))w). \end{aligned}$$

Hence the function $t \mapsto B(\rho(\exp(tX))v, \rho(\exp(tX))w)$ is constant, so that $B(\rho(\exp(X))v, \rho(\exp(X))w) = B(v, w)$. Since G is connected, it is generated by elements of the form $\exp(X)$ with $X \in \mathfrak{g}$, so that B is G-invariant as desired.

12. *Let G be a connected Lie group, $\phi\colon \mathfrak{g} \to \mathrm{End}(V)$ be a representation of its Lie algebra $\mathfrak{g}$. Show that the bilinear form on $\mathfrak{g}$ defined by*

$$B(X, Y) = \mathrm{Tr}(\phi(X)\phi(Y))$$

is G-invariant (with respect to the adjoint action).

Since G and its universal covering group are isogeneous, we may replace G by its universal covering group $\tilde{G}$ if necessary because the images of $\mathrm{Ad}|_G\colon G \to \mathrm{Gl}(\mathfrak{g})$ and $\mathrm{Ad}|_{\tilde{G}}\colon \tilde{G} \to \mathrm{Gl}(\mathfrak{g})$ are the same. Thus we can assume without loss of generality that G is connected and simply connected. Hence the Lie algebra representation ϕ can be lifted to a Lie group representation $\Phi\colon G \to \mathrm{Gl}(V)$. If we write Ad for the adjoint representations of both G and $\mathrm{Gl}(V)$, we have $\mathrm{Ad}\circ\phi = \Phi\circ\mathrm{Ad}$, i.e. $\phi(\mathrm{Ad}(g)X) = \mathrm{Ad}(\Phi(g))X = \Phi(g)X\Phi(g)^{-1}$ for all $g\in G$, $X\in\mathfrak{g}$ (see the proof of exercise 5). Hence

$$\begin{aligned} B(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) &= \mathrm{Tr}(\phi(\mathrm{Ad}(g)X)\phi(\mathrm{Ad}(g)Y)) \\ &= \mathrm{Tr}(\Phi(g)\phi(X)\Phi(g^{-1})\Phi(g)\phi(Y)\Phi(g^{-1})) \\ &= \mathrm{Tr}(\phi(X)\phi(Y)\Phi(g^{-1})\Phi(g)) = B(X,Y) \end{aligned}$$

for all $g\in G$, $X,Y\in\mathfrak{g}$, i.e. B is G-invariant as desired.

13. *Let G be a connected simply connected Lie group with Lie algebra $\mathfrak{g}$. Prove that its commutator G' (the subgroup generated by $ghg^{-1}h^{-1}$ for all $g,h\in G$) is a closed connected subgroup with Lie algebra $\mathfrak{D}\mathfrak{g}$.*

Let $G^\dagger$ be the Lie subgroup in G generated by $\mathfrak{D}\mathfrak{g}$. This subgroup is connected and normal because $\mathfrak{D}\mathfrak{g}$ is an ideal in $\mathfrak{g}$ and hence closed because G is simply connected. We have to show that $G^\dagger = G'$. The commutator can be characterized as the smallest normal subgroup H such that G/H is Abelian; thus if H is any normal subgroup and G/H is Abelian, then $G'\subset H$. The group $G/G^\dagger$ is connected and has Lie algebra $\mathfrak{g}/\mathfrak{D}\mathfrak{g}$. The latter is trivially Abelian, so that $G/G^\dagger$ is Abelian. This yields the inclusion $G'\subset G^\dagger$. If we show that G' is an open subgroup of $G^\dagger$, we will be done because $G^\dagger$ is connected. It suffices to show that G' contains a neighborhood of e in $G^\dagger$. This is what we are going to do.

First, choose $X_1,\dots,X_n\in\mathfrak{g}$ such that the spaces $\mathrm{Im}(\mathrm{ad}\,X_k)$ span $\mathfrak{D}\mathfrak{g}$ and let $g_k = \exp(X_k)$. Let $Y_k\in\mathfrak{g}$ be arbitrary, $k=1,\dots,n$, and let γ_k be curves in G representing Y_k, i.e. $\gamma_k(0)=0$, $\gamma_k'(0)=Y_k$. Then

$$\alpha(t) = \prod_{k=1}^n (g_k\gamma_k(t)g_k^{-1}\gamma_k(t)^{-1})$$

is a curve in G' with $\alpha(0)=e$. We have

$$\begin{aligned} \alpha'(0) &= \sum_{k=1}^n \mathrm{Ad}(g_k)(Y_k) - Y_k = \sum_{k=1}^n (\mathrm{Ad}(\exp X_k) - \mathrm{id}_{\mathfrak{g}})Y_k \\ &= \sum_{k=1}^n (\exp(\mathrm{ad}\,X_k) - \mathrm{id}_{\mathfrak{g}})Y_k. \end{aligned}$$

The last exp is simply the exponential of the linear operator $\mathrm{ad}\,X_k\in\mathrm{Gl}(\mathfrak{g})$. Bringing $\mathrm{ad}\,X_k$ into Jordan normal form, we see that the image of $\exp(\mathrm{ad}\,X_k) - \mathrm{id}_{\mathfrak{g}}$ is precisely the image of $\mathrm{ad}\,X_k$. By assumption, the images of $\mathrm{ad}\,X_k$ span $\mathfrak{D}\mathfrak{g}$. Hence given $Z\in\mathfrak{D}\mathfrak{g}$, we can find Y_k such that $\alpha'(0)=Z$. Consider the mapping $\phi\colon G^n\to G'\subset G^\dagger$, $\phi(h_1,\dots,h_n) = g_1h_1g_1^{-1}h_1^{-1}\cdots g_nh_ng_n^{-1}h_n^{-1}$. Our above computation shows that $D\phi|_e$ maps *onto* $\mathfrak{D}\mathfrak{g} = \mathrm{T}_eG^\dagger$. Hence a neighborhood of e in $G^\dagger$ is contained in the image of ϕ, which is what we wanted to show.

14. *Let H be the group of affine transformations of the plane, $\mathfrak{h}$ its Lie algebra. Determine $\mathfrak{D}\mathfrak{h}$, $\mathrm{Rad}(\mathfrak{h})$, and $\mathfrak{h}_{\mathrm{ss}}$. Show that H is isomorphic to the group of automorphisms of the Heisenberg Lie algebra.*

Every affine transformation can be written as the composition of a translation and a linear transformation (fixing the origin). The translations form a normal subgroup T isomorphic to $\mathbb{R}^2$, and the linear transformations form a subgroup isomorphic to $\mathrm{Gl}_2(\mathbb{R})$. Hence H is the semidirect product of T and $\mathrm{Gl}_2(\mathbb{R})$. Consequently, $\mathfrak{h}$ is the semidirect sum of $\mathfrak{t}\cong\mathbb{R}^2$ (Abelian) and $\mathfrak{gl}_2(\mathbb{R})$. By definition, $\mathfrak{t}$ is contained in the radical. $\mathfrak{h}/\mathfrak{t}\cong\mathfrak{gl}_2(\mathbb{R})$ has center $\langle\mathrm{id}_{\mathbb{R}^2}\rangle$. Hence the Lie subalgebra generated by $\mathrm{id}_{\mathbb{R}^2}$ and $\mathfrak{t}$ is a solvable ideal and thus contained in the radical. We claim that $\mathrm{Rad}(\mathfrak{h}) = \langle\mathrm{id}_{\mathbb{R}^2}\rangle\oplus\mathfrak{t}$. Thus $\mathfrak{h}_{\mathrm{ss}} = \mathfrak{h}/\mathrm{Rad}(\mathfrak{h})\cong\mathfrak{gl}_2(\mathbb{R})/Z(\mathfrak{gl}_2(\mathbb{R}))\cong\mathfrak{sl}_2(\mathbb{R})$, because $\mathrm{Gl}_2(\mathbb{R})$ is the direct product of its center and $\mathrm{Sl}_2(\mathbb{R})$. We have seen in the lecture that $\mathfrak{sl}_2(\mathbb{C})$ is semisimple. If $\mathfrak{sl}_2(\mathbb{R})$ contained some non-trivial solvable ideal $\mathfrak{i}$, then $\mathfrak{i}\otimes\mathbb{C}$ would be a solvable ideal in $\mathfrak{sl}_2(\mathbb{C})$, which is impossible. Hence $\mathfrak{sl}_2(\mathbb{R})$ is semisimple. Thus the radical cannot be larger, and we have seen that it cannot be smaller either.

By exercise 13, $\mathfrak{D}\mathfrak{g}$ is the Lie algebra belonging to the commutator of H. If τ_a denotes translation by $a\in\mathbb{R}^2$ and $M\in\mathrm{Gl}_2(\mathbb{R})$, then $M\tau_aM^{-1}\tau_a^{-1} = \tau_{Ma}\tau_{-a} = \tau_{a-Ma}$. If we take, say, $M = 2\mathrm{id}_{\mathbb{R}^2}$, we see that every translation can be written as a commutator, so that $T\subset H'$ and $\mathfrak{t}\subset\mathfrak{D}\mathfrak{h}$. We have seen in the lecture that $\mathfrak{D}\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$. Using $\mathfrak{sl}_2(\mathbb{R})\otimes\mathbb{C} = \mathfrak{sl}_2(\mathbb{C})$, we obtain $\mathfrak{D}\mathfrak{sl}_2(\mathbb{R}) = \mathfrak{sl}_2(\mathbb{R})$. Hence $\mathfrak{sl}_2(\mathbb{R})\subset\mathfrak{D}\mathfrak{h}$. This implies that either $\mathfrak{D}\mathfrak{h} = \mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{t}$ or $\mathfrak{D}\mathfrak{h} = \mathfrak{h}$. The latter is impossible because it would imply $H' = H$. However, the commutator of two affine transformation is easily seen to preserve the volume, so that H' must be a proper subgroup of H. Therefore, $\mathfrak{D}\mathfrak{h} = \mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{t}$.

In order to identify H with the automorphism group of the Heisenberg Lie algebra, we choose the following (physically motivated) representation of the latter one. Let $\mathfrak{g}$ be the linear space of affine functions from $\mathbb{R}^2 = \langle p,q\rangle$ to $\mathbb{R}$. Thus $\mathfrak{g}$ is spanned by the functions 1, p, and q. On $\mathbb{R}^2$, we have the canonical 2-form $dp\wedge dq$, and we define the bracket on $\mathfrak{g}$ by $[\alpha,\beta] = dp\wedge dq(D\alpha, D\beta)$, where $D\sqcup$ denotes the derivative (at any point), considered as an element of $\mathbb{R}^2$. Obviously, this defines a skew-symmetric bilinear functional with $[1, \sqcup] = 0$, $[p, q] = 1$, so that we have obtained the Heisenberg Lie algebra.

We define $\rho\colon H \to \mathrm{Gl}(\mathfrak{g})$ by $\rho(g)\alpha = \det(Dg)\alpha \circ g^{-1}$ for $g \in H$, $\alpha \in \mathfrak{g}$. Here Dg denotes the derivative of g (at any point). Since the determinant is multiplicative, this is a representation of H. We claim that the image is contained in $\mathrm{Aut}(\mathfrak{g})$, i.e. preserves the bracket on $\mathfrak{g}$.

Indeed, using that $dp \wedge dq$ is the determinant of a pair of vectors in $\mathbb{R}^2$, we obtain

$$\begin{aligned} dp \wedge dq(\rho(g)\alpha, \rho(g)\beta) &= \det(Dg)^2 dp \wedge dq((D\alpha) \circ (Dg)^{-1}, (D\beta) \circ (Dg)^{-1}) \\ &= \det(Dg)^2 \det(Dg^{-1}) dp \wedge dq(D\alpha, D\beta) \\ &= \det(Dg)[\alpha, \beta] = \rho(g)[\alpha, \beta], \end{aligned}$$

because $[\alpha, \beta]$ is always a constant function, so that $[\alpha, \beta] \circ g^{-1} = [\alpha, \beta]$. Thus $\rho(H) \subset \mathrm{Aut}(\mathfrak{g})$.

Moreover, the kernel of ρ is trivial: If $\rho(g) = \mathrm{id}_{\mathfrak{g}}$, then $\rho(g)1 = 1$ implies $\det(Dg) = 1$, and $\alpha \circ g^{-1} = \alpha$ for all affine functions α then implies that $g^{-1} = \mathrm{id}_{\mathbb{R}^2}$ because affine functions separate the points of $\mathbb{R}^2$. We claim that the image of ρ is all of $\mathrm{Aut}(\mathfrak{g})$. Therefore, let a be an automorphism of $\mathrm{Aut}(\mathfrak{g})$. Then a is determined by $a(p)$ and $a(q)$ because $a(1) = [a(p), a(q)]$. Given affine functions $\alpha, \beta \in \mathfrak{g}$, there obviously is an affine map $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $p \circ L = \alpha$, $q \circ L = \beta$. If α and β are linearly independent, then L is invertible and hence an element of H. Hence we can find $L \in H$ with $\rho(L^{-1})(p) = a(p)$, $\rho(L^{-1})q = a(q)$, so that $\rho(L) \circ a$ fixes both p and q and thus must be the identity map. This proves that ρ is surjective, so that indeed H is isomorphic to the automorphism group of $\mathfrak{g}$.

15. *Classify complex 4-dimensional Lie algebras $\mathfrak{g}$ with* $\dim \mathfrak{D}\mathfrak{g} = 1$.

Let $\mathfrak{D}\mathfrak{g} = \langle Z \rangle$. We have to distinguish two cases. The first is that the rank of $\operatorname{ad} Z$ is 0, the second is that this rank is 1.

In the first case, we have $[Z, \mathfrak{g}] = 0$, but there must be $X, Y \in \mathfrak{g}$ with $[X, Y] = Z$. It is evident that X, Y, and Z must be linearly independent. Choose $U \in \mathfrak{g}$ such that $\{X, Y, Z, U\}$ is a basis for $\mathfrak{g}$. Then $[X, U] = aZ$, $[Y, U] = bZ$. Replace U by $U - aY + bX$. Then $\{X, Y, Z, U\}$ is still a basis for $\mathfrak{g}$, and now $[X, U] = aZ - a[X, Y] = 0$ and $[Y, U] = bZ + b[Y, X] = 0$. This implies that $\mathfrak{g}$ is the direct sum of $\langle U \rangle$ and $\langle X, Y, Z \rangle$. The first summand is one-dimensional and Abelian, the second summand satisfies the rules $[X, Y] = Z$ and $[X, Z] = [Y, Z] = 0$ and hence is isomorphic to the Heisenberg Lie algebra.

In the second case, we have $[Z, X] = Z$ for some $X \in \mathfrak{g}$. The kernel of $\operatorname{ad} Z$ has dimension 3 and contains Z. Hence we can find $U, V \in \operatorname{Ker}(\operatorname{ad} Z)$ such that $\{U, V, Z\}$ is a basis for $\operatorname{Ker}(\operatorname{ad} Z)$. By assumption, we have $[Z, U] = [Z, V] = 0$. Moreover, $[X, U] = aZ$. Thus if we replace U by $U + aZ$, we obtain $[X, U] = aZ + a[X, Z] = 0$ and still have a basis for $\operatorname{Ker}(\operatorname{ad} Z)$. Similarly, we can achieve $[X, V] = 0$. We have $[U, V] = cZ$ for some $c \in \mathbb{C}$ because $[U, V] \in \mathfrak{D}\mathfrak{g}$. Since $cZ = [[U, V], X] = [[V, X], U] + [[X, U], V] = 0$ by the Jacobi identity, we have $c = 0$. Thus $\operatorname{ad} U = \operatorname{ad} V = 0$, i.e. $\mathfrak{g}$ is the direct sum of the Abelian Lie algebras $\langle U \rangle \cong \mathbb{C}$ and $\langle V \rangle \cong \mathbb{C}$ and $\langle X, Z \rangle$. The latter Lie algebra is given by $[X, Z] = Z$. We have seen in the lecture that such a Lie algebra exists.

Hence there are two 4-dimensional complex Lie algebras with $\dim \mathfrak{D}\mathfrak{g} = 1$, and they are both direct sums lower dimensional Lie algebras.

16. *Show that the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module $V(n)$ has a non-degenerate $\mathfrak{sl}_2(\mathbb{C})$-invariant form, which is symmetric for even n and skew-symmetric for odd n. Show that such a form is unique up to multiplication by a scalar.*

A bilinear form B on $V(n)$ is $\mathfrak{sl}_2(\mathbb{C})$-invariant, if it is invariant under $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ as an element of $V(n) \otimes V(n)$. This means that $\mathbb{C} \cdot B$ is a $\mathfrak{g}$-invariant subspace of $V(n) \otimes V(n)$. Thus B corresponds to a summand $V(0)$ in the direct sum decomposition of $V(n) \otimes V(n)$. We have seen in the lecture that there is precisely one such summand, so that there is a $\mathfrak{g}$-invariant form $B \neq 0$ on $V(n)$ such that all other invariant forms are multiples of B. The kernel of B, i.e. the space of all $v \in V$ with $B(v, w) = 0$ for all $w \in V(n)$ is a $\mathfrak{g}$-submodule because $B(Xv, w) = -B(v, Xw) = 0$ for all $v \in \operatorname{Ker} B$, $w \in V(n)$, $X \in \mathfrak{g}$ by invariance of B. Since $V(n)$ has no nontrivial proper submodules and $\operatorname{Ker} B \neq V(n)$ because $B \neq 0$, the form B must be non-degenerate. In order to see that B is symmetric or skew-symmetric, we have to show that the corresponding summand $V(0)$ lies in $S^2(V(n))$ or in $\Lambda^2(V(n))$. These two modules are decomposed in exercise 17, and we see there that $V(0) \subset S^2(V(n))$ if n is even and $V(0) \subset \Lambda^2(V(n))$ if n is odd.

17. *Decompose the following $\mathfrak{sl}_2(\mathbb{C})$-modules into a sum of irreducible components: $S^2(V(n))$; $\Lambda^2(V(n))$; $S^n(V(2))$.*

We have seen in the lecture that a representation of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ is uniquely determined by the dimensions of the eigenspaces of H. Moreover, $V(n)$ has the H-eigenbasis v_i, $i \in \{-n, -n+2, \dots, n\} = I_n$ with

$Hv_i = iv_i$. The elements $s_{ij} = v_i \otimes v_j + v_j \otimes v_i$ with $i \le j$, $i,j \in I_n$, form a basis for $S^2(V(n))$. Furthermore, $H(s_{ij}) = Hv_i \otimes v_j + v_i \otimes Hv_j + Hv_j \otimes v_i + v_j \otimes Hv_i = (i+j)s_{ij}$, so that we have an H-eigenbasis for $S^2(V(n))$. Let $\alpha_k = \dim \operatorname{Ker}(H - k \cdot \mathrm{id})$. A simple computation now shows that $\alpha_{2n} = 1$, $\alpha_{2n-2} = 1$, $\alpha_{2n-4} = 2$, $\alpha_{2n-6} = 2$, $\alpha_{2n-8} = 3$, ... The same dimensions are obtained for the $\mathfrak{g}$-module

$$V(2n) \oplus V(2n-4) \oplus V(2n-8) \oplus \cdots \oplus V(2n - 4[n/2]),$$

where $[x]$ is the smallest integer not greater than x. Hence $S^2(V(n))$ is isomorphic to the above direct sum. Since $V(n) \otimes V(n) = V(2n) \oplus V(2n-2) \oplus \cdots V(0)$ by the lecture and $\Lambda^2(V) + S^2(V) = V \otimes V$, we obtain

$$\Lambda^2(V(n)) \cong V(2n-2) \oplus V(2n-6) \oplus \cdots \oplus V(2n-2-4[(n-1)/2]).$$

We have proven in the lecture that $S^n(V(m)) \cong S^m(V(n))$. Especially,

$$S^n(V(2)) \cong S^2(V(n)) \cong V(2n) \oplus V(2n-4) \oplus V(2n-8) \oplus \cdots \oplus V(2n-4[n/2]).$$

18. *Let $\mathfrak{g}$ be a Lie algebra and $B(X,Y)$ an invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$. Let $\{E_1, \dots, E_n\}$ and $\{F_1, \dots, F_n\}$ be two bases in $\mathfrak{g}$, satisfyingg the condition $B(E_i, F_j) = \delta_{ij}$.*

(a): *Let $X \in \mathfrak{g}$ and $\operatorname{ad} X(E_i) = \sum_{i=1}^n c_{ij} E_j$, $\operatorname{ad} X(F_i) = \sum_{i=1}^n d_{ij} F_j$. Show that $c_{ij} = -d_{ji}$.*

(b): *Let V be any $\mathfrak{g}$-module and $\rho_B = \sum_{i=1}^n E_i F_i$. Show that $\rho_B \in \operatorname{End}_{\mathfrak{g}}(V)$.*

The $\mathfrak{g}$-invariance of B means that $B(\operatorname{ad} X(Y), Z) + B(Y, \operatorname{ad} X(Z)) = 0$. By definition,

$$B(\operatorname{ad} X(E_i), F_j) = c_{ij} \qquad \text{and} \qquad B(E_i, \operatorname{ad} X(F_j)) = d_{ji}.$$

Thus we obtain $c_{ij} + d_{ji} = B(\operatorname{ad} X(E_i), F_j) + B(E_i, \operatorname{ad} X(F_j)) = 0$, which proves (a). We have to show that $X\rho_B = \rho_B X$ for all $X \in \mathfrak{g}$, if we view elements of $\mathfrak{g}$ as endomorphisms of V. This is a straightforward computation:

$$\begin{aligned} X\rho_B - \rho_B X &= \sum_{i=1}^n X E_i F_i - E_i X F_i + E_i X F_i - E_i F_i X \\ &= \sum_{i=1}^n (\operatorname{ad}(X) E_i) F_i + E_i (\operatorname{ad}(X) F_i) \\ &= \sum_{i,j=1}^n c_{ij} E_j F_i - c_{ji} E_i F_j = 0. \end{aligned}$$

19. *Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $B(X_1, X_2) = \operatorname{Tr}(X_1 X_2)$. Show that B is nondegenerate and write the expression for ρ_B in the basis $\{H, X, Y\}$. Determine the action of ρ_B on $V(n)$ for all $n \in \mathbb{N}$.*

In the basis $\{H, X, Y\}$, the bilinear form B is given by the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and thus is non-degenerate. This means that $E = (H, X, Y)$, $F = (H/2, Y, X)$ satisfy $B(E_i, F_j) = \delta_{ij}$. Hence $\rho_B = H^2/2 + XY + YX$. If V is an irreducible $\mathfrak{g}$-module, then $\operatorname{End}_{\mathfrak{g}}(V)$ is a finite dimensional division ring over $\mathbb{C}$ and hence isomorphic to $\mathbb{C}$. Therefore, ρ_B acts on $V(n)$ as a multiple of the identity. If we identify $V(n)$ with the $\mathfrak{g}$-module of homogeneous polynomials of degree n, then

$$\rho_B(x^n) = \frac{n^2}{2} x^n + 0 + n x^n = \frac{n(n+2)}{2} x^n$$

implies that $\rho_B|_{V(n)} = \frac{n(n+2)}{2} \mathrm{id}_{V(n)}$.

20. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ be a semi-simple ideal. Show that there is an ideal $\mathfrak{u} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{u}$.*

Let $\mathfrak{u} = \mathfrak{h}^\perp$ be the orthogonal complement of $\mathfrak{h}$ with respect to the Killing form $B_{\mathfrak{g}}$. Since $\mathfrak{h}$ is a $\mathfrak{g}$-submodule of $\mathfrak{g}$ and $B_{\mathfrak{g}}$ is $\mathfrak{g}$-invariant, $\mathfrak{u}$ is a $\mathfrak{g}$-submodule, i.e. an ideal of $\mathfrak{g}$. The intersection $\mathfrak{h} \cap \mathfrak{h}^\perp$ is zero because it is a solvable subideal of $\mathfrak{h}$ and $\mathfrak{h}$ is assumed to be semi-simple. Moreover, the codimension of $\mathfrak{h}^\perp$ is at most the dimension of $\mathfrak{h}$, so that we must have $\mathfrak{h} + \mathfrak{u} = \mathfrak{g}$. Moreover, $[\mathfrak{h}, \mathfrak{u}] \subset \mathfrak{h} \cap \mathfrak{u} = 0$, so that indeed $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{u}$ as a Lie algebra.

21. *Let $\mathfrak{g}$ be a semi-simple Lie algebra.*

(a): *Show that* $\operatorname{Der} \mathfrak{g} \cong \mathfrak{g}$.

(b): *Show that* $(\operatorname{Aut} \mathfrak{g})^0$ *(the connected component of 1 of the automorphism group) is isomorphic to $G/Z(G)$, where G is the simply connected group with Lie algebra $\mathfrak{g}$.*

(a): Exercise 7b yields that $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is a homomorphism of Lie algebras, even if $\mathfrak{g}$ is not semi-simple. The kernel of ad is the center of $\mathfrak{g}$, which must vanish for semi-simple $\mathfrak{g}$, so that ad is injective. It remains to show that ad is surjective.

If $d \in \mathrm{Der}(\mathfrak{g})$, $X \in \mathfrak{g}$, then $[d, \mathrm{ad}\, X](Y) = d([X,Y]) - [X, d(Y)] = [d(X), Y]$, so that $\mathrm{ad}(\mathfrak{g})$ is an ideal in $\mathrm{Der}(\mathfrak{g})$. Since $\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}$ is semi-simple, there is an ideal $\mathfrak{u} \subset \mathrm{Der}(\mathfrak{g})$ such that $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}) \oplus \mathfrak{u}$ by exercise 20. Hence $[d(X), Y] = [d, \mathrm{ad}\, X](Y) = 0$ for all $d \in \mathfrak{u}$, $X, Y \in \mathfrak{g}$, so that $d(\mathfrak{g}) \subset Z(\mathfrak{g}) = \{0\}$ and hence $d = 0$. Thus $\mathfrak{u} = \{0\}$ and consequently $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$.

(b): We have seen in exercise 8 that $\mathrm{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathrm{Der}(\mathfrak{g}) \cong \mathfrak{g}$ by part a. Consider $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$. The kernel of Ad is $Z(G)$, which is discrete because otherwise $\mathfrak{g}$ would have a non-trivial center. Hence Ad is an immersion, so that its image is a Lie subgroup of dimension $\dim \mathfrak{g} = \dim \mathrm{Aut}(\mathfrak{g})$. Hence $\mathrm{Ad}(G)$ contains the connected component of 1 in $\mathrm{Aut}(\mathfrak{g})$, and it cannot be larger because G is connected.

22. *Let $\mathfrak{g}$ be a semi-simple complex Lie algebra and let V be a $\mathfrak{g}$-module. Show that the Casimir operator $C_V = \sum_{i=1}^{\dim \mathfrak{g}} X_i X_i^\vee$ does not depend on the choice of the basis $\{X_i\}$ of $\mathfrak{g}$.*

Let $V = \oplus_{i=1}^n V_i$, where the V_i are irreducible $\mathfrak{g}$-modules. This decomposition is possible by Weyl's theorem because $\mathfrak{g}$ is semi-simple. Then $C_V(V_i) \subset V_i$ because C_V is a polynomial in elements of $\mathfrak{g}$. Thus we will be finished if we show that $C_V|_{V_i}$ does not depend on the choice of the basis X_i. Since V_i is irreducible, we have $C_V|_{V_i} = \lambda_i \mathrm{id}_{V_i}$ for some $\lambda_i \in \mathbb{C}$ by Schur's lemma. Moreover,

$$\lambda_i \dim V_i = \mathrm{Tr}\, C_V|_{V_i} = \sum_{i=1}^{\dim \mathfrak{g}} \mathrm{Tr}(X_i X_i^\vee) = \sum_{i=1}^{\dim \mathfrak{g}} B_{\mathfrak{g}}(X_i, X_i^\vee) = \sum_{i=1}^{\dim \mathfrak{g}} 1 = \dim \mathfrak{g}.$$

Thus λ_i does not depend on the choice of the basis X_i.

23. *Show that the following conditions on a Lie algebra $\mathfrak{g}$ are equivalent:*

a. $\mathrm{Rad}\, \mathfrak{g} = Z(\mathfrak{g})$*;*
b. $\mathfrak{g} \cong \mathfrak{g}_{\mathrm{ss}} \oplus Z(\mathfrak{g})$*;*
c. $\mathfrak{D}\mathfrak{g}$ *is semi-simple;*
d. *The adjoint representation of $\mathfrak{g}$ is semi-simple, i.e. $\mathfrak{g}$ is a completely reducible $\mathfrak{g}$-module.*

Such a Lie algebra is called reductive*.*

All these conditions are not changed if we go over to the complexification of $\mathfrak{g}$. Hence we may assume that $\mathfrak{g}$ is a complex Lie algebra.

If (2) is satisfied, then trivially $\mathfrak{D}\mathfrak{g} = \mathfrak{D}\mathfrak{g}_{\mathrm{ss}} \oplus \mathfrak{D}Z(\mathfrak{g}) = \mathfrak{g}_{\mathrm{ss}} \oplus \{0\}$, so that $\mathfrak{D}\mathfrak{g}$ is semi-simple. Thus (2) implies (3). Moreover, since $Z(\mathfrak{g}) \subset \mathrm{Rad}\, \mathfrak{g}$ and $\mathfrak{g}/Z(\mathfrak{g}) \cong \mathfrak{g}_{\mathrm{ss}}$ is semi-simple, we have $\mathrm{Rad}\, \mathfrak{g} = Z(\mathfrak{g})$, so that (2) implies (1) as well.

If (1) is true, then $\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}/Z(\mathfrak{g})$ is semi-simple, so that the $\mathfrak{g}/Z(\mathfrak{g})$-module $\mathfrak{g}$ (with operation induced by ad) is completely reducible by Weyl's theorem. Hence $\mathfrak{g}$ is also completely reducible as a $\mathfrak{g}$-module because the center does not operate at all on $\mathfrak{g}$. Hence (1) implies (4).

If $\mathfrak{D}\mathfrak{g}$ is semi-simple, then by exercise 21 there is an ideal $\mathfrak{u} \subset \mathfrak{g}$ with $\mathfrak{g} = \mathfrak{D}\mathfrak{g} \oplus \mathfrak{u}$, because $\mathfrak{D}\mathfrak{g}$ is an ideal. $[\mathfrak{u}, \mathfrak{g}] \subset \mathfrak{D}\mathfrak{g} \cap \mathfrak{u} = \{0\}$ implies that $\mathfrak{u} \subset Z(\mathfrak{g})$. On the other hand, the semi-simple Lie algebra $\mathfrak{D}\mathfrak{g} \cong \mathfrak{g}/\mathfrak{u}$ has trivial center, so that the center of $\mathfrak{g}$ must be contained in $\mathfrak{u}$. Therefore, $\mathfrak{u} = Z(\mathfrak{g})$, so that (3) implies (2).

If (4) is true, we can find a $\mathfrak{g}$-submodule V such that $\mathfrak{g} = Z(\mathfrak{g}) \oplus V$, because $Z(\mathfrak{g})$ is certainly a $\mathfrak{g}$-submodule. Let W be any Abelian ideal contained in V. Then $V = W \oplus W_2$ for some submodule W_2 and hence $\mathfrak{g} = W \oplus W_2 \oplus Z(\mathfrak{g})$. We have $[W, W] = 0$ because W is Abelian, $[W, W_2] \subset W \cap W_2 = 0$ and $[W, Z(\mathfrak{g})] = 0$ be definition of the center, and hence $[W, \mathfrak{g}] = 0$, so that $W \subset Z(\mathfrak{g})$. Thus $W = 0$, i.e. V contains no Abelian ideals and hence is semi-simple. Thus $V \cong \mathfrak{g}_{\mathrm{ss}}$ and hence $\mathfrak{g} \cong \mathfrak{g}_{\mathrm{ss}} \oplus Z(\mathfrak{g})$, so that (4) implies (2). This finishes the proof.

24. *Let $\Gamma_{a,b}$ be the irreducible representation of $\mathfrak{sl}_3(\mathbb{C})$ with highest weight $a\epsilon_1 - b\epsilon_2$. Show that $\dim \Gamma_{a,b} = (a+1)(b+1)(a+b+2)/2$.*

$\Gamma_{0,0} \cong \mathbb{C}$ is the trivial representation of dimension 1, which is in accord with the above formula. The standard representation V of $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ coincides with $\Gamma_{1,0}$ and has dimension 3. Let x_1, x_2, x_3 be the canonical basis for V with $E_{ij} x_k = \delta_{jk} x_i$, and let $\check{x}_1, \check{x}_2, \check{x}_3$ be the corresponding dual basis. Consider the operation of $\mathfrak{g}$ on $\mathrm{Sym}^a(V)$ and $\mathrm{Sym}^b(V)^*$. It is not difficult to see that, if we write $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$,

$$E_{ij} x^\alpha = x_i \frac{\partial}{\partial x_j} x^\alpha = \frac{\alpha_j x_i}{x_j} x^\alpha, \qquad E_{ij} \check{x}^\beta = -\check{x}_j \frac{\partial}{\partial \check{x}_i} \check{x}^\beta = -\frac{\beta_i \check{x}_j}{\check{x}_i} \check{x}^\beta.$$

The first follows by making explicit the inclusion $\mathrm{Sym}^a(V) \subset V^{\otimes a}$, and the second follows because $X \in \mathfrak{g}$ operates on V^* like $-X^t$. A trivial computation shows that the only highest vector of $\mathrm{Sym}^a(V)$ is x_1^a, with weight $a\epsilon_1$, and the only highest vector of $\mathrm{Sym}^b(V)^*$ is $\check{x}_3^b$ with weight $-b\epsilon_3$. Hence $\Gamma_{a,0} \cong \mathrm{Sym}^a(V)$ and $\Gamma_{0,b} \cong \mathrm{Sym}^b(V)^*$.

The vectors $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} = x^\alpha$ with $|\alpha| = \alpha_1+\alpha_2+\alpha_3 = a$ form a basis for $\Gamma_{a,0}$. Thus the dimension of $\Gamma_{a,0}$ is

$$\dim \Gamma_{a,0} = \sum_{\alpha_1=0}^{a}\sum_{\alpha_2=0}^{a-\alpha_1} 1 = \sum_{\alpha_1=0}^{a} a+1-\alpha_1 = \sum_{k=1}^{a+1} k = \frac{1}{2}(a+1)(a+2),$$

which is in accord with the claimed formula. Moreover, $\dim \Gamma_{0,b} = \frac{1}{2}(b+1)(b+2)$. We claim that

$$\Gamma_{a,0}\otimes\Gamma_{0,b} \cong \Gamma_{a-1,0}\otimes\Gamma_{0,b-1}\oplus\Gamma_{a,b} \cong \Gamma_{a,b}\oplus\Gamma_{a-1,b-1}\oplus\Gamma_{a-2,b-2}\oplus\cdots. \tag{3}$$

The second equality clearly follows from the first by induction. Provided this equation is true, we obtain

$$\begin{aligned}\dim\Gamma_{a,b} &= \dim\Gamma_{a,0}\dim\Gamma_{0,b} - \dim\Gamma_{a-1,0}\dim\Gamma_{0,b-1}\\ &= \frac{1}{4}\big((a+1)(a+2)(b+1)(b+2) - (a+1)a(b+1)b\big)\\ &= \frac{1}{4}(a+1)(b+1)(ab+2a+2b+4-ab) = \frac{1}{2}(a+1)(b+1)(a+b+2).\end{aligned}$$

Hence it only remains to show (3). Clearly, $h_0 = x_1^a\otimes\check{x}_3^b$ is a highest vector of $W = \Gamma_{a,0}\otimes\Gamma_{0,b}$ with weight $a\epsilon_1 - b\epsilon_3$. In the lecture, we have given a surjective linear map $\Phi\colon W \to \Gamma_{a-1,0}\otimes\Gamma_{0,b-1}$ such that $h_0 \in \mathrm{Ker}\,\Phi$. Thus we have to show that the kernel of Φ is irreducible. Since (by induction) $\Gamma_{a-1,0}\otimes\Gamma_{0,b-1}$ is a direct sum of the $r = \min(a,b)$ irreducible $\mathfrak{g}$-modules $\Gamma_{a-k,b-k}$, $k = 0,\ldots,r$, we only have to show that W has at most $r+1$ irreducible submodules. Every irreducible submodule of W is generated by a highest vector. The linear span of the highest vectors in W is contained in $K = \mathrm{Ker}\,E_{12}\cap\mathrm{Ker}\,E_{23}$. Hence we will be done if we show that the dimension of K is at most $r+1$. We assume in the following that $b \le a$ because the argument for $b < a$ is symmetrical.

We have

$$E_{ij}(x^\alpha\otimes\check{x}^\beta) = \frac{\alpha_j x_i}{x_j}x^\alpha\otimes\check{x}^\beta - x^\alpha\otimes\frac{\beta_i\check{x}_j}{\check{x}_i}\check{x}^\beta,$$

where the fractions are interpreted as 0 if $\alpha_j = 0$ or $\beta_i = 0$. Let $v = \sum c_{\alpha\beta}x^\alpha\otimes\check{x}^\beta \in K$, $c_{\alpha\beta}\in\mathbb{C}$ for all α,β with $|\alpha| = a$, $|\beta| = b$. In the following, we adopt the convention that $c_{\alpha\beta} = 0$ if some of the entries of (α,β) are negative. $E_{12}v = E_{23}v = 0$ means (compare the coefficient of $x^\alpha\otimes\check{x}^\beta$)

$$(\alpha_2+1)c_{\alpha_1-1,\alpha_2+1,\alpha_3,\beta} = (\beta_1+1)c_{\alpha,\beta_1+1,\beta_2-1,\beta_3}, \tag{4}$$

$$(\alpha_3+1)c_{\alpha_1,\alpha_2-1,\alpha_3+1,\beta} = (\beta_2+1)c_{\alpha,\beta_1,\beta_2+1,\beta_3-1}. \tag{5}$$

Using (4) and (5) repeatedly, we obtain

$$\begin{aligned}c_{\alpha\beta} &= \frac{\beta_2+1}{\alpha_3}c_{\alpha_1,\alpha_2+1,\alpha_3-1,\beta_1,\beta_2+1,\beta_3-1} = \frac{(\beta_2+1)(\beta_2+2)\cdots(\beta_2+\alpha_3)}{\alpha_3(\alpha_3-1)\cdots 1}c_{\alpha_1,\alpha_2+\alpha_3,0,\beta_1,\beta_2+\alpha_3,\beta_3-\alpha_3}\\ &= \binom{\beta_2+\alpha_3}{\beta_2}\frac{\beta_1+1}{\alpha_2+\alpha_3}c_{\alpha_1+1,\alpha_2+\alpha_3-1,0,\beta_1+1,\beta_2+\alpha_3-1,\beta_3-\alpha_3}\\ &= \binom{\beta_2+\alpha_3}{\beta_2}\binom{\beta_1+\alpha_2+\alpha_3}{\beta_1}c_{a,0,0,a+\beta_1-\alpha_1,\beta_2-\alpha_2,\beta_3-\alpha_3}.\end{aligned}$$

Moreover, if $\beta_2 \neq 0$, then (5) yields

$$c_{a,0,0,\beta_1,\beta_2,\beta_3} = \frac{1}{\beta_2}c_{a,-1,1,\beta_1,\beta_2-1,\beta_3+1} = 0.$$

Thus the $b+1$ numbers $c_{a,0,0,\beta,0,b-\beta}$, $\beta = 0,\ldots,b$, completely determine v. This implies that $\dim K \le b+1 = r+1$ for $b \le a$. If $b < a$, repeated application of equation (4) implies that $c_{a,0,0,\beta,0,b-\beta} = 0$ for $\beta > a$. This can also be seen because $E_{11}v \ge 0$ for all highest vectors. A better idea is to express all $c_{\alpha\beta}$ in terms of $c_{a-\alpha,0,\alpha,0,0,b}$ instead, if $b > a$. The simplest argument, however, is that if W has at most r irreducible components, then the same holds for W^*, so that the case $a < b$ can be reduced to the case $a > b$.

25. *Let $\Gamma_{a,b}$ be the same as in problem 1. Show that $\Gamma_{a,b}^* = \Gamma_{b,a}$. Show that there exists a non-zero bilinear $\mathfrak{sl}_3(\mathbb{C})$-invariant form on $\Gamma_{a,b}$ iff $a = b$. Determine when this form is symmetric (skew-symmetric).*

We have seen in the lecture that $\Gamma_{a,0} = \mathrm{Sym}^a(V)$, where V is the standard 3-dimensional representation of $\mathfrak{sl}_3(\mathbb{C})$, and $\Gamma_{0,a} = \mathrm{Sym}^a(V)^*$ is its dual. Hence $\Gamma_{a,0}^* \cong \Gamma_{0,a}$ for all $a \in \mathbb{Z}_+$. Moreover,

$$\begin{aligned}
&\Gamma_{a,b}^* \oplus \Gamma_{a-1,b-1}^* \oplus \Gamma_{a-2,b-2}^* \oplus \cdots \cong (\Gamma_{a,b} \oplus \Gamma_{a-1,b-1} \oplus \Gamma_{a-2,b-2} \oplus \cdots)^* \\
&\cong (\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V)^*)^* \cong (\mathrm{Sym}^a(V)^* \otimes (\mathrm{Sym}^b(V)^*)^* \cong \mathrm{Sym}^a(V)^* \otimes \mathrm{Sym}^b(V) \\
&\cong \mathrm{Sym}^b(V) \otimes \mathrm{Sym}^a(V)^* \cong \Gamma_{b,a} \oplus \Gamma_{b-1,a-1} \oplus \Gamma_{b-2,a-2} \oplus \cdots .
\end{aligned}$$

By induction over b, this implies $\Gamma_{a,b}^* \cong \Gamma_{b,a}$.

In order to prove the second part, we first prove the following theorem:

Theorem 2. *Let V be a finite-dimensional $\mathfrak{g}$-module, where $\mathfrak{g}$ is any Lie algebra. There is a non-degenerate $\mathfrak{g}$-invariant form on V if and only if $V \cong V^*$. If, in addition, V is irreducible, any $\mathfrak{g}$-invariant, non-zero bilinear form is non-degenerate and two invariant forms are linearly dependent; every invariant bilinear form is either symmetric or skew-symmetric.*

Proof: Let $B\colon V \times V \to \mathbb{C}$ be a non-degenerate $\mathfrak{g}$-invariant form. Define $\rho\colon V \to V^*$ by $\rho(v)\colon w \mapsto B(v, w)$ for $v, w \in V$. For $X \in \mathfrak{g}$, $v, w \in V$,

$$\big(X\rho(v)\big)(w) = -\big(\rho(v)\big)(Xw) = -B(v, Xw) = B(Xv, w) = \big(\rho(Xv)\big)(w),$$

i.e. ρ is $\mathfrak{g}$-linear. Since B is non-degenerate, $\operatorname{Ker}\rho = \{0\}$, and since $\dim V = \dim V^* < \infty$, ρ is even an isomorphism. Conversely, given a $\mathfrak{g}$-linear isomorphism $\rho\colon V \to V^*$, we can define a bilinear form $B\colon V \times V \to \mathbb{C}$ by $B(v, w) = \big(\rho(v)\big)(w)$, and a trivial computation as above shows that B is $\mathfrak{g}$-invariant. This proves the first claim.

If B is any invariant form, the kernel of B is a $\mathfrak{g}$-submodule. If V is irreducible, the kernel of B can only be all of V, i.e. B is zero, or $\{0\}$, i.e. B is non-degenerate. If B_1 and B_2 are $\mathfrak{g}$-invariant bilinear forms and, say, $B_2 \neq 0$, then $z \mapsto \det(B_1 - zB_2)$ is a non-constant polynomial and hence has a zero λ. The form $B_1 - \lambda B_2$ is invariant and degenerate, hence 0, i.e. $B_1 = \lambda B_2$. If B is an invariant form, so are its symmetric and skew-symmetric parts $(v, w) \mapsto B(v, w) \pm B(w, v)$. Hence the symmetric and skew-symmetric part are linearly dependent, which is possible only if one of them vanishes, i.e. B is either symmetric or skew-symmetric. $\square$

Hence there is a non-zero $\mathfrak{sl}_3(\mathbb{C})$-invariant bilinear form on $\Gamma_{a,b}$ if and only if $\Gamma_{a,b} \cong \Gamma_{a,b}^* \cong \Gamma_{b,a}$, because $\Gamma_{a,b}$ is irreducible. Since the highest weight of an irreducible $\mathfrak{sl}_3(\mathbb{C})$-module is unique, this is true if and only if $(a, b) = (b, a)$, i.e. $a = b$. Consequently, there is a non-zero invariant bilinear form on $\Gamma_{a,b}$ if and only if $a = b$. Moreover, we already know that the bilinear form on $\Gamma_{a,a}$ is uniquely determined up to scalar multiples and either symmetric or skew-symmetric. We claim that it is always symmetric. In order to show this, we look at the canonical isomorphism $\Gamma_{a,a} \to \Gamma_{a,a}^*$. This is the restriction of the canonical isomorphism $\rho\colon \mathrm{Sym}^a(V) \otimes \mathrm{Sym}^a(V)^* \to (\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^a(V)^*)^*$ given by

$$\rho(x_1^{\alpha_1} \cdots x_3^{\alpha_3} \otimes \check{x}_1^{\beta_1} \cdots \check{x}_3^{\beta_3})(x_1^{\tilde{\alpha}_1} \cdots x_3^{\tilde{\alpha}_3} \otimes \check{x}_1^{\tilde{\beta}_1} \cdots \check{x}_3^{\tilde{\beta}_3}) = \delta_{\alpha\tilde{\beta}} \cdot \delta_{\tilde{\alpha}\beta},$$

where $x_1, \dots, x_3$ is a fixed basis of $\mathbb{C}^3$ and $\delta_{\alpha\beta} = \delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}$. The bilinear form induced by this is obviously symmetric.

26. *Find the irreducible components of the $\mathfrak{sl}_3(\mathbb{C})$-module $W = \Gamma_{a,b} \otimes \Gamma_{1,0}$. (Hint: Show that this tensor product has no more than 3 components and use the dimension formula.)*

If $a = 0$, we obtain $\Gamma_{0,b} \otimes \Gamma_{1,0} \cong \Gamma_{1,b} \oplus \Gamma_{0,b-1}$ by (3). Hence we assume in the following that $a > 0$. The highest vector of $\Gamma_{a,b} \subset \mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V)^*$ is $v_0 = x_1^a \otimes \check{x}_3^b$ with weight $a\epsilon_1 - b\epsilon_3$. A trivial computation shows that $w_0 = (x_1^a \otimes \check{x}_3^b) \otimes x_1$ is a highest vector of W with weight $(a+1)\epsilon_1 - b\epsilon_3$. Since $a > 0$, $a^{-1}E_{21}v_0 = x_1^{a-1}x_2 \otimes \check{x}_3^b \in \Gamma_{a,b}$, and

$$w_1 = x_1^{a-1}x_2 \otimes \check{x}_3^b \otimes x_1 - x_1^a \otimes \check{x}_3^b \otimes x_2$$

is a non-zero vector in W with $E_{12}w_1 = E_{23}w_1 = 0$, $E_{11}w_1 = aw_1$, $E_{22}w_1 = w_1$, and $E_{33}w_1 = -bw_1$. Thus w_1 is a highest vector with weight $a\epsilon_1 + \epsilon_2 - b\epsilon_3 = (a-1)\epsilon_1 - (b+1)\epsilon_3$ because $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$. If $b = 0$, then

$$\begin{aligned}
\dim \Gamma_{a+1,0} + \dim \Gamma_{a-1,1} &= \frac{1}{2}(a+3)(a+2) + \frac{1}{2}2a(a+2) = \frac{3}{2}(a+1)(a+2) \\
&= \dim \Gamma_{a,0} \otimes \Gamma_{1,0}
\end{aligned}$$

by exercise 25, so that we obtain $\Gamma_{a,0} \otimes \Gamma_{1,0} \cong \Gamma_{a+1,0} \oplus \Gamma_{a-1,1}$. Hence we assume in the following that $b > 0$.

$-b^{-1}E_{32}v_0 = x_1^a \otimes \check{x}_2\check{x}_3^{b-1}$ and $-b^{-1}E_{31}v_0 = x_1^a \otimes \check{x}_1\check{x}_3^{b-1}$ are elements of $\Gamma_{a,b}$. Consider

$$w_2 = x_1^a \otimes \check{x}_1\check{x}_3^{b-1} \otimes x_1 + x_1^a \otimes \check{x}_2\check{x}_3^{b-1} \otimes x_2 + x_1^a \otimes \check{x}_3\check{x}_3^{b-1} \otimes x_3 \in W.$$

An easy calculation shows that $E_{12}w_2 = E_{23}w_2 = 0$, $E_{11}w_2 = aw_2$, $E_{22}w_2 = 0$, $E_{33}w_2 = -(b-1)w_2$, so that w_2 is a highest vector with weight $a\epsilon_1 - (b-1)\epsilon_2$.

Thus $\Gamma_{a+1,b} \oplus \Gamma_{a-1,b+1} \oplus \Gamma_{a,b-1}$ is a $\mathfrak{g}$-submodule of W. We claim that it is everything, and for this purpose we only have to compare the dimensions, using exercise 25:

$$\begin{aligned} & 2(\dim \Gamma_{a+1,b} + \dim \Gamma_{a-1,b+1} + \dim \Gamma_{a,b-1} - 3 \dim \Gamma_{a,b}) \\ = \;& (a+2)(b+1)(a+b+3) + a(b+2)(a+b+2) + (a+1)b(a+b+1) - 3(a+1)(b+1)(a+b+2) \\ = \;& (a+b+2)(ab+2b+a+2+ab+2a+ab+b-3ab-3b-3a-3) + (a+2)(b+1) - (a+1)b \\ = \;& (a+b+2)(-1) + ab + 2b + a + 2 - ab - b = 0. \end{aligned}$$

27. *Describe the root decompositions of the Lie algebras $\mathfrak{so}_5(\mathbb{C})$ and $\mathfrak{sp}_4(\mathbb{C})$ and show that these two algebras are isomorphic. (Hint: construct a 5-dimensional representation of $\mathfrak{sp}_4(\mathbb{C})$ as a direct summand in $\Lambda^2(V)$, where V is the representation in column vectors.)*

$\mathfrak{sp}_4(\mathbb{C}) \subset \mathfrak{gl}_4(\mathbb{C})$ operates on $\Lambda^2\mathbb{C}^4$ in a canonical way, namely by $\tilde{\rho}(X)M = X^tM + MX$ for $X \in \mathfrak{gl}_4(\mathbb{C})$, $M \in \Lambda^2\mathbb{C}^4$. In fact, $\mathfrak{sp}_4(\mathbb{C})$ is the stabilizer of some skew-invariant form $Q \in \Lambda^2\mathbb{C}^4$.

If we choose the skew-symmetric bilinear form given by the matrix

$$Q = \begin{pmatrix} 0 & \mathrm{id}_{\mathbb{C}^2} \\ -\mathrm{id}_{\mathbb{C}^2} & 0 \end{pmatrix},$$

then $\mathfrak{sp}_4(\mathbb{C}) = \{X \in \mathbb{M}_4 \mid X^tQ + QX = 0\}$. If we write X in block form, this means

$$\begin{pmatrix} -C & A \\ -D & B \end{pmatrix} + \begin{pmatrix} C^t & D^t \\ -A^t & -B^t \end{pmatrix} = 0,$$

i.e. $B = B^t$, $D = -A^t$, and $C = C^t$. Thus

$$\mathfrak{sp}_4(\mathbb{C}) = \left\{ X(A,B,C) = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \,\middle|\, A, B, C \in \mathbb{M}_2,\ B = B^t,\ C = C^t \right\}.$$

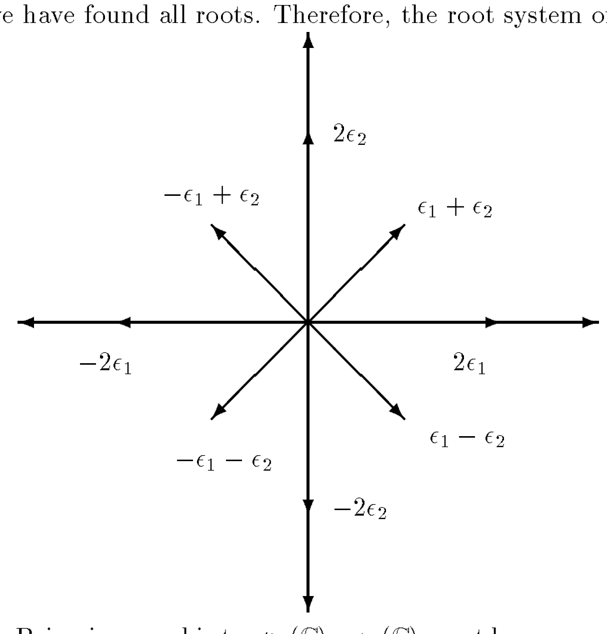
Consequently, $\dim \mathfrak{sp}_4(\mathbb{C}) = 4 + 3 + 3 = 10$.

As Cartan subalgebra, we take the diagonal matrices. This yields the two-dimensional subalgebra

$$\mathfrak{h} = \{\mathrm{diag}(a_1, a_2, -a_1, -a_2) \mid a_1, a_2 \in \mathbb{C}\}.$$

Choose the basis $h_1 = \mathrm{diag}(1,0,-1,0)$, $h_2 = \mathrm{diag}(0,1,0,-1)$ for $\mathfrak{h}$ and let $\epsilon_1, \epsilon_2 \in \mathfrak{h}^*$ be the dual basis.

Let $1 \le i \le j \le 2$. Then $U_{ij} = X(0, E_{ij} + E_{ji}, 0)$ satisfies $[h_k, U_{ij}] = (\delta_{ki} + \delta_{kj})U_{ij}$, so that $\mathbb{C} \cdot U_{ij}$ is a root space for the root $\epsilon_i + \epsilon_j$. Moreover, $V_{ij} = X(0, 0, E_{ij} + E_{ji})$ satisfies $[h_k, V_{ij}] = -(\delta_{ki} + \delta_{kj})V_{ij}$, so that $\mathbb{C} \cdot V_{ij}$ is a root space for the root $-\epsilon_i - \epsilon_j$. This yields 6 root spaces. Finally, let $i, j \in \{1, 2\}$, $i \ne j$, and let $W_{ij} = X(E_{ij}, 0, 0)$. Then $[h_k, W_{ij}] = (\delta_{ki} - \delta_{kj})W_{ij}$, i.e. $\mathbb{C} \cdot W_{ij}$ is a root space for the root $\epsilon_i - \epsilon_j$. This yields two further root spaces. Together with $\mathfrak{h}$, these sum up to the dimension of $\mathfrak{g}$, so that we have found all roots. Therefore, the root system of $\mathfrak{g}$ looks like this:



Being isomorphic to $\mathfrak{sp}_4(\mathbb{C})$, $\mathfrak{so}_5(\mathbb{C})$ must have an equivalent root system. Now we give an isomorphism between $\mathfrak{so}_5(\mathbb{C})$ and $\mathfrak{sp}_4(\mathbb{C})$.

Since Q is $\mathfrak{sp}_4(\mathbb{C})$-invariant, it spans a one-dimensional $\mathfrak{sp}_4(\mathbb{C})$-submodule of $\Lambda^2\mathbb{C}^4$. Moreover, we have the canonical multiplication $\mu\colon \Lambda^2\mathbb{C}^4 \times \Lambda^2\mathbb{C}^4 \to \Lambda^4\mathbb{C}^4 \cong \mathbb{C}$. The last congruence is an equivalence of $\mathfrak{sp}_4(\mathbb{C})$-modules because $\mathfrak{sp}_4(\mathbb{C}) = \mathfrak{D}\mathfrak{sp}_4(\mathbb{C})$ implies $\mathfrak{sp}_4(\mathbb{C}) \subset \mathfrak{sl}_4(\mathbb{C})$, which operates trivially on $\Lambda^4\mathbb{C}^4$. Hence μ yields an invariant bilinear form on $\Lambda^2\mathbb{C}^4$, which is trivially non-degenerate. The multiplication μ and hence the resulting bilinear form is symmetric (because we always do two transpositions at a time if we exchange the factors).

Since $\mu(Q,Q) = Q\wedge Q \neq 0$, the orthogonal complement $Q^\perp$ of Q is a five-dimensional $\mathfrak{sp}_4(\mathbb{C})$-submodule of $\Lambda^2\mathbb{C}^4$ not containing Q, so that $Q \oplus Q^\perp = \Lambda^2\mathbb{C}^4$. Moreover, the restriction of the bilinear form to $Q^\perp$ remains non-degenerate and of course symmetric. Hence the image of $\rho\colon \mathfrak{sp}_4(\mathbb{C}) \to \mathrm{End}(Q^\perp)$ is contained in $\mathfrak{o}_5(\mathbb{C})$ and even in $\mathfrak{so}_5(\mathbb{C})$ because $\mathfrak{D}\mathfrak{sp}_4(\mathbb{C}) = \mathfrak{sp}_4(\mathbb{C})$. Clearly, $\rho \neq 0$, and since $\mathfrak{sp}_4(\mathbb{C})$ is simple, ρ is injective. Since $\dim \mathfrak{sp}_4(\mathbb{C}) = \dim \mathfrak{so}_5(\mathbb{C}) = 10$, ρ is an isomorphism.

28. *Show that a semi-simple Lie algebra is simple iff its root system is irreducible.*

Let $\mathfrak{g}$ be a semi-simple Lie algebra and let R be the root system of $\mathfrak{g}$. Assume first that $\mathfrak{g}$ is not simple, say, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. If $\mathfrak{h}_k$ is a Cartan subalgebra of $\mathfrak{g}_k$, $k = 1,2$, then $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ is a Cartan subalgebra of $\mathfrak{g}$. Moreover, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are orthogonal with respect to the Killing form because $\operatorname{ad} H_1 \operatorname{ad} H_2(\mathfrak{g}) \subset [H_1, \mathfrak{g}_2] = \{0\}$ for $H_k \in \mathfrak{h}_k$, i.e. $\operatorname{ad} H_1 \operatorname{ad} H_2 = 0$. If $\alpha \in R$, $X \in \mathfrak{g}_\alpha = \mathbb{C}\cdot X$, then $\alpha(H_k)X = [H_k, X] \in \mathfrak{g}_k$ for $H_k \in \mathfrak{h}_k$. Hence $X \in \mathfrak{g}_k$ or $\alpha(H_k) = 0$. Thus all root spaces are contained in $\mathfrak{g}_1$ or $\mathfrak{g}_2$, and every root α either annihilates $\mathfrak{h}_1$ (if $\mathfrak{g}_\alpha \subset \mathfrak{g}_2$) or $\mathfrak{h}_2$. Hence if $E_k = \mathfrak{h}_k^\perp$, $k = 1,2$, then $R = (R \cap E_1) \amalg (R \cap E_2)$. Moreover, since $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are orthogonal, so are E_1 and E_2. Thus R is reducible.

Now assume that R is reducible, i.e. $E = E_1 \oplus E_2$, where E_1 and E_2 are orthogonal and $R = (R \cap E_1) \amalg (R \cap E_2)$. Let $\mathfrak{h}$ be any Cartan subalgebra, so that $E = \mathfrak{h}^*$. Let $\mathfrak{h}_k = E_k^\perp$ be the subspace annihilated by E_k. Then $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. Let

$$\mathfrak{g}_1 = \mathfrak{h}_1 \cup \bigcup_{\alpha \in R \cap E_2} \mathfrak{g}_\alpha.$$

We claim that $\mathfrak{g}_1$ is an ideal in $\mathfrak{g}$. We have to show that if $X \in \mathfrak{h}_1$ or $X \in \mathfrak{g}_\alpha$ with $\alpha \in R \cap E_2$, and if $Y \in \mathfrak{h}$ or $Y \in \mathfrak{g}_\beta$ for some $\beta \in R$, then $[Y,X] \in \mathfrak{g}_1$. First, if $X \in \mathfrak{h}_1$, then $[Y,X] = 0$ for $Y \in \mathfrak{h}$ and $[Y,X] = -\beta(X)Y$ for $Y \in \mathfrak{g}_\beta$. If $\beta(X) \neq 0$, then $\beta \notin E_1$ and hence $\beta \in R \cap E_2$, so that $Y \in \mathfrak{g}_1$ and thus $[Y,X] \in \mathfrak{g}_1$. Secondly, if $X \in \mathfrak{g}_\alpha$, then $[Y,X] = \alpha(Y)X \in \mathfrak{g}_1$ for $Y \in \mathfrak{h}$ and $[Y,X] \in \mathfrak{g}_{\alpha+\beta}$ for $Y \in \mathfrak{g}_\beta$. If $\beta \in R \cap E_2$, then $\alpha + \beta \in E_2$, so that $[Y,X] \in \mathfrak{g}_1$ (if $\alpha + \beta \notin R$, then $[Y,X] = 0$). If $\beta \in R \cap E_1$, then $\alpha + \beta \notin R$ and hence $[Y,X] = 0 \in \mathfrak{g}_1$. Consequently, $\mathfrak{g}_1$ is a nontrivial ideal in $\mathfrak{g}$, so that $\mathfrak{g}$ is not simple.

29. *For each classical root system A_n, B_n, C_n, and D_n, find a base and the Dynkin diagram.*

In the following, let $\{\epsilon_i\}$ be an orthonormal basis of some Euclidean Space E containing the root system.

The root system A_n has rank n,

$$A_n = \{\epsilon_i - \epsilon_k \mid i,k = 1,2,\dots,n+1, i \neq k\}.$$

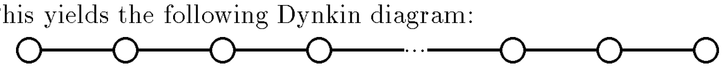
The linear span of A_n is the hyperplane $E = \{x_1 + \cdots + x_{n+1} = 0\}$ in $\mathbb{R}^{n+1}$. We call the root $\epsilon_i - \epsilon_k$ positive iff $i < k$. Since

$$\epsilon_i - \epsilon_k = (\epsilon_i - \epsilon_{i+1}) + (\epsilon_{i+1} - \epsilon_{i+2}) + \cdots + (\epsilon_{k-1} - \epsilon_k),$$

the roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $i = 1,\dots,n$, form a base for the root system A_n. We have

$$\langle \alpha_i, \alpha_k \rangle = \begin{cases} 0, & \text{if } |i-k| > 1, \\ -1, & \text{if } |i-k| = 1, \\ 2, & \text{if } i = k. \end{cases}$$

This yields the following Dynkin diagram:



A_n

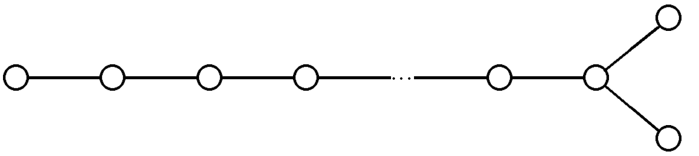
The root system D_n has rank n as well and is given by

$$D_n = A_{n-1} \cup \{\pm(\epsilon_i + \epsilon_k) \mid i,k = 1,\dots,n, i \neq k\}.$$

We call the roots in A_{n-1} positive if they are positive according to the above condition, and we call $\epsilon_i + \epsilon_k$ positive for $i,k = 1,\dots,n$. We have

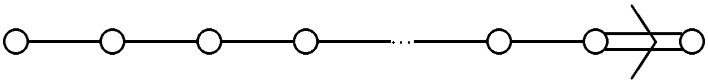
$$\epsilon_i + \epsilon_k = (\epsilon_i - \epsilon_{n-1}) + (\epsilon_k - \epsilon_n) + (\epsilon_{n-1} + \epsilon_n).$$

This implies that the roots $\delta_i = \alpha_i$, $i = 1,\dots,n-1$, as above, and $\delta_n = \epsilon_{n-1} + \epsilon_n$ are a base for D_n. We have $\langle \delta_n, \delta_n \rangle = 2$ and $\langle \delta_n, \alpha_i \rangle = -\delta_{i,n-2}$, where $\delta_{i,n-2}$ is a Kronecker δ. Hence the Dynkin diagram of D_n is



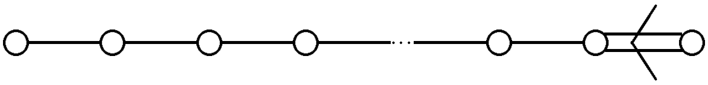
D_n

The root system B_n is obtained from D_n by adding the roots $\pm\epsilon_i$, $i = 1, \dots, n$. We call $+\epsilon_i$ positive. Since $\delta_n = \alpha_{n-1} + 2\epsilon_n$, the roots $\beta_i = \alpha_i$, $i = 1, \dots, n-1$, $\beta_n = \epsilon_n$ form a base for B_n. Since $\langle \beta_n, \beta_n \rangle = 1$, $\langle \beta_n, \beta_i \rangle = -\delta_{i,n-1}$, we obtain the Dynkin diagram



B_n

In the case of C_n, we add the roots $\pm 2\epsilon_i$, $i = 1, \dots, n$, to D_n. The same argument as above shows that $\gamma_i = \alpha_i$, $i = 1, \dots, n-1$, and $\gamma_n = 2\epsilon_n$ is a base for C_n, so that we obtain the Dynkin diagram



C_n

In all these diagrams, the subscript n is the rank of the root system and hence the number of nodes in the Dynkin diagram.

30. *Show that the Weyl groups of B_n and C_n are isomorphic. Prove that this Weyl group is isomorphic to a semi-direct product of the permutation group S_n and $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n times). Describe the Weyl group of D_n as a subgroup of the Weyl group of B_n.*

The Weyl groups of a root system depend only on the directions of the roots, not on their lengths. Hence B_n and C_n have the same Weyl groups because they have the same roots up to scalar multiples. The reflection in the root $\epsilon_i - \epsilon_j$ operates on $\mathbb{R}^n$ by exchanging the i-th and j-th component of a vector. The reflection in $\pm\epsilon_i$ (or $\pm 2\epsilon_i$) operates by changing the sign of the i-th component of a vector. Hence the roots in A_{n-1} generate a group isomorphic to the symmetric group S_n. The roots of the form $\pm\epsilon_i$ generate a group isomorphic to $\mathbb{Z}_2^n$. Clearly, if we first exchange some coordinates, then change the sign of some coordinate and then exchange the coordinates backwards, we only have changed the sign of one coordinate. Hence the group $\mathbb{Z}_2^n$ is normal in the Weyl group. This implies that the Weyl group is indeed a semi-direct product of S_n and $\mathbb{Z}_2^n$. The action of S_n is given simply by exchanging the factors in $\mathbb{Z}_2^n$.

The reflection in $\epsilon_i + \epsilon_k$ changes the sign of the i-th and k-th coordinate and exchanges these two coordinates. Since we can reverse this exchange of coordinates by reflecting in the root $\epsilon_i - \epsilon_k \in D_n$, the Weyl group of D_n is generated by the symmetric group S_n, generated by the reflections in A_{n-1} as above, and those sign changes of the coordinates that involve an even number of signs changed. Thus D_n is the semi-direct product of S_n and $\mathbb{Z}_2^{n-1}$, and $\mathbb{Z}_2^{n-1} \subset \mathbb{Z}_2^n$ is the subgroup with an even number of non-zero entries.

31. *(The root system F_4. Let Q be a cube in 4-dimensional Euclidean space with the center at the origin. Let R be the following collection of vectors:*

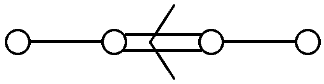
a. *vectors joining the origin with the centers of the 3-dimensional faces of Q.*
b. *vectors joining the origin with the centers of the 2-dimensional faces of Q.*
c. *halves of vectors joining the origin with the vertices of Q.*

Show that R is a root system and draw its Dynkin diagram.

Let $(1,1,1,1)$ be an edge of the cube, so that $w = \frac{1}{2}(1,1,1,1)$ is in R. A typical root of the form described in (1) is $(1,0,0,0)$, a typical root of the form described in (2) is $(1,1,0,0)$. A list of all roots of R looks like this: Roots of type (1) are all roots of the form $\pm(1,0,0,0)$ and those that can be obtained from this by exchanging the coordinates. This yields 8 roots. Roots of type (2) are all roots of the form $(\pm 1, \pm 1, 0, 0)$ and those that can be obtained from this by exchanging the coordinates. This yields 24 roots. There are 16 roots of type (3), given by $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ (every $\pm$ yields two possibilities). Hence the lengths of roots are either 1 or $\sqrt{2}$. Since exchanges of coordinates and sign changes are symmetries of Q, the reflections generated by roots of type (1) and (2) are symmetries of Q (compare the analysis of the Weyl group of D_n). Clearly, the root system R is invariant under symmetries of the cube Q. One of the roots of the form described in (3) is w and all others can be obtained from w by symmetries of the cube. Hence the corresponding reflections are conjugate with respect to the symmetries of Q. Thus if we show that the reflection in w maps R into itself, we obtain that R is closed under reflections in roots in R.

The reflection in w maps $x = (x_1, \dots, x_4)$ to $x - \frac{1}{2}(x_1 + \cdots + x_4)(1,1,1,1)$. We now check that this maps $(1,0,0,0)$, $(1,1,0,0)$, $(1,-1,0,0)$, $\frac{1}{2}(1,1,1,-1)$ and $\frac{1}{2}(1,1,-1,-1)$ into vectors in R. Since exchanging the coordinates and multiplying all coordinates by -1 commutes with reflection in w, this implies that R is closed under reflection in w.

It is trivial to check that $2(\alpha, \beta)/(\beta, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in R$, using that $(\beta, \beta) = 1$ unless β is of type (2). We claim that the roots $\phi_1 = \frac{1}{2}(1, -1, -1, -1)$, $\phi_2 = (0, 0, 0, 1)$, $\phi_3 = (0, 0, 1, -1)$, and $\phi_4 = (0, 1, -1, 0)$ are a base for R, where we call $\alpha \in R$ positive if $l(\alpha) > 0$ for $l(x) = 1000x_1 + 100x_2 + 10x_3 + x_4$. It is trivial to check that none of the roots ϕ_i can be written as a sum of two positive ones. Computing the scalar products of the roots ϕ_i yields that R has the Dynkin diagram



$R = F_4$

Let $R \subset E$ be a reduced root system with a fixed subset of positive roots R^+ and the simple roots $\alpha_1, \dots, \alpha_n$. Let W be the Weyl group, $\{s_1, \dots, s_n\}$ be the set of the simple reflections.

32. *Show that $(s_i s_j)^{m(i,j)} = 1$ for any two simple reflections s_i and s_j and an appropriate minimal integer $m(i,j)$. How to find $m(i,j)$ from the Dynkin diagram?*

The existence of $m(i,j)$ follows already because the Weyl group is finite. We want to determine the minimal possible $m(i,j)$, i.e. the order of $s_i s_j$ in W. If $i = j$, we have $s_i s_j = s_i^2 = 1$, i.e. $m(i,j) = 1$. Thus we may assume in the following that $i \neq j$. The reflection s_i fixes the hyperplane $\alpha_i^\perp$. Hence any product containing only the factors s_i and s_j fixes the 2-codimensional subspace $\alpha_i^\perp \cap \alpha_j^\perp = V^\perp$, where $V = \operatorname{lin}\{\alpha_i, \alpha_j\}$. Hence it suffices to consider the operation of such products on V: We have $(s_i s_j)^m = 1$ if and only if $(s_i|_V s_j|_V)^m = \mathrm{id}_V$. Since $R \cap V$ is the root system generated by α_i and α_j, we can reduce our assertion to root systems of rank 2, so that there are only finitely many cases to check.

If there is no edge joining the nodes α_i and α_j in the Dynkin diagram, then $s_i s_j|_V = -\mathrm{id}$, so that $m(i,j) = 2$. In case there is one edge, we are in the root system A_2 which has Weyl group S_3, and $(12)(23) = (123)$ has order 3, so that $m(i,j) = 3$. In case there are two edges, we are in the root system $B_2 \cong C_2$, and $s_i s_j$ is a rotation by the angle $-90°$, so that we obtain $m(i,j) = 4$. In case there are three edges, we have the root system G_2 and $s_i s_j$ is a rotation by the angle $-60°$, so that we obtain $m(i,j) = 6$.

33. *For each $w \in W$ define the* length $l(w)$ *as the number of simple reflections in a reduced decomposition of w.*

a. *Show that $l(w) = \#(w(R^+) \cap R^-)$;*
b. *Show that $l(wu) \leq l(w) + l(u)$ for any $w, u \in W$;*
c. *Show that there exists a unique element of maximal length in W and that its length is equal to the number of positive roots.*

a. Every simple reflection makes only one positive root negative. Hence a product of l simple reflections can make no more than l positive roots negative, i.e. $l(w) \geq \#(w(R^+) \cap R^-) = l'(w)$. Conversely, every $w \in W$ can be written as $w = s_i w_1$ for some $i \in \{1, \dots, n\}$, $w_1 \in W$ with $l'(w_1) = l'(w) - 1$, if we pick i such that $-\alpha_i \in w(R^+)$. Iterating this process, we finally arrive at $w = s_{i_1} \cdots s_{i_l} \cdot \tilde{w}$, where $l = l'(w)$ and $l'(\tilde{w}) = 0$, i.e. $\tilde{w}(R^+) = R^+$. This implies that $\tilde{w}(\Pi) = \Pi$, where $\Pi = \{\alpha_1, \dots, \alpha_n\}$ is a base for R^+ and hence $\tilde{w} = 1$ because the Weyl group operates simply transitively on the set of bases of R. Hence $l'(w) \geq l(w)$ as well, which proves (a).
b. Write w and u as products of $l(w)$ and $l(u)$ simple reflections, then wu can be written as a product of $l(w) + l(u)$ simple reflections, i.e. $l(wu) \leq l(w) + l(u)$.
c. Since the Weyl group operates simply transitively on the set of bases of R, there is a unique $w \in W$ that maps Π to $-\Pi$. Since this w is a linear map, it maps $\mathbb{Z}_+\Pi$ to $-\mathbb{Z}_+\Pi$ and hence R^+ to R^-. Furthermore, every $w \in W$ with the latter property must map Π to $-\Pi$ by the definition of a base, so that w is unique with this property. By (a), w has length $\#R^+$, and this length is the maximal possible length. Every element with maximal length must map R^+ to R^- and hence be equal to w.

34. *The subset*

$$\mathcal{W} = \{\epsilon \in E \mid (\epsilon, \alpha_i) \geq 0 \text{ for all simple roots } \alpha_1, \dots, \alpha_n\}$$

is called a Weyl chamber*.*

a. *Show that the W-orbit of any point $\epsilon \in E$ meets $\mathcal{W}$ in exactly one point;*
b. *Draw a Weyl chamber for each root system of rank 2;*
c. *Show that $\mathcal{W}$ is a convex cone in E with n faces.*

a. Fix $\epsilon \in E$, let $S = \{\alpha \in R \mid (\epsilon, \alpha) \geq 0\}$. If ϵ is in general position, i.e. $(\epsilon, \alpha) \neq 0$ for all $\alpha \in R$, then this determines a set of positive roots in R. There is a unique base $\Pi_S \subset S$ for S and hence a unique $w \in W$ with $w(S) = R$ because the Weyl group acts simply transitively on the set of bases.

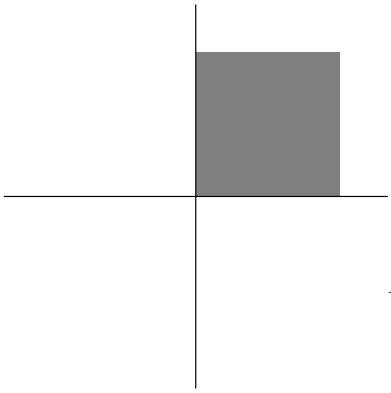
Hence $(w(\epsilon), R) = (w(\epsilon), w(S)) = (w, S) \subset \mathbb{R}_+$, i.e. $w(\epsilon) \in \mathcal{W}$. Moreover, if $w \in W$, $w(S) \neq R$, then $w(\epsilon) \notin \mathcal{W}$. Thus $w(\epsilon)$ is the only point in the W-orbit of ϵ that lies in $\mathcal{W}$.

If ϵ is not in general position, we can find a sequence (ϵ_n) of points in general position converging towards ϵ. There are unique $w_n \in W$ with $w_n(\epsilon_n) \in \mathcal{W}$. Since the Weyl group is finite, by going over to a subsequence we can achieve that $w(\epsilon_n) \in \mathcal{W}$ for all $n \in \mathbb{N}$, some fixed $w \in W$. Since $\mathcal{W}$ is closed, we obtain $w(\epsilon) \in \mathcal{W}$. Thus every W-orbit meets the Weyl chamber. We remark that $x \in \mathcal{W}$ implies that $(x, \alpha) \geq 0$ for all $\alpha \in R^+$ (and hence $(x, \alpha) \leq 0$ for all $\alpha \in R^-$) because every such α is a positive linear combination of the simple roots α_i.

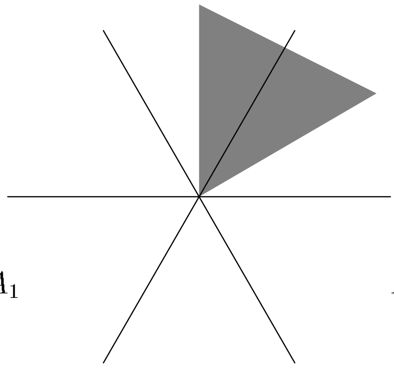
If the W-orbit of some $\epsilon \in \mathcal{W}$ meets $\mathcal{W}$ in two points, then we can find $w \in W$ of shortest length such that $w(\epsilon) \in \mathcal{W} \setminus \{\epsilon\}$. Since $w \neq 1$, there must be some $i \in \{1, \dots, n\}$ with $w(\alpha_i) \in R^-$.

Thus $0 \geq \big(w(\epsilon), w(\alpha_i)\big) = (\epsilon, \alpha_i) \geq 0$, i.e. $\epsilon \perp \alpha_i$, because $w\epsilon \in \mathcal{W}$. Hence $s_i(\epsilon) = \epsilon$, so that $ws_i\epsilon = w\epsilon$. Moreover, the length of ws_i is shorter than the length of w by exercise 34a, because $ws_i(R^+) = w(R^+) \cup \{-w\alpha_i\} \setminus \{w\alpha_i\}$ and because $w\alpha_i$ is negative. This contradicts the minimality of the length of w.

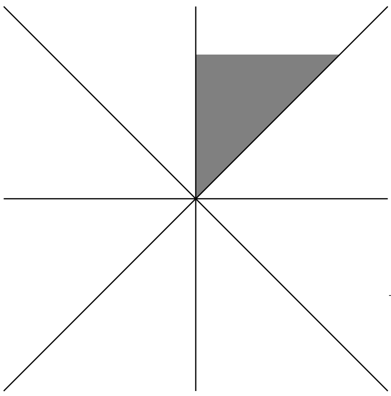
b. The following pictures do not show the true lengths of the roots since this is not important for the determination of the Weyl chamber, anyway.



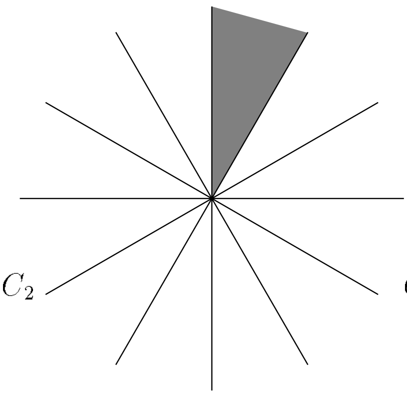
$A_1 + A_1$



A_2



$B_2 = C_2$



G_2

c. By definition, $\mathcal{W} = \bigcap_{i=1}^n H_{\alpha_i}$, where $H_\alpha = \{\epsilon \in E \mid (\epsilon, \alpha) \geq 0\}$ is a closed half-space. Hence $\mathcal{W}$ is a convex cone as an intersection of convex cones. The subsets $F_i = \partial H_{\alpha_i} \cap \mathcal{W}$ are faces of $\mathcal{W}$. They are all different because Π is a base for E. Moreover, if $\epsilon \in \partial\mathcal{W}$, then necessarily $(\epsilon, \alpha_i) = 0$ for some $i \in \{1, \dots, n\}$, i.e. $\epsilon \in F_i$. Hence there are no other $n-1$-dimensional faces besides the faces F_i, so that $\mathcal{W}$ is a convex cone with n faces.

35. *Let $\mathfrak{g}$ be a semi-simple Lie algebra, $X \in \mathfrak{g}$ be a semi-simple element. Show that the centralizer $c(X)$ of X is a reductive subalgebra of $\mathfrak{g}$. (Hint: choose a Cartan subalgebra containing X and consider the root decomposition.)*

Let $\mathfrak{h} \subset \mathfrak{g}$ be a maximal Abelian semi-simple Lie algebra containing X. By definition $\mathfrak{h}$ is a Cartan subalgebra. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

be the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $V \subset E = \mathfrak{h}^*$ be the orthogonal complement of X. Then $\mathfrak{g}_\alpha \subset c(X)$ if and only if $\alpha(X) = 0$. Clearly, $\mathfrak{h} \subset c(X)$, and since $c(X)$ is a Lie subalgebra of $\mathfrak{g}$, it is invariant under the action of $\mathfrak{h}$ and hence completely reducible as an $\mathfrak{h}$-module. Thus we obtain

$$c(X) = \mathfrak{h} \oplus \bigoplus_{\alpha \in R \cap V} \mathfrak{g}_\alpha.$$

Let $\mathfrak{i}$ be an Abelian ideal in the Lie algebra $c(X)$. We can view $\mathfrak{i}$ as an $\mathfrak{h}$-submodule, and since $c(X)$ is completely reducible, so is $\mathfrak{i}$, i.e.

$$\mathfrak{i} = (\mathfrak{i} \cap \mathfrak{h}) \oplus \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha$$

for some $S \subset R$. If $\mathfrak{g}_\alpha \subset \mathfrak{i}$, then the $\mathfrak{sl}_2(\mathbb{C})$-subalgebra L_α generated by $\mathfrak{g}_\alpha$ must be contained in $\mathfrak{i}$ because $\mathfrak{i} \cap L_\alpha$ is a non-zero ideal in the simple Lie algebra $L_\alpha \cong \mathfrak{sl}_2(\mathbb{C})$. Since $\mathfrak{sl}_2(\mathbb{C})$ is not Abelian, $S = \emptyset$, i.e. $\mathfrak{i} \subset \mathfrak{h}$. Since R spans $E = \mathfrak{h}^*$, $R \cap V$ spans the set V of all functionals on $\mathfrak{h}$ annihilating X. For every $Y \in \mathfrak{h} \setminus \mathbb{C}X$ there is $\alpha \in V$ with $\alpha(Y) \neq 0$. Hence there is even $\alpha \in R \cap V$ with this property. Therefore, $[Y, \mathfrak{g}_\alpha] \supset \mathfrak{g}_\alpha$. Since $\mathfrak{g}_\alpha \cap \mathfrak{i} = \{0\}$ and $\mathfrak{i}$ is an ideal, $Y \notin \mathfrak{i}$. Hence $\mathfrak{i} \subset \mathbb{C}X$. Since by construction X is in the center of $c(X)$, we obtain that $\mathfrak{i}$ is contained in the center of $c(X)$, which coincides with $\mathbb{C}X$. Therefore, $\mathrm{Rad}\big(c(X)\big) = Z\big(c(X)\big)$, i.e. $c(X)$ is reductive.

Let $\mathbb{J}$ be the space of Hermitian 3×3-matrices with coefficients in $\mathbb{O}$. Define the operation $\circ$ on $\mathbb{J}$ by the formula

$$A \circ B = (AB + BA)/2.$$

Let $\mathrm{Der}\,\mathbb{J}$ be the algebra of derivations of $\mathbb{J}$. We are going to prove that $\mathrm{Der}\,\mathbb{J}_\mathbb{C}$ is the simple Lie algebra F_4.

36. a. *Show that if $d \in \mathrm{Der}\,\mathbb{J}$ then $\mathrm{Tr}(dX) = 0$ for any $X \in \mathbb{J}$;*

b. *Show that if A is a skew-Hermitian 3×3-matrix and $\mathrm{Tr}\,A = 0$ then d_A, given by $d_A(X) = AX - XA$, is a derivation of $\mathbb{J}$.*

Let $e_0 = 1$, e_j, $j = 1, \dots, 7$ be the canonical basis of $\mathbb{O}$. Let $t_1 = \mathrm{diag}(1,0,0)$, $t_2 = \mathrm{diag}(0,1,0)$, $t_3 = \mathrm{diag}(0,0,1)$,

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is convenient to define a (non-associative) multiplication $\mathbb{O} \times \mathbb{J} \to \mathbb{J}$ as follows:

$$x \cdot (a_{ij}) = \begin{pmatrix} (x,1)a_{11} & \overline{xa_{21}} & xa_{13} \\ xa_{21} & (x,1)a_{22} & \overline{xa_{32}} \\ \overline{xa_{13}} & xa_{32} & (x,1)a_{33} \end{pmatrix}.$$

Let $e_{ji} = e_j E_i$. Thus the set of all t_i and e_{ji} $i = 1,2,3$, $j = 0, \dots, 7$ is a basis of $\mathbb{J}$ over $\mathbb{R}$.

Clearly, the multiplication in $\mathbb{J}$ is commutative and distributive, but not associative. We have, $t_i \circ e_{ji} = 0$ for all $i = 1,2,3$, $j = 0, \dots, 7$, $t_i \circ e_{jk} = \frac{1}{2} e_{jk}$ for all $j = 0, \dots, 7$, $k \neq i$. Moreover, $xE_i \circ yE_i = (x,y)(1 - t_i)$ and $x_i E_i \circ x_j E_j = \frac{1}{2} \overline{x_i} \cdot \overline{x_j} E_k$, if i, j, k is an even permutation.

a. We have to show that the trace is a multiplicative invariant. For a generic real-valued matrix, the trace can be viewed as the sum of its three distinct eigenvalues. The eigenvalues are the zeros of the uniquely determined minimal polynomial and hence invariant under automorphisms. In exercise 39 we will show that there is an open set $S \subset \mathbb{J}$ such that for every $X \in S$ there is a trace-preserving automorphism ϕ_X such that $\phi_X(X)$ is a generic diagonal matrix. Let ψ be any not necessarily trace-preserving automorphism of $\mathbb{J}$, $X \in U$ such that $\psi(X) \in U$. Then

$$\mathrm{Tr}\,\psi(X) = \mathrm{Tr}\big(\phi_{\psi(X)} \circ \psi \circ \phi_X^{-1}(\phi_X(X))\big).$$

The automorphism $\hat\psi = \phi_{\psi(X)} \circ \psi \circ \phi_X^{-1}$ maps the generic diagonal matrix $\phi_X(X)$ onto another generic diagonal matrix. Since the minimal polynomial is unique, $\hat\psi(\phi_X X)$ has the same eigenvalues as $\phi_X(X)$ and thus the same trace. Thus $\mathrm{Tr}\,\psi(X) = \mathrm{Tr}\,\phi_X(X) = \mathrm{Tr}\,X$ because ϕ_X is assumed to preserve the trace. Hence the two linear functionals $\mathrm{Tr} \circ \psi$ and Tr coincide on the open set $U \cap \psi^{-1}(U)$. Hence if the latter is not empty, then these two functionals coincide everywhere by linearity. Thus all automorphisms of $\mathbb{J}$ sufficiently close to the identity preserve the trace. Since a connected topological group is generated by any neighborhood of the identity, all automorphisms in the connected component of the identity preserve the trace. This implies the relation $\mathrm{Tr}\,dX = 0$ for all $d \in \mathrm{Der}\,\mathbb{J}$ because $\mathrm{Der}\,\mathbb{J}$ is the Lie algebra of $\mathrm{Aut}\,\mathbb{J}$.

For the time being, we redefine $\mathrm{Der}\,\mathbb{J}$ to be those derivations with $\mathrm{Tr}\,dX = 0$ and $\mathrm{Aut}\,\mathbb{J}$ to be those automorphisms leaving the trace invariant.

b. For matrices with entries in $\mathbb{O}$ we have

$$\overline{(XY)_{ij}}^t = \sum_k \overline{X_{jk}Y_{ki}} = \sum_k (\overline{Y}^t)_{ik}(\overline{X}^t)_{kj} = (\overline{Y}^t\overline{X}^t)_{ij}.$$

Hence if A is skew-Hermitian and X is Hermitian, then

$$\overline{AX - XA}^t = \overline{X}^t\overline{A}^t - \overline{A}^t\overline{X}^t = -XA + AX,$$

i.e. $d_A(X) \in \mathbb{J}$.

First assume that all entries of A are real numbers. This implies that $(AX)Y = A(XY)$, etc. for all $X, Y \in \mathbb{J}$, i.e. we can leave all brackets in prducts with three factors. Thus

$$\begin{aligned}2\big(d_A(X \circ Y) - d_A(X) \circ Y - X \circ d_A(Y)\big) &= A(XY + YX) - (XY + YX)A \\ &- (AX - XA)Y - Y(AX - XA) - X(AY - YA) - (AY - YA)X = 0,\end{aligned}$$

i.e. d_A is a derivation. (The trace condition does not appear because $d_I = 0$ anyway). Hence it remains to show the assertion for matrices with entries in $\mathbb{O}_{\rm im}$.

If B is real-valued and A is any skew-Hermitian matric with trace 0, then

$$\begin{aligned}d_{[A,B]}(X) &= (AB - BA)X - X(AB - BA) \\ &= A(BX - XB) - (BX - XB)A - B(AX - XA) + (AX - XA)B \\ &= d_A d_B(X) - d_B d_A(X),\end{aligned}$$

where $[A, B] = AB - BA$. Hence if d_A is a derivation, so is $d_{[A,B]}$. Let $a \in \mathbb{O}_{\rm im}$,

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

then

$$[A, B] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & -a & 0 \end{pmatrix}.$$

Conjugation by permutation matrices π is a symmetry of $\mathbb{O}^{3\times 3}$. Hence if we show that d_A is a derivation for the above A, it follows first that d_C is a derivation for all diagonal skew-Hermitian matrices of trace 0, because we can obtain them as a sum of A and $\pi^{-1}A\pi$ for suitable π. Moreover, we can obtain skew-Hermitian matrices with entries in $\mathbb{O}_{\rm im}$ and 0 on the diagonal as sums of $\pi^{-1}[A, B]\pi$ for suitable permutation matrices π. Hence it remains to show that d_A is a derivation.

We also use the permutation matrix transposing the first two coordinates, which transforms A into $-A$. This again reduces the set of equations to check. We have

$$d_A(X) = \begin{pmatrix} 0 & ax_{12} + x_{12}a & ax_{13} \\ \overline{a}x_{21} + x_{21}\overline{a} & 0 & \overline{a}x_{23} \\ x_{31}\overline{a} & x_{32}a & 0 \end{pmatrix}.$$

Clearly, $d_A(t_i \circ xE_i) = d_A(0) = 0 = t_i \circ d_A(xE_i) + d_A(t_i) \circ (xE_i)$, because $d_A(t_i) = 0$ and $d_A(xE_i)$ is an $\mathbb{O}$-multiple of E_i. Furthermore,

$$\begin{aligned} d_A(t_k \circ xE_1) &= \frac{1}{2}d_A(xE_1) = \frac{1}{2}xaE_1 = t_k \circ d_A(xE_1) = d_A(t_k) \circ xE_1 + t_k \circ d_A(xE_1), \ k = 2, 3, \\ d_A(t_1 \circ xE_3) &= \frac{1}{2}d_A(xE_3) = \frac{1}{2}(\overline{a}x + x\overline{a})E_3 = t_1 \circ d_A(xE_3) = d_A(t_1) \circ xE_3 + t_1 \circ d_A(xE_3), \\ d_A(xE_1 \circ yE_2) &= \frac{1}{2}(\overline{a} \cdot \overline{x} \cdot \overline{y} + \overline{x} \cdot \overline{y} \cdot \overline{a})E_3 = d_A(xE_1) \circ yE_2 + xE_1 \circ d_A(yE_2). \end{aligned}$$

The remaining equations are less obvious and require some computations with octonions. We have $d_A(\overline{x}E_1 \circ \overline{y}E_3) = \frac{1}{2}a(yx)E_2$ and $d_A(\overline{x}E_1) \circ \overline{y}E_3 + \overline{x}E_1 \circ d_A(\overline{y}E_3) = \frac{1}{2}\big(-y(ax) + (ay)x + (ya)x\big)E_2$, and this follows from the rule

$$a(bc) + b(ac) = (ab)c + (ba)c$$

for all $a \in \mathbb{O}_{\rm im}$, $b, c \in \mathbb{O}$. Furthermore, $d_A(xE_i \circ yE_i) = 0$ and

$$\begin{aligned} 2d_A(xE_1) \circ yE_1 + 2xE_1 \circ d_A(yE_1) &= \big((xa, y) + (x, ya)\big)(1 - t_1) = 0, \\ 2d_A(xE_3) \circ yE_3 + 2xE_3 \circ d_A(yE_3) &= \big((x\overline{a} + \overline{a}x, y) + (x, \overline{a}y + y\overline{a})\big) = 0. \end{aligned}$$

In the latter equations we used that right/left multiplication by octonions of norm 1 leaves the scalar product invariant, so that $(xa, y) + (x, ya) = 0$, etc. for $a \in \mathbb{O}_{\rm im}$. All other equations to be checked for the derivation property follow by using permutation matrices.

Moreover, $\operatorname{Tr} d_A(X) = 0$. Since $\operatorname{Tr} d_B(X) = 0$ as well for the matrix B above, we obtain $\operatorname{Tr} d_{[A,B]}(X) = 0$ for all $X \in \mathbb{J}$. Hence all derivations generated by skew-Hermitian matrices are trace-preserving.

37. *Let $D \subset \mathbb{J}$ be the subspace of the diagonal matrices, let*

$$\mathfrak{D} = \{d \in \operatorname{Der} \mathbb{J} \mid d(D) = 0\}.$$

Show that $\mathfrak{D}$ is a Lie algebra isomorphic to $\mathfrak{so}_8(\mathbb{R})$.

Define the bilinear form $(A, B) = \operatorname{Tr}(A \circ B)$. This form is invariant under trace-preserving automorphisms of $\mathbb{J}$. If $A = (a_{ij})$, $B = (b_{ij})$, then

$$2(A, B) = \sum_{i,k} (a_{ik} b_{ki} + b_{ik} a_{ki}) = \sum_{i,k} (a_{ik} \overline{b_{ik}} + b_{ik} \overline{a_{ik}}) = \sum_{i,k} 2(a_{ik}, b_{ik}),$$

where (a_{ik}, b_{ik}) is the invariant scalar product in $\mathbb{O}$. Hence (A, B) defines a positive definite $\operatorname{Der}\mathbb{J}$-invariant form on $\mathbb{J}$. The orthogonal complement of D clearly $\mathfrak{D}$-invariant.

Let $d \in \mathfrak{D}$. Since $d(t_i) = 0$, we have $t_i \circ d(X) = d(t_i \circ X)$ for all $X \in \mathbb{J}$. Hence $t_i \circ X$ implies $t_i \circ d(X) = 0$. Since $t_i \circ X = 0$, $X \perp D$ is equivalent to $X \in \mathbb{O} E_i$, the spaces E_i, $i = 1, 2, 3$ are d-invariant. Hence the map $\rho\colon d \mapsto d|_{\mathbb{O} \cdot E_1}$, say, is a homomorphism of Lie algebras $\operatorname{Der} \mathbb{J} \to \mathfrak{so}_8(\mathbb{R})$. We have to show that it is an isomorphism.

The skew-Hermitian diagonal matrices with trace 0 form a 14-dimensional subspace V of $\mathfrak{D}$. Furthermore, the automorphisms of $\mathbb{O}$ operate on $\mathfrak{D}$ componentwise and trivially preserve the trace. This yields that the simple Lie algebra $\mathfrak{g}_2$ with root system G_2 is contained in $\mathfrak{D}$ as well. Moreover, $(\operatorname{Ker} \rho + V) \cap \mathfrak{g}_2 = \{0\}$, because the operation of $\operatorname{diag}(a_1, a_2, a_3)$ on $\mathbb{O}E_1$ is given by $x \mapsto a_2 x - x a_3$. If this were a derivation, then $1 \mapsto 0$ implies $a_2 = a_3$. Moreover, $a_3 \in \mathbb{O}_{\text{im}}$, so that there is an automorphism of $\mathbb{O}$ mapping a_3 to a multiple of e_1. Hence without loss of generality, $x \mapsto e_1 x - x e_1$ would be a derivation of $\mathbb{O}$, but this is not the case. Hence the image of ρ has dimension 28, so that ρ is surjective.

In order to show that ρ is injective, pick $d \in \operatorname{Ker} \rho$. We already know $d(E_i) = \alpha_i E_i$ for some $\alpha_i \in \mathbb{O}$, $\alpha_1 = 0$. We have

$$d(1 - t_i) = d(E_i \circ E_i) = 2E_i \circ d(E_i) = 2(\alpha_i, 1)(1 - t_i),$$

i.e. $\alpha_i \in \mathbb{O}_{\text{im}}$. Moreover,

$$d(xE_3) = d(2\overline{x}E_1 \circ E_2) = 2d(\overline{x}E_1) \circ E_2 + 2\overline{x}E_1 \circ d(E_2) = x\overline{\alpha_2}E_3$$

and similarly $d(xE_2) = \overline{\alpha_3} x E_2$. Thus

$$\begin{aligned} -(xy)\alpha_2 E_3 (xy)\overline{\alpha_2} E_3 &= d(xyE_3) = d(2\overline{x}E_1 \circ \overline{y}E_2) \\ &= 2d(\overline{x}E_1) \circ \overline{y}E_2 + 2\overline{x}E_1 \circ d(\overline{y}E_2) = 2\overline{x}E_1 \circ \overline{\alpha_3} \cdot \overline{y}E_2 = x(y\alpha_3)E_2 \end{aligned}$$

for all $x, y \in \mathbb{O}$. For $x = 1$, this implies $\alpha_3 = -\alpha_2$. Thus $(xy)\alpha_3 = x(y\alpha_3)$ for all $x, y \in \mathbb{O}$. However, $\mathbb{O}$ is not associative, and its automorphism group is transitive on spheres in $\mathbb{O}_{\text{im}}$. Hence if $\alpha_3 \neq 0$, there must be $x, y \in \mathbb{O}_{\text{im}}$ with $(xy)\alpha_3 \neq x(y\alpha_3)$. This yields $\alpha_3 = 0$ and thus $d = 0$, so that ρ is injective.

38. *Let X be a generic element in D, M be the orbit of X under the action of* $\operatorname{Aut}\mathbb{J}$. *Show that* $\dim M = 24$. *(Hint: show that the Lie algebra of the stabilizer of X is $\mathfrak{D}$ and* $\dim \operatorname{Der}\mathbb{J}(X) = 24$.*) Calculate the dimension of* $\operatorname{Aut}\mathbb{J}$.

Let $X = \operatorname{diag}(x_1, x_2, x_3) \in D$ have three different diagonal entries. Then D is the linear span of X^k, $k = 0, \dots, 2$. The multiplication in D is just the usual multiplication of matrices. If $d(X) = 0$, then $d(X^2) = 2Xd(X) = 0$, and $d(1) = 0$ anyway, so that $d(D) = 0$, i.e. $d \in \mathfrak{D}$. By exercise 38, the Lie algebra of the stabilizer of X is isomorphic to $\mathfrak{so}_8(\mathbb{R})$ and hence has dimension 28. If we show that $\dim M = 24$, it follows that $\dim \operatorname{Aut}\mathbb{J} = 52$.

If $A = (a_{ij})$ is skew-Hermitian with zeros on the diagonal, then a straightforward calculation shows

$$d_A(X) = \begin{pmatrix} 0 & (x_1 - x_2)a_{12} & (x_1 - x_3)a_{13} \\ (x_1 - x_2)\overline{a_{12}} & 0 & (x_2 - x_3)a_{23} \\ (x_1 - x_3)\overline{a_{13}} & (x_2 - x_3)\overline{a_{23}} & 0 \end{pmatrix}.$$

This is not zero unless $A = 0$. Since $d_A(X)$ is a derivation of $\mathbb{J}$, we obtain that the dimension of M is at least the dimension 24 of the space of skew-Hermitian matrices with zeros on the diagonal. Moreover, we obtain that the tangent spaces of D and M at X span a 27-dimensional space, i.e. all of $\mathbb{J}$, so that M and D intersect transversally in X. Hence the dimension of M is at least 24.

If $\phi \in \operatorname{Aut}\mathbb{J}$ maps X to $\phi(X) \in D$, then X and $\phi(X)$ must have the same minimal polynomial and hence the same eigenvalues. Since all eigenvalues of X are different and $\phi(X)$ is diagonal, there are only finitely many possibilities. Thus $M \cap D$ is discrete. Since the intersection is transverse, we must have $\dim \mathbb{J} = \dim M + \dim D$, i.e. $\dim M = 24$.

Consider the mapping $h\colon (GD) \times \operatorname{Aut}\mathbb{J} \to \mathbb{J}$, $(X, \phi) \mapsto \phi(X)$, where $GD \subset D$ is the set of generic diagonal matrices. We have just seen that every point in $GD \times \{\mathrm{id}_{\mathbb{J}}\}$ is a regular point of h, i.e. the derivative of h at these points is surjective. Consequently, the image of h contains a neighborhood of each such point. Especially, the image of h contains some open set, which is what we needed for exercise 37a (recall that all automorphisms occuring preserve the trace). It would be nicer to have the more global result that every element of $\mathbb{J}$ is the $\operatorname{Aut}\mathbb{J}$-image of a diagonal matrix, but this would require some work that is not necessary for our purpose.

39. *Show that the subspace $\mathbb{J}_0 \subset \mathbb{J}$ of the traceless matrices is an irreducible 26-dimensional module over* $\operatorname{Der}\mathbb{J}$.

Since $\operatorname{Aut}\mathbb{J}$ operates orthogonally on $\mathbb{J}$, $\mathbb{J}_0 \subset \mathbb{J}$ is completely reducible. Take a generic $X \in D$ and let $M = \exp(\operatorname{Der}\mathbb{J})X$. Then $\exp(\operatorname{Der}\mathbb{J})$ operates transitively on M, so that the linear span of M is an irreducible $\operatorname{Der}\mathbb{J}$-module. Since $\dim M = 24$ and M is contained in the sphere of radius $(X, X)^{1/2}$, the linear span of M has at least dimension 25. Thus if $\mathbb{J}_0$ is not irreducible, it must be of the form $\operatorname{lin}\{M\} \oplus \operatorname{lin}\{Y\}$ for some $Y \in \mathbb{J}_0$. Thus $d(Y) = 0$ for all derivations of $\mathbb{J}$, so that Y commutes with all skew-Hermitian matrices with trace 0. Especially, Y commutes with all real-valued skew-Hermitian matrices. This implies that all the off-diagonal entries of Y are zero and that all the diagonal entries are the same as in the case of real or complex matrices. Hence $Y = cI$ for some $c \in \mathbb{R}$ and since $3c = \operatorname{Tr}(Y) = 0$, we have $Y = 0$, contradiction. Hence $\mathbb{J}_0$ is irreducible.

40. *Show that* $\operatorname{Der}\mathbb{J}_{\mathbb{C}}$ *is a semi-simple Lie algebra of rank 4 and dimension 52. Deduce from this that* $\operatorname{Der}\mathbb{J}_{\mathbb{C}}$ *has the root system F_4.*

We claim that $\mathbb{J}_0 \otimes \mathbb{C}$ is an irreducible $\operatorname{Der}\mathbb{J}_{\mathbb{C}}$-module. The only possibility for it to decompose would be into two 13-dimensional (complex) submodules. But the $\operatorname{Der}\mathbb{J}_{\mathbb{C}}$-image of a generic $X \in D$ is a complex submanifold of $\mathbb{J} \otimes \mathbb{C}$ of (complex) dimension 24 because it contains M as a real submanifold. Since $\exp(\operatorname{Der}\mathbb{J}_{\mathbb{C}})$ operates transitively on this submanifold, $\mathbb{J}_0$ cannot decompose.

Assume that $\mathfrak{i} \subset \operatorname{Der}\mathbb{J}_{\mathbb{C}}$ is a solvable ideal. Then $\mathfrak{i}$ operates by scalars on the irreducible $\operatorname{Der}\mathbb{J}_{\mathbb{C}}$-module $\mathbb{J}_0$. But the map $X \mapsto c(X - \operatorname{Tr} X)$ is not a derivation on $\mathbb{J}$ for any $c \in \mathbb{C}$ and is the only candidate for a derivation whose restriction to $\mathbb{J}_0$ is scalar. Thus $\operatorname{Der}\mathbb{J}_{\mathbb{C}}$ is semi-simple. We already computed its dimension to be 52. Since $\mathfrak{so}_8(\mathbb{R}) \subset \operatorname{Der}\mathbb{J}$ by exercise 38, we have $\mathfrak{so}_8(\mathbb{C}) \subset \operatorname{Der}\mathbb{J}_{\mathbb{C}}$. Since we can extend a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{so}_8(\mathbb{C})$ to a Cartan subalgebra of $\operatorname{Der}\mathbb{J}_{\mathbb{C}}$, the rank of $\operatorname{Der}\mathbb{J}_{\mathbb{C}}$ is at least 4.

If the rank were greater, there would be some H in a Cartan subalgebra of $\operatorname{Der}\mathbb{J}_{\mathbb{C}}$ orthogonal to the roots of $\mathfrak{D}$, which span a four-dimensional space. Thus $[H, \mathfrak{D}] = 0$. This means that $\mathfrak{D}$ vanishes on the space HD. Since $\mathfrak{D}$ has no common zeros outside D by the proof of exercise 38, this means $HD \subset D$. Since $\operatorname{Der}\mathbb{J}$ is spanned by $\mathfrak{D}$ and the skew-symmetric matrices of trace 0 by the proof of exercise 39, we have $H = d_A + d$, $d \in \mathfrak{D}$, A skew-Hermitian with zeros on the diagonal. By definition, $d(D) = 0$, and d_A does not map D into itself by the proof of exerise 3 (unless $A = 0$). This contradiction shows that the rank must be 4.

The root system of $\operatorname{Der}\mathbb{J}_{\mathbb{C}}$ cannot be A_4, B_4, C_4, or D_4 because the corresponding Lie algebras have dimensions 24, 28, 28, or 36, respectively. Since $\mathfrak{so}_8(\mathbb{C}) \subset \operatorname{Der}\mathbb{J}_{\mathbb{C}}$ already has a connected Dynkin diagram, $\operatorname{Der}\mathbb{J}_{\mathbb{C}}$ must be simple, since it has rank 4. Hence the only remaining case for the root system of $\operatorname{Der}\mathbb{J}_{\mathbb{C}}$ is F_4.

41. *Let $\mathfrak{g}$ be a Lie algebra, $U(\mathfrak{g})$ its universal enveloping algebra.*

a. *Put* $\operatorname{ad} X(Z) = XZ - ZX$ *for* $X \in \mathfrak{g}$, $Z \in U(\mathfrak{g})$. *Show that* ad *defines a representation of* $\mathfrak{g}$ *on* $U(\mathfrak{g})$ *(it is called the adjoint representation);*
b. *Show that* $U(\mathfrak{g}) \cong \sum_{n=0}^{\infty} S^n(\mathfrak{g})$ *as the adjoint* $\mathfrak{g}$*-module.*

a. We have to check that $\operatorname{ad}[X, Y](Z) = [\operatorname{ad} X, \operatorname{ad} Y](Z)$ for all $X, Y \in \mathfrak{g}$, $Z \in U(\mathfrak{g})$. This is a straightforward calculation:

$$\begin{aligned}
\operatorname{ad}[X,Y](Z) - [\operatorname{ad} X, \operatorname{ad} Y](Z) &= [X,Y]Z - Z[X,Y] - \operatorname{ad} X(YZ - ZY) + \operatorname{ad} Y(XZ - ZX) \\
&= (XY - YX)Z - Z(XY - YX) - X(YZ - ZY) + (YZ - ZY)X \\
&\quad + Y(XZ - ZX) - (XZ - ZX)Y = 0.
\end{aligned}$$

b. Let $U_0 = \mathbb{C} \subset U_1 \subset U_2 \subset \cdots$ be the canonical filtration of $U(\mathfrak{g}) = T(\mathfrak{g})/\!\sim$. By the Poincaré-Birkhoff-Witt theorem, $U_n/U_{n-1} \cong \operatorname{Sym}^n(\mathfrak{g})$ and hence $U(\mathfrak{g}) \cong \bigoplus_{n=1}^{\infty} \operatorname{Sym}^n(\mathfrak{g})$ as a vector space.

Let $Z = Z_1 \cdots Z_k \in U_n$, $k \leq n$, $X \in \mathfrak{g}$. Then

$$\begin{aligned} \operatorname{ad} X(Z) &= XZ_1 \cdots Z_k - Z_1 \cdots Z_k X = [X, Z_1]Z_2 \cdots Z_k + Z_1 X Z_2 \cdots Z_k - Z_1 \cdots Z_k X \\ &= [X, Z_1]Z_2 \cdots Z_k + Z_1[X, Z_2]Z_3 \cdots Z_k + \cdots + Z_1 \cdots Z_{k-1}[X, Z_k]. \end{aligned}$$

This implies $\operatorname{ad} X(U_n) \subset U_n$ and that $\operatorname{ad} X$ operates on U_n as on $\bigoplus_{k=1}^n \operatorname{Sym}^k(\mathfrak{g})$ where the action on $\operatorname{Sym}^k \mathfrak{g})$ is generated by the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$.

42. *Let $\mathfrak{g}$ be a semi-simple Lie algebra, $Z(\mathfrak{g})$ be the center of its universal enveloping algebra $U(\mathfrak{g})$.*

a. *Show that $Z(\mathfrak{g})$ is isomorphic to the ring of polynomials on $\mathfrak{g}$ invariant under the adjoint action of the group G (Hint: Use problem 1);*
b. *Show that the Casimir operator $C = \sum_{i=1}^n E_i F_i$ (where $\{E_i\}$ and $\{F_i\}$ are dual bases with respect to the Killing form) belongs to $Z(\mathfrak{g})$;*
c. *Show that if $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, then $Z(\mathfrak{g})$ is generated by C and is isomorphic to the ring of polynomials of one variable. (Hint: Use the decomposition of $S^n\big(V(2)\big)$).*

a. If $Z \in Z(\mathfrak{g})$, then $\operatorname{ad} X(Z) = XZ - ZX = 0$ for all $X \in \mathfrak{g} \subset U(\mathfrak{g})$. Conversely, since the algebra $U(\mathfrak{g})$ is generated by $\mathfrak{g}$, if $XZ = ZX$ for all $X \in \mathfrak{g}$, then $Z \in Z(\mathfrak{g})$. Hence the center of $U(\mathfrak{g})$ is the set of common zeros of the adjoint action of $\mathfrak{g}$. The action of $\mathfrak{g}$ can be integrated to an action of the associated connected simply connected Lie group G because $U(\mathfrak{g})$ is a direct sum of finite-dimensional $\mathfrak{g}$-modules. The set of common zeros of $\operatorname{ad}\mathfrak{g}$ is precisely the set of fixed points of the group action. By exercise 41b, $U(\mathfrak{g}) \cong \bigoplus_{k=1} \operatorname{Sym}^k(\mathfrak{g})$. Viewing $\operatorname{Sym}^k(\mathfrak{g})$ as the space of homogeneous polynomials of degree k on $\mathfrak{g}$, we obtain that $U(\mathfrak{g})$ is isomorphic to the ring of polynomials on $\mathfrak{g}$ as a $\mathfrak{g}$-module. This proves (a).
b. The Killing form B on $\mathfrak{g}$ is non-degenerate and $\mathfrak{g}$-invariant, i.e. $B([X,Y],Z) + B(Y,[X,Z]) = 0$ for all $X, Y, Z \in \mathfrak{g}$. This implies, for all $X \in \mathfrak{g}$, $[X, E_i] = -\sum_{j=1}^n B(E_i, [X, F_j])E_j$, $[X, F_j] = -\sum_{i=1}^n B([X, E_i], F_j)F_i$, and hence

$$\begin{aligned} XC - CX &= \sum_{i=1}^n XE_iF_i - E_iF_iX = \sum_{i=1}^n [X, E_i]F_i + E_i[X, F_i] = \sum_{i=1}^n [X, E_i]F_i + \sum_{j=1}^n E_j[X, F_j] \\ &= -\sum_{i,j=1}^n B(E_i, [X, F_j])E_jF_i + E_jB([X, E_i], F_j)F_i = 0. \end{aligned}$$

c. The ring of polynomials on $\mathfrak{g}$ generated by C is contained in $Z(\mathfrak{g})$. It is isomorphic to the ring of polynomials in one variable. We have to show that it is all of $\mathfrak{g}$.

The adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$ is $V(2)$. We have proved in homework 5, exercise 42 that

$$\operatorname{Sym}^n\big(V(2)\big) \cong V(2n) \oplus V(2n-4) \oplus V(2n-8) \oplus \cdots \oplus V(2n - 4[n/2]).$$

Hence $\operatorname{Sym}^n\big(V(2)\big)$ contains a submodule isomorphic to $V(0)$ if and only if n is even, and then only of multiplicity 1. Since $C^{n/2}$ generates such a submodule for even n, the only invariant polynomials are the powers of C. Thus $Z(\mathfrak{g})$ is generated by C.

43. *Let $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$, V be the representation of $\mathfrak{g}$ in the $2n$-dimensional space of column vectors. Decompose $\Lambda^k(V)$ into a sum of irreducible modules for all $k \leq n$.*

First, we fix some terminology. We choose the basis $(e_1, \dots, e_{2n})$ of V such that $Q(e_i, e_j) = 0$ for $i \neq j + n$ and $j \neq i + n$ and $Q(e_i, e_{i+n}) = 1$. Let E_{ij} be the $2n \times 2n$-matrix with (i,j)-th entry 1 and 0 otherwise. Then the elements $H_i = E_{ii} - E_{i+n,i+n}$, $i = 1, \dots, n$, form a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let (L_i) be the corresponding dual basis of $\mathfrak{h}^*$. Let $X_{ij} = E_{ij} - E_{j+n,i+n}$, $Y_{ij} = E_{i,n+j} + E_{j,n+i}$, $Z_{ij} = E_{n+i,j} + E_{n+j,i}$ for $i, j = 1, \dots, n$, $i \neq j$; $U_i = E_{i,n+i}$, $V_i = E_{n+i,i}$ for $i = 1, \dots, n$. This is a weight basis of $\mathfrak{g}$, and the positive roots correspond to the X_{ij}, Y_{ij} with $i < j$ ($Y_{ij} = Y_{ji}$), and U_i. The vectors e_i are a weight basis of V: $He_i = L_i(H)e_i$ for $i = 1, \dots, n$, $He_i = -L_i(H)e_i$ for $i = n+1, \dots, 2n$.

Let Γ_λ be the irreducible representation with highest weight λ for $\lambda \in P_+$, where P_+ is the set of positive weights in the weight lattice. We claim that

$$\Lambda^k(V) \cong \Gamma_{L_1 + \cdots + L_k} \oplus \Gamma_{L_1 + \cdots + L_{k-2}} \oplus \cdots.$$

In order to show this, we have to find all highest vectors in $\Lambda^k(V)$.

The vectors $e_I = e_{i_1} \wedge \cdots e_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq 2n$, form a weight basis of $\Lambda^k(V)$ with $He_I = (\sum L_{i_l})e_I$, where we put $L_{n+i} = -L_i$ for $i = 1, \dots, n$. The positive weights are of the form $a_1L_1 + \cdots a_nL_n$ with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. The only positive weights occurring for the vectors e_I are $\lambda_s = L_1 + \cdots L_s$ for some $s \in 1, \dots, k$ with $2 \mid k - s$. Fix such an s, let $k - s = 2r$. We have to show that up to scalar multiples there is precisely one highest vector with weight λ_s. Then we will be finished.

A generic element of $\Lambda^k(V)$ in the weight space of λ_s is of the form

$$w = e_1 \wedge \cdots e_s \wedge \sum_{s<I,\#I=r} a_I e_{i_1} \wedge e_{i_1+n} \wedge \cdots \wedge e_{i_r} \wedge e_{i_r+n}.$$

In this sum, we assume the indices $i_1, \ldots, i_r$ to be in ascending order. The following relations are immediate: $U_i w = 0$ for $i = 1, \ldots, n$; $X_{ij} w = Y_{ij} w = 0$ for $i < j$, $i \leq s$. It remains to check when $X_{ij} w = 0$ and $Y_{ij} w = 0$ for $s < i < j \leq n$. For fixed $s < i < j \leq n$, the first relation becomes $a_{K \cup \{i\}} = a_{K \cup \{j\}}$ for all $K \subset \{s+1, \ldots, n\}$ with $\#K = r - 1$, $i, j \notin K$. This means $a_I = a_J$ for all $s < I, J$, $\#I = \#J = r$. Thus up to scalar multiples there can be at most one highest vector with weight λ_s, namely

$$w_s = e_1 \wedge \cdots e_s \wedge \sum_{s<I,\#I=r} e_{i_1} \wedge e_{i_1+n} \wedge \cdots \wedge e_{i_r} \wedge e_{i_r+n}.$$

It is straightforward to check that $Y_{ij} w_w = 0$. Hence w_s is a highest vector with weight λ_s, and there are no further highest vectors.

44. *Let $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$.*

a. *Show that if n is odd then the spinor representation of $\mathfrak{g}$ is self-dual, i.e. it is isomorphic to its dual.*
b. *Show that if n is even then for $n = 4k$ both spinor representations are self-dual and for $n = 4k + 2$ two spinor representations are dual to each other.*

In case (a), the Weyl group, operating on the weight space, contains the operator $-\mathrm{id}$ sending each weight to its negative. Since the character χ of any representation ρ is invariant under the Weyl group, it is symmetric with respect to $-\mathrm{id}$. However, the character χ^* of the dual representation ρ^* is $\chi \circ (-\mathrm{id})$, i.e. the weight space ρ^*_μ for the weight μ has the same dimension as $\rho_{-\mu}$. Therefore, every representation of $\mathfrak{g}$ is self-dual. Especially, the spinor representation is self-dual.

In case (b), $n = 2r$, the Weyl group also contains $-\mathrm{id}$ if r is even, because the Weyl group contains all matrices $\mathrm{diag}(\pm 1, \ldots, \pm 1)$ with an even number of sign changes, and the same reasoning as above yields that both spinor representations are self-dual. If r is odd, the Weyl group contains $w = \mathrm{diag}(-1, -1, \ldots, -1, 1)$. This maps the highest weight $\omega_{r-1} = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{r-1} - \epsilon_r)$ of one spinor representation to the negative of the highest weight $\omega_r = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_r)$ of the other spinor representation. Thus $V(\omega_{r-1})^*$ contains $V(\omega_r)$. Since $V(\omega_{r-1})^*$ is irreducible, we even have $V(\omega_{r-1})^* = V(\omega_r)$, i.e. the two spin representations are dual to each other.

45. *In the situation of problem 2 and a self-dual spinor representation determine when the invariant bilinear form on it is symmetric and when it is skew-symmetric (depending on n).*

If $n = 2r = 4k$, let $S = S^+ \oplus S^-$ be the direct sum of the two spinor representations. We have $S \otimes S = S^+ \otimes S^+ \oplus S^- \otimes S^- \oplus S^+ \otimes S^- \oplus S^+ \otimes S^-$. Since $S^\pm$ are two different irreducible, self-dual representations of $\mathfrak{so}_n(\mathbb{C})$, the trivial representation occurs once in $S^+ \otimes S^+$ and in $S^- \otimes S^-$ and not in $S^+ \otimes S^- \cong S^+ \otimes (S^-)^*$, because there is no intertwining operator $S^- \to S^+$. Hence if $\beta^\pm$ are non-zero invariant bilinear forms on $S^\pm$ respectively, then the invariant bilinear forms on S are of the form $c_+\beta^+ + c_-\beta^-$. Such a form is non-degenerate if and only if $c_+, c_- \neq 0$. The $\mathfrak{so}_n(\mathbb{C})$-module S can be given a $\mathfrak{so}_{n+1}(\mathbb{C})$-module structure, in fact $S^\pm$ are defined as the two irreducible components of the restriction of the spinor representation S of $\mathfrak{so}_{n+1}(\mathbb{C})$ to $\mathfrak{so}_n(\mathbb{C})$. By exercise 46, there is a non-degenerate $\mathfrak{so}_{n+1}(\mathbb{C})$-invariant bilinear form β on S. This form is a fortiori $\mathfrak{so}_n(\mathbb{C})$-invariant and hence of the form $c_+\beta^+ + c_-\beta^-$. The upshot of this is that if β is symmetric, then so are $\beta^\pm$, and if β is skew-symmetric, then so are $\beta^\pm$. Hence we can reduce our analysis to the case of odd n, $n = 2r + 1$. The result will be that the canonical bilinear form on S is symmetric for $r \bmod 4 \in \{0, 3\}$ and skew-symmetric for $r \bmod 4 \in \{1, 2\}$. Consequently, the $\mathfrak{so}_{2r}(\mathbb{C})$-invariant forms on $S^\pm$ are both symmetric if $r \bmod 4 = 0$ and skew-symmetric if $r \bmod 4 = 2$. (If r is odd, the subspaces S^+ and S^- of S are isotropic, so that there is no non-zero invariant form on either of them, but they are duals of each other because of the form β on their direct sum.)

Let Q be the invariant symmetric form on $\mathbb{C}^{2r+1}$, for which we form $\mathfrak{so}_{2r+1}(\mathbb{C})$. Let $\mathbb{C}^{2r+1} = W \oplus W' \oplus U$, where W and W' are maximal isotropic subspaces, $\dim U = 1$, and $U \perp W \oplus W'$. Let $e_0 \in U$ be a vector with $Q(e_0, e_0) = 1$, and let $e_1, \ldots, e_r$ and $e_{r+1}, \ldots, e_{2r}$ be bases of W and W' respectively. The spinor representation of $\mathfrak{so}_n(\mathbb{C})$ is obtained by embedding $\mathfrak{so}_n(\mathbb{C})$ in the Clifford algebra $C(n)$, which can be identified with $\mathrm{End}(S)$ for $S = \Lambda^\cdot(W)$. Thus $\Lambda^\cdot(W)$ has a canonical left-$C(n)$-module structure.

The first step is to show that $\Lambda^\cdot(W)$ is isomorphic to some ideal in $C(n)$. Define $\rho \colon \Lambda^\cdot(W) \to C(n)$ by

$$\rho(e_{i_1} \wedge \cdots e_{i_k}) = (1 + (-1)^k e_0) e_{i_1} \cdots e_{i_k} \cdot e_{r+1} \cdot e_{r+2} \cdots e_{2r}.$$

We claim that this is a $C(n)$-module isomorphism. Since $e_ie_j + e_je_i = 2Q(e_i, e_j)$ in $C(2n)$ [following the convention in Fulton-Harris], this map is well-defined. Moreover, using the relations $e_ie_j = -e_je_i$, $e_ie_{n+j} = 2 - e_{n+j}e_i$ for $i, j = 1, \dots, n$, and $e_0e_i = -e_ie_0$ for all $i = 1, \dots, 2r$, $e_0^2 = 1$, we obtain

$$\begin{aligned} e_0 \cdot \rho(e_{i_1} \wedge \cdots \wedge e_{i_k}) &= (e_0 + (-1)^k)e_{i_1} \cdots e_{i_k} e_{r+1} \cdots e_{2r} = (-1)^k \rho(e_{i_1} \wedge \cdots \wedge e_{i_k}) \\ &= \rho(e_0 \bullet e_{i_1} \wedge \cdots \wedge e_{i_k}), \\ e_j \cdot \rho(e_{i_1} \wedge \cdots \wedge e_{i_k}) &= (1 + (-1)^{k+1}e_0)e_j \cdot e_{i_1} \cdots e_{i_k} \cdot e_{r+1} \cdots e_{2r} = \rho(e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}) \\ &= \rho(e_j \bullet e_{i_1} \wedge \cdots \wedge e_{i_k}) \end{aligned}$$

for $i_1, \dots, i_k, j \in \{1, \dots, r\}$. If $j \notin \{i_1, \dots, i_k\}$, we have $e_{n+j}e_{i_1} \cdots e_{i_k} = (-1)^k e_{i_1} \cdots e_{i_k} e_{n+j}$ and hence $e_{n+j}\rho(e_{i_1} \wedge \cdots \wedge e_{i_k}) = 0$ because $e_{n+j}^2 = 0$. However, if $j \in \{i_1, \dots, i_k\}$, say $j = i_1$, then $e_{n+j}e_{i_1} \cdots e_{i_k} = (2 - e_{i_1}e_{n+j})e_{i_2} \cdots e_{i_k}$, and the same reasoning shows that $e_{n+j}\rho(e_{i_1} \wedge \cdots \wedge e_{i_k}) = 2\rho(e_{i_2} \wedge \cdots \wedge e_{i_k}) = \rho(e_{n+j} \bullet e_{i_1} \wedge \cdots \wedge e_{i_k})$. Thus $\rho(\Lambda^{\cdot}(W))$ is an ideal in $C(n)$ and ρ is an embedding of left-$C(n)$-modules.

There are two canonical anti-isomorphisms of $C(2n)$, given by

$$\tau_+(v_1 \cdots v_k) = v_k \cdots v_1, \qquad \tau_-(v_1 \cdots v_k) = (-1)^k v_k \cdots v_1.$$

Our second step is to show that for even r, $(\tau_+ \circ \rho)(s) \cdot \rho(t) = 2\beta(s,t)\rho(1)$ and for odd r, $\tau_- \circ \rho(s) \cdot \rho(t) = 2\beta(s,t)\rho(1)$ for some bilinear form β. The third step will be to show that this form is invariant.

We may assume $s = e_{i_1} \wedge \cdots \wedge e_{i_k}$, $t = e_{j_1} \wedge \cdots \wedge e_{j_l}$. We have to show $\tau_\pm \circ \rho(s) \cdot \rho(t) = c\rho(1)$ for some $c \in \mathbb{C}$, where we take the sign $(-1)^r$. Let $I = \{i_m\}$, $J = \{j_n\}$. There are several trivial cases. If $I \cap J \neq \emptyset$ or $I \cup J \neq \{1, \dots, n\}$, then $\tau_\pm \circ \rho(s) \cdot \rho(t) = 0$. Hence we may assume $k + l = r$ in the following. Let σ be the permutation in S_r given by $\sigma(1, \dots, n) = (i_k, \dots, i_1, j_1, \dots, j_l)$.

Now assume first that r is even. Then

$$\tau_+\rho(s)\rho(t) = e_{2r} \cdots e_{r+1} e_{i_k} \cdots e_{i_1}(1 + (-1)^k e_0)(1 + (-1)^l e_0)e_{j_1} \cdots e_{j_l} \cdot e_{r+1} \cdots e_{2r}.$$

If $k \neq l \pmod 2$, then this is 0 because $(1 + e_0)(1 - e_0) = 0$. If $k = l \pmod 2$, then $(1 + (-1)^k e_0)(1 + (-1)^l e_0) = 2(1 + (-1)^k e_0)$. Hence we can rewrite the above expression as

$$\begin{aligned} &2(1 + (-1)^r e_0)e_{2r} \cdots e_{r+1} e_{i_k} \cdots e_{i_1} e_{j_1} \cdots e_{j_l} \cdot e_{r+1} \cdots e_{2r}. \\ &= 2(-1)^\sigma(1 + e_0)e_{2r} \cdots e_{r+1} e_1 \cdots e_r e_{r+1} \cdots e_{2r} = 2(-1)^\sigma(1 + e_0)e_{r+1} \cdots e_{2r} = 2(-1)^\sigma \rho(1), \end{aligned}$$

which is what we want. In this computation we used that r is even.

Now assume that r is odd. Then

$$\tau_-\rho(s)\rho(t) = e_{2r} \cdots e_{r+1} e_{i_k} \cdots e_{i_1}\big((-1)^{r+k} + (-1)^{r+k+1}e_0\big)(1 + (-1)^l e_0)e_{j_1} \cdots e_{j_l} \cdot e_{r+1} \cdots e_{2r}.$$

As above, this is 0 if $l = k \pmod 2$. If $l \neq k \pmod 2$, this expression becomes $2(-1)^{1+k}(-1)^\sigma(1 + e_0)e_{r+1} \cdots e_{2r} = 2(-1)^{1+k}(-1)^\sigma\rho(1)$. This completes the second step.

The third step is to show that the bilinear form β is $\mathfrak{so}_n(\mathbb{C})$ invariant. For this purpose, we use the characterization of $\mathrm{Spin}(n) \subset C(2n) \cong \mathrm{End}(\Lambda^{\cdot}(W))$ given in Fulton-Harris, p. 308, (20.27). In that equation, $x^* = \tau_+(x) = \tau_-(x)$ because the degree of x is even. The group representation of $\mathrm{Spin}(n)$ on $\rho(\Lambda^{\cdot}(W))$ is given simply by left multiplication. If $x \in \mathrm{Spin}(n)$, then $xx^* = 1$, i.e. $x^* = x^{-1}$. Therefore, $x^* \in \mathrm{Spin}(n)$ as well and $x^*(x^*)^* = \tau_\pm(x)x = 1$. Hence

$$\begin{aligned} 2\beta(x \bullet s, x \bullet t)\rho(1) &= \tau_\pm \circ \rho(x \bullet s) \cdot \rho(x \bullet t) = \tau_\pm\big(x\rho(s)\big) \cdot x \cdot \rho(t) = \tau_\pm\big(\rho(s)\big) \cdot \tau_\pm(x) \cdot x \cdot \rho(t) \\ &= \tau_\pm \circ \rho(s) \cdot \rho(t) = 2\beta(s,t)\rho(1), \end{aligned}$$

which proves that β is $\mathrm{Spin}(n)$-invariant.

The final step is to determine when β is symmetric or skew-symmetric. If r is even, then $\beta(s,t) = (-1)^\sigma$ in the above notation, and $\beta(t,s) = (-1)^\sigma(-1)^\pi$, where $\pi \in S_r$ is given by $\pi(1, \dots, r) = (r, \dots, 1)$. Thus $(-1)^\pi = (-1)^{r(r-1)/2}$, so that $\beta(t,s) = (-1)^{r(r-1)/2}\beta(s,t)$. Thus β is symmetric for $r \equiv 0 \pmod 4$ and skew-symmetric for $r \equiv 2 \pmod 4$. If r is odd, then $\beta(s,t) = (-1)^\sigma(-1)^{1+k}$. If we exchange s and t, we replace k by $l = r - k \equiv 1 + k \pmod 2$, which gives an additional sign change, so that $\beta(t,s) = (-1)^{r(r-1)/2+1}$. Thus β is skew-symmetric for $r \equiv 1 \pmod 4$ and symmetric for $r \equiv 3 \pmod 4$.

46. *For any $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ define*

$$S_a(x_1, \dots, x_n) = |x_j^{a_i+n-i}| / |x_j^{n-i}|.$$

a. *Show that S_a is a symmetric polynomial of $x_1, \dots, x_n$. It is called a Schur polynomial.*
b. *Show that for $a_1 \geq a_2 \geq \cdots \geq a_n$ the Schur polynomial S_a gives the character of the irreducible $\mathfrak{sl}_n(\mathbb{C})$ representation $V(a_1\epsilon_1 + \cdots + a_n\epsilon_n)$ after the substitution $x_i = e^{\epsilon_i}$.*

a. The denominator in the fraction defining S_a is the discriminant $\Delta = \prod_{j<k}(x_j - x_k)$ because we compute the determinant of a Vandermode matrix. We have to show that Δ divides $A = |x_j^{a_i+n-i}|$ in the ring $\mathbb{Q}[x_1, \dots, x_n]$. Since the latter is a unique factorization domain, it suffices to show that $x_j - x_k$ divides A for all $j < k$. To do this, write $A = (x_j - x_k)B + C$, $B \in \mathbb{Q}[x_1, \dots, x_n]$, $C \in \mathbb{Q}[x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n]$. This is always possible. If we substitute $x_j = x_k$, we have $A|_{x_j=x_k} = 0$ because two columns of the matrix $(x_j^{a_i+n-i})$ are equal. This implies $C = 0$, i.e. $x_j - x_k$ divides A in $\mathbb{Q}[x_1, \dots, x_n]$. Thus S_a is a polynomial in the variables $x_1, \dots, x_n$. If we transpose two of the variables x_j both Δ and A change their sign, so that their quotient remains unchanged. Hence S_a is symmetric.

b. The Weyl group W of $\mathfrak{sl}_n(\mathbb{C})$ is the symmetric group S_n on n elements. Let $a_1 \geq a_2 \geq \cdots \geq a_n$, $\lambda = a_1\epsilon_1 + \cdots + a_n\epsilon_n$. The positive roots are $\epsilon_i - \epsilon_j$, $1 \leq i < j \leq n$, with $\epsilon_1 + \cdots \epsilon_n = 0$. Thus the weight ρ occurring in the Weyl character formula has the form

$$\rho = \frac{1}{2}\sum_{i<j} \epsilon_i - \epsilon_j = \frac{1}{2}\sum_{i=1}^{n}\big((n-i) - (i-1)\big)\epsilon_i = \frac{1}{2}\sum_{i=1}^{n-1}(2n-2i)\epsilon_i = \sum_{i=1}^{n}(n-i)\epsilon_i.$$

Thus if we substitute $x_j = e^{\epsilon_j} = \exp(\epsilon_j)$,

$$\begin{aligned}
|x_j^{a_i+n-i}| &= \sum_{\sigma\in W}(-1)^\sigma x_{\sigma(1)}^{a_1+n-1}\cdots x_{\sigma(n)}^{a_n+n-n} \\
&= \sum_{\sigma\in W}(-1)^\sigma \exp\big((a_1+n-1)\epsilon_{\sigma(1)}\big)\cdots\exp\big((a_n+n-n)\epsilon_{\sigma(n)}\big) \\
&= \sum_{\sigma\in W}(-1)^\sigma \exp\left(\sum_{i=1}^{n}(a_i+n-i)\sigma\epsilon_i\right) = \sum_{\sigma\in W}(-1)^\sigma \exp\big(\sigma(\lambda+\rho)\big), \\
|x_j^{n-i}| &= \sum_{\sigma\in W}(-1)^\sigma \exp\big(\sigma(\rho)\big).
\end{aligned}$$

Thus the quotient S_a of these two expressions is the character of the irreducible representation with highest weight λ according to the Weyl character formula.

47. *Let $\mathfrak{g}$ be a semi-simple Lie algebra with Weyl group W. Let $\omega_1, \dots, \omega_n$ be the fundamental weights, $R_\mathfrak{g} = \mathbb{Z}[e^{\pm\omega_1}, \dots, e^{\pm\omega_n}]$. Let $R_\mathfrak{g}^W$ be the subring of polynomials in $R_\mathfrak{g}$ invariant under the natural W-action. Show that $R_\mathfrak{g}^W$ coincides with the ring of formal characters of finite dimensional representations of $\mathfrak{g}$. (Hint: Show that $\operatorname{ch} V(\lambda)$ form a basis of $R_\mathfrak{g}^W$.)*

Every finite-dimensional representation is uniquely determined by its character, which is a polynomial $\chi = \sum_{\lambda\in\Lambda_W} a_\lambda e^\lambda$, where Λ_W is the weight lattice. Moreover, χ is invariant under the action of the Weyl group. The weights ω_i form a basis of the weight lattice, and every $\lambda \in \Lambda_W$ with $a_\lambda \neq 0$ must be an integer linear combination of these fundamental weights. Hence χ is a polynomial in the variables $e^{\pm\omega_i}$, $i = 1, \dots, n$, and can therefore be considered as an element of $R_\mathfrak{g}^W$. It remains to show that every $f \in R_\mathfrak{g}^W$ is a linear combination of characters of finite-dimensional representations of $\mathfrak{g}$.

Let $f = \sum_{\lambda\in\Lambda_W} a_\lambda e^\lambda$ be invariant under the Weyl group. Every element of $R_\mathfrak{g}^W$ can be written in this form, with only finitely many non-zero a_λ, by viewing $(e^{\pm\omega_1})^{\lambda_1}\cdots(e^{\pm\omega_n})^{\lambda_n} = e^{\pm\lambda_1\omega_1\pm\cdots\pm\lambda_n\omega_n}$. Since f is W-invariant, there are $\lambda \geq 0$ with $a_\lambda \neq 0$ (unless $f = 0$), and we can pick a maximal $\lambda \in \Lambda_W$, $\lambda \geq 0$, with $a_\lambda \neq 0$. Let χ_λ be the character of the irreducible representation with highest weight λ. Then $f_2 = f - a_\lambda\chi_\lambda \in R_\mathfrak{g}^W$.

Furthermore, f_2 does not contain the weight λ. Let $C(f)$ be the convex hull of the positive weights λ with $a_\lambda \neq 0$. Then $C(f_2) \subset C(f)$ because $C(\chi_\lambda) \subset C(f)$, and $\lambda \notin C(f_2)$. Hence the set $C(f_2) \cap \Lambda_W$ has fewer elements than the (finite) set $C(f) \cap \Lambda_W$. Iteration of this process yields a sequence $f_2, f_3, \dots$ of elements of $R_\mathfrak{g}^W$ with $\#C(f_{i+1}) \cap \Lambda_W < \#C(f_i) \cap \Lambda_W$. After finitely many steps, this sequence must end, i.e. $f_i = 0$ for some i. This gives us a decomposition of f as a linear combination of characters of irreducible representations, which is what we want.

48. *Find the centers of the simply connected Lie groups of type F_4, E_6, E_7, and E_8.*

The order of the center of the simply connected Lie group for a certain Dynkin diagram is the determinant of the Cartan matrix associated to the diagram. The Cartan matrices and their determinants are:

$$F_4\colon \quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad \#Z(F_4) = 1$$

$$E_6\colon \quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad \#Z(E_6) = 3$$

$$E_7\colon \quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad \#Z(E_7) = 2$$

$$E_8\colon \quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad \#Z(E_8) = 1$$

Thus the order of the center is either 1,2, or 3. All Abelian groups of these orders are cyclic. Hence $Z(F_4) = \{1\}$, $Z(E_8) = \{1\}$, $Z(E_7) = \mathbb{Z}_2$, $Z(E_6) = \mathbb{Z}_3$.

49. *Let G be a real Lie group, $H = \exp(Xt)$ be a one-parameter subgroup. Show that either H is closed or its closure $\overline{H}$ is compact. In the latter case show that $\overline{H}$ is isomorphic to a torus. (Hint: look at the normal form of* $\operatorname{ad} X$*.)*

Trivially, H is Abelian and connected, and thus $\overline{H}$ is Abelian and connected as well. It is well-known that, as a closed subgroup of a Lie group, $\overline{H}$ is a Lie group as well. Let $\mathfrak{h}$ be the Lie algebra of $\overline{H}$, then $\mathfrak{h}$ is Abelian. Hence the connected simply connected Lie group with Lie algebra $\mathfrak{h}$ is $\mathbb{R}^N$ with addition, $N = \dim \mathfrak{h}$. Since $\overline{H}$ is connected and has Lie algebra $\mathfrak{h}$, there is a covering map $\mathbb{R}^N \to \overline{H}$, which identifies $\overline{H}$ with $\mathbb{R}^N/\Lambda$ for some lattice $\Lambda \subset \mathbb{R}^N$. The mapping $\exp\colon \mathfrak{h} \to \mathbb{R}^N$ is linear, hence $H \subset \overline{H}$ is the image of a line L in $\mathbb{R}^N$. Since H is supposed to be dense in $\overline{H}$, $L + \Lambda$ is dense in $\mathbb{R}^N$. Suppose Λ has rank less than N. If $L \subset \mathbb{R} \cdot \Lambda$, then the closure of $L + \Lambda \subset \mathbb{R} \cdot \Lambda$ has dimension less than N and cannot be dense in $\mathbb{R}^N$. If $L \not\subset \mathbb{R} \cdot \Lambda$, we can choose a basis $v_0, v_1, \dots, v_{N-1}$ in $\mathbb{R}^N$ such that $L = \mathbb{R} \cdot v_0$, $\Lambda = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_k$, $k \leq N-1$. Unless $N = 1$, the vector $\frac{1}{2}v_1 \in \mathbb{R}^N$ is not in the closure of $L + \Lambda$. Hence either $N = 1$ or $\Lambda \cong \mathbb{Z}^N$. In the first case, $H = \overline{H}$, i.e. H is closed. In the second case, $\overline{H} \cong \mathbb{R}^N/\Lambda \cong (\mathbb{R}/\mathbb{Z})^N$ is compact and isomorphic to a torus.

50. a. *Show that any complex connected semi-simple Lie group has a faithful finite-dimensional representation.*

b. *Give an example of a connected complex Lie group which does not have a finite-dimensional faithful representation.*

a. Let G be a complex connected semi-simple Lie group, let $\mathfrak{g}$ be its complex semi-simple Lie algebra, let $\tilde{G}$ be the corresponding connected simply connected Lie group with Lie algebra $\mathfrak{g}$. Then $G = \tilde{G}/N$ for some discrete subgroup $N \subset Z(\tilde{G})$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $T = \exp \mathfrak{h} \subset \tilde{G}$ be the corresponding maximal torus in $\tilde{G}$. Then $N \subset T$. Let $\Lambda = \frac{1}{2\pi i} \exp^{-1}(N) \subset \mathfrak{h}$. Then $P^\vee \subset \Lambda \subset Q^\vee$, where P and Q are the weight lattice and the root lattice, respectively. A representation ρ of $\mathfrak{g}$ induces a representation of G iff N operates trivially in the induced representation of $T \subset \tilde{G}$ iff all (highest) weights of ρ lie in $\Lambda^\vee$. The quotient $\Lambda^\vee/Q \subset P/Q$ is a finite group. Choose positive weights $\lambda_1, \dots, \lambda_N \in \Lambda^\vee$ with $\Lambda^\vee = \bigcup(\lambda_k + Q)$ (we can always achieve positivity by adding appropriate elements of Q). Let ρ be the direct sum of the corresponding representations $V(\lambda_k)$ of $\mathfrak{g}$ and the adjoint representation of $\mathfrak{g}$. Then ρ induces a representation $\hat\rho$ of G because all highest weights λ_k lie in $\Lambda^\vee$, and the adjoint representation is even a representation of $\tilde G/Z(\tilde G)$.

We claim that $\hat\rho$ is faithful. To see this, let $t \in \tilde G$ be in the kernel of $\hat\rho \circ \pi$, where $\pi\colon \tilde G \to G$ is the universal covering map. Since the adjoint representation of G is contained in $\hat\rho$, $t \in Z(\tilde G) \subset T$, say, $t = \exp(2\pi i H)$. We have to show $t \in N$, and this means $H \in \Lambda$. Therefore, it suffices to show that $(\lambda, H) \in \mathbb{Z}$ for all $\lambda \in \Lambda^\vee$. Since $\pi(t) \in \operatorname{Ker}\hat\rho$, $(\lambda_k, H) \in \mathbb{Z}$ for $k = 1, \ldots, N$. Moreover, $t \in Z(\tilde G)$ implies $(\lambda, H) \in \mathbb{Z}$ for all $\lambda \in Q$. Hence $(\lambda + \lambda_k, H) \in \mathbb{Z}$ for $k = 1, \ldots, N$, $\lambda \in Q$. Since elements of this form are all of $\Lambda^\vee$, we have $H \in \Lambda$ as claimed.

b. We have seen in the lecture that $G = \mathrm{Sl}_2(\mathbb{R})$ is connected but not simply connected. Let $\tilde G$ be its universal covering group and let $\pi\colon \tilde G \to G$ be the covering map. $\tilde G$ is a connected, simply connected, semi-simple Lie group, and we claim that it has no faithful finite-dimensional representation. To see this assume that $\rho\colon \tilde G \to \mathrm{Aut}(V)$ is such a representation, V a finite-dimensional complex vector space. Then the derivative $D\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra representation of $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{R})$. It induces a Lie algebra representation $\widehat{D\rho}$ of $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. This can be integrated to a representation $\hat\rho\colon \mathrm{Sl}_2(\mathbb{C}) \to \mathrm{Aut}(V)$ of $\mathrm{Sl}_2(\mathbb{C})$ because $\mathrm{Sl}_2(\mathbb{C})$ is simply connected. Let $i\colon \mathrm{Sl}_2(\mathbb{R}) \to \mathrm{Sl}_2(\mathbb{C})$ be the canonical embedding. The representations $\hat\rho \circ i \circ \pi$ and ρ of $\tilde G$ both have the same derivative $D\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. Hence these two representations coincide. Since ρ is assumed to be injective, π must be injective, which is absurd. Hence $\tilde G$ has no faithful finite-dimensional representation.

51. *Let W be the spinor representation of the Lie algebra $\mathfrak{so}_{2n+1}(\mathbb{C})$. Decompose $S^2(W)$ and $\Lambda^2(W)$ into a sum of irreducible components.*

Let V be the standard representation of $\mathfrak{so}_{2n+1}(\mathbb{C})$. Then $\Lambda^s(V)$ is an irreducible representation of $\mathfrak{so}_{2n+1}(\mathbb{C})$ with highest weight $\lambda_s = L_1 + \cdots + L_s$ for $s = 1, \ldots, n$. We want to show that

$$\begin{aligned}
\mathrm{Sym}^2(W) &= \Lambda^n(V) \oplus \Lambda^{n-3}(V) \oplus \Lambda^{n-4}(V) \oplus \Lambda^{n-3-4}(V) \oplus \Lambda^{n-8}(V) \oplus \Lambda^{n-3-8} \oplus \cdots \\
\Lambda^2(W) &= \Lambda^{n-1}(V) \oplus \Lambda^{n-2}(V) \oplus \Lambda^{n-1-4}(V) \oplus \Lambda^{n-2-4}(V) \oplus \Lambda^{n-1-8}(V) \oplus \Lambda^{n-2-8} \oplus \cdots.
\end{aligned}$$

Therefore, we have to find all highest vectors in $W \otimes W$. For $I = (i_1, \ldots, i_k)$, let $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$. Then

$$\{e_I \otimes e_J \mid 1 \le i_1 < \cdots < i_k \le n, 1 \le j_1 < \cdots < j_l \le n\}$$

is a basis of $W \otimes W$. Moreover, it is a weight basis with respect to the standard Cartan subalgebra of $\mathfrak{so}_{2n+1}(\mathbb{C})$. The weight of $e_I \otimes e_J$ is

$$\sum_{i\in I} L_i + \sum_{j \in J} L_j - \sum_{k=1}^n L_k.$$

The positive weights are of the form $a_1 L_1 + \cdots a_n L_n$ with $a_1 \ge a_2 \ge \cdots a_n \ge 0$. Hence the only positive weights in $W \otimes W$ are $\lambda_s = L_1 + \cdots + L_s$. We have to show that for each such weight there is precisely one highest vector, and to determine when it lies in $\mathrm{Sym}^2(W)$ or in $\Lambda^2(W)$. Let $S = (1, \ldots, s)$, $L = (s+1, \ldots, n)$. A generic weight vector for the weight λ_s is of the form

$$w = \sum_{I \subset L} a_I e_S \wedge e_I \otimes e_S \wedge e_{L\setminus I}.$$

Considering $\mathfrak{so}_{2n+1}(\mathbb{C}) \subset C(2n+1)$, the root spaces for positive roots are spanned by $e_0 e_i$, $i = 1, \ldots, n$; $e_{n+j} e_i$, $i, j = 1, \ldots, n$, $i < j$; $e_i e_j$, $i, j = 1, \ldots, n$. These expressions operate on W by $e_0 \bullet e_I = (-1)^{\#I} e_I$, $e_i \bullet e_I = e_i \wedge e_I$ for $i = 1, \ldots, n$, and $e_{n+i} \bullet (e_i \wedge e_I) = e_I$, $e_{n+i} \bullet e_I = 0$ if $i \notin I$, $i = 1, \ldots, n$. Since $e_0^2 = 1$, if w is a highest vector then $e_i w = 0$ for all $i = 1, \ldots, n$. However, this in turn implies that w is a highest vector. Hence we have to find a condition for $e_i w = 0$, $i = 1, \ldots, n$.

If $i \in S$, then trivially $e_i w = 0$. Assume $i \notin S$. Then

$$\begin{aligned}
e_i w &= \sum_{I \subset L} a_I e_i \wedge e_S \wedge e_I \otimes e_S \wedge e_{L\setminus I} + \sum_{I\subset L} a_I e_S \wedge e_I \otimes e_i \wedge e_S \wedge e_{L \setminus I} \\
&= (-1)^s \sum_{I \subset L, i \notin I} a_I e_S \wedge e_i \wedge e_I \otimes e_S \wedge e_{L\setminus I} + (-1)^s \sum_{I \subset L, i \in I} a_I e_S \wedge e_I \otimes e_S \wedge e_i \wedge e_{L\setminus I} \\
&= (-1)^s \sum_{I \subset L, i\notin I} \left((-1)^{\#\{j<i | j \in L\setminus I\}} a_I + (-1)^{\#\{j<i|j\in I\}} a_{I\cup\{i\}} \right) e_S \wedge e_i \wedge e_I \otimes e_S \wedge e_i \wedge e_{L\setminus I\setminus\{i\}}
\end{aligned}$$

In order for this expression to be 0, we must have $(-1)^{\#\{j<i|j\in L\setminus I\}} a_I + (-1)^{\#\{j<i|j\in I\}} a_{I\cup\{i\}} = 0$, i.e.

$$a_{I\cup\{i\}} = -(-1)^{\#\{j|j<i\}} a_I = (-1)^i a_I.$$

By iteration, this rule implies $a_I = (-1)^{\sum I} a_\emptyset$. Conversely, this is also sufficient for w to be a highest vector. Hence up to scalar multiplies there is precisely one highest vector with weight λ_s, namely

$$w = \sum_{I \subset L} (-1)^{\sum I} e_S \wedge e_I \otimes e_S \wedge e_{L \setminus I}.$$

Finally, we have to determine whether w is symmetric or skew-symmetric. w is symmetric iff $(-1)^{\sum I} = (-1)^{\sum L \setminus I}$ for all $I \subset L$ iff $(-1)^{\sum L} = 1$ iff $\frac{1}{2}(n-s)(n-s+1) = \sum L \equiv 0 \pmod 2$. Therefore, $w \in \operatorname{Sym}^2(W)$ if and only if $s \equiv n \pmod 4$ or $s \equiv n+1 \pmod 4$. Otherwise $w \in \Lambda^2(W)$. This proves the above formulas for $\operatorname{Sym}^2(W)$ and $\Lambda^2(W)$.

52. *Let $\mathfrak{g}$ be a semi-simple Lie algebra with root lattice Q, $V(\lambda)$ be a finite-dimensional irreducible representation, $m(\mu)$ be the multiplicity of the weight μ in $V(\lambda)$. Show that $m(\mu)$ is a piecewise polynomial function on $\lambda + Q$ with degree of pieces not greater than $N - n$, and the number of pieces not greater than $F(n)$. Here N is the number of positive roots, n is the rank of $\mathfrak{g}$, and F is an appropriate function.*

53. *Let $\mathfrak{g}$ be a simple complex Lie algebra. A non-zero integral dominant weight $\lambda \in P_+$ is called a minuscule if $\lambda(H_\alpha) \le 1$ for each positive root α.*

a. *Show that λ is a minuscule if and only if the set of weights of $V(\lambda)$ coincides with the orbit of λ under the Weyl group action.*
b. *Show that if λ is a minuscule then λ is a fundamental weight, and find all minuscule weights for all simple Lie algebras.*

Let R be the root system of $\mathfrak{g}$, $H_\alpha, X_\alpha, Y_\alpha$ be the standard generators, where α runs through a set of primitive roots Π in R. Let W be the Weyl group of $\mathfrak{g}$. Let P be the weight and Q be the root lattice, and P^+ and Q^+ be the subsets of positive weights in P and Q, respectively. Let $\lambda \in P^+$, let $V(\lambda)$ be the irreducible representation of $\mathfrak{g}$ with highest weight λ and let $v \in V(\lambda)$ be a highest vector for $V(\lambda)$.

a. Let λ be a minuscule weight. Every weights of $V(\lambda)$ is in the W-orbit of some positive weight. Hence what we have to show is that λ is the only positive weight of $V(\lambda)$. To see this, note that all weights of $V(\lambda)$ are in $\lambda - Q^+$, because the weight spaces of $V(\lambda)$ are spanned by elements of the form $Y_{\alpha_1} \cdots Y_{\alpha_k} v$. Hence it suffices to show that $P^+ \cap (\lambda - Q^+) = \{\lambda\}$. Since $\alpha(H_\alpha) = 2$ for all $\alpha \in \Pi$, we have $(\lambda - \alpha)(H_\alpha) \le 1 - 2 < 0$ for all $\alpha \in \Pi$, i.e. $\lambda - \alpha \notin P^+$. A fortiori, $\lambda - \sum n_\alpha \alpha \notin P^+$ if $n_\alpha \ge 0$, $n_{\alpha_0} \ne 0$ for some $\alpha_0 \in \Pi$. Thus $(\lambda - Q^+) \cap P^+ = \{\lambda\}$.

   Conversely, if λ is not a minuscule weight, say, $\lambda(H_\alpha) \ge 2$, then $\lambda - \alpha = \mu \in P^+$, because $\alpha(H_\beta) < 0$ for $\alpha \ne \beta$ and $\mu(H_\alpha) \ge 0$ by assumption. If $Y_\alpha v = 0$, then $H_\alpha v = [X_\alpha, Y_\alpha] v = X_\alpha Y_\alpha v - Y_\alpha X_\alpha v = 0$, i.e. $\lambda(H_\alpha) = 0$ contrary to our assumption. Hence $Y_\alpha v \ne 0$ and $Y_\alpha v \in V(\lambda)_\mu$. Thus μ is indeed a weight of $V(\lambda)$. Since $\mu, \lambda \ge 0$, μ does not lie in the W-orbit of λ because the Weyl chamber is a fundamental domain for the action of W.
b. We will exhibit the minuscule weights of all simple Lie algebras and find that they are all fundamental weights. If the Lie algebra is not simple, then minuscule weights are not necessarily fundamental weights. For example, the semi-simple Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ has the minuscule $\omega_1 + \omega_2$, where ω_1, ω_2 are the fundamental weights of the two $\mathfrak{sl}_2(\mathbb{C})$-summands.

   A_n: We have
   $$R^+ = \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n+1\}.$$
   Let (L_i) be the dual basis of (ϵ_i). A weight is given by a sum $\lambda = \sum_{i=1}^n a_i L_i$. λ is a minuscule iff $\lambda(\epsilon_i - \epsilon_j) = a_i - a_j \in \{0,1\}$ for all $1 \le i < j \le n$ and $\lambda(\epsilon_i - \epsilon_{n+1}) = a_i \in \{0,1\}$ for all $i = 1, \dots, n$. These conditions are equivalent to $1 \ge a_1 \ge a_2 \ge \cdots \le a_n \ge 0$, $a_i \in \{0,1\}$. This shows that the minuscules are
   $$\omega_s = L_1 + \cdots L_s, \qquad s = 1, \dots, n.$$
   Especially, all minuscule weights are fundamental weights.

   B_n: We can assume $n > 1$ and have
   $$R^+ = \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n\} \cup \{\epsilon_i + \epsilon_j \mid 1 \le i < j \le n\} \cup \{\epsilon_i \mid 1 \le i \le n\}.$$
   Hence $\lambda = \sum a_i L_i$ is a minuscule if and only if $a_i - a_j, a_i + a_j \in \{0,1\}$ for all $1 \le i < j \le n$, and $a_i \in \{0,1\}$ for $i = 1, \dots, n$. Using the results for A_n, the only candidates are $L_1 + \cdots L_s$, $s = 1, \dots, n$. But for $s \ne 1$, $(L_1 + \cdots L_s)(\epsilon_1 + \epsilon_2) = 2$, so that only $\omega_1 = L_1$ can be a minuscule. In fact, it is easy to see that ω_1 is a minuscule. Again, all minuscule weights are fundamental weights.

   C_n: We can assume $n > 1$ and have
   $$R^+ = \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n\} \cup \{\epsilon_i + \epsilon_j \mid 1 \le i < j \le n\} \cup \{2\epsilon_i \mid 1 \le i \le n\}.$$

Hence $\lambda = \sum a_i L_i$ is a minuscule if and only if $a_i - a_j, a_i + a_j \in \{0,1\}$ for all $1 \leq i < j \leq n$, $2a_i \in \{0,1\}$ for $i = 1,\ldots,n$. Thus $\frac{1}{2} \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Since the difference of two different numbers in $[0,1/2]$ is never an integer, we must have $a_1 = \cdots = a_n = 1/2$, i.e. the only possible minuscule is $\omega_n = \frac{1}{2}(L_1 + \cdots + L_n)$. Again, it is obvious that this is in fact a minuscule, and it also is a fundamental weight.

D_n: We can assume $n > 3$ and have

$$R^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n\}$$

Hence $\lambda = \sum a_i L_i$ is a minuscule if and only if $a_i - a_j, a_i + a_j \in \{0,1\}$ for all $1 \leq i < j \leq n$. Thus

$$a_i = \frac{1}{2}\big((a_i - a_{i+1}) + (a_i + a_{i+1})\big) \in \{0, 1/2, 1\}$$

for $i = 1,\ldots,n-1$. If a_i is 1, then all a_j with $j \neq i$ must be 0. Since also $a_j \geq a_i > 0$ for $j < i$, we must have $i = 1$, i.e. $\lambda = \omega_1 = L_1$. Clearly, this is a minuscule and a fundamental weight. Otherwise, there is some i with $a_i = 1/2$. Then $a_j = 0$ is impossible for $j = 1,\ldots,n-1$, so that $\lambda = \frac{1}{2}(L_1 + \cdots + L_{n-1} + 2a_n L_n)$. We have $a_n + 1/2 \in \{0,1\}$, which leaves open the possibilities $a_n = \pm 1/2$. It is easy to check that this gives in fact minuscule weights $\omega_n = \frac{1}{2}(L_1 + \cdots L_n)$ and $\omega_{n-1} = \frac{1}{2}(L_1 + \cdots L_{n-1} - L_n)$, which are also fundamental weights.

G_2: We have

$$R^+ = \{\alpha, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\},$$

where α, β are primitive roots, $(\alpha,\alpha) = 3$, $(\beta,\beta) = 1$, $(\alpha,\beta) = -3/2$. If $L(\alpha), L(\beta) \in \{0,1\}$, then $L(2\alpha + 3\beta) > 1$ unless $L = 0$. Hence there are no minuscule weights for G_2.

F_4: We have described the root system F_4 in a previous exercise. We saw that the roots $\alpha_1 = \frac{1}{2}(1,-1,-1,-1)$, $\alpha_2 = (0,0,0,1)$, $\alpha_3 = (0,0,1,-1)$, and $\alpha_4 = (0,1,-1,0)$ can be taken as a set of primitive roots. Moreover,

$$\beta = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 = (1,1,0,0)$$

is a root as well. Therefore, if $L(\alpha_i) \in \{0,1\}$, then $L(\beta) \notin \{0,1\}$ unless $L(\alpha_i) = 0$ for $i = 1,\ldots,4$, i.e. $L = 0$. Hence F_4 has no minuscule weights either.

E_8: We have seen in the lecture that the root system $E_8 \subset \mathrm{lin}\{\epsilon_i \mid i = 1,\ldots,8\}$ can be described as

$$E_8 = D_8 \cup \left\{ \frac{1}{2}\sum_{i=1}^{8} \sigma_i \epsilon_i \;\middle|\; \sigma_i \in \{\pm 1\}, \frac{1}{4}\sum_{i=1}^{8} \sigma_i \in \mathbb{Z} \right\}.$$

The roots $\alpha_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \cdots - \epsilon_7 + \epsilon_8)$, $\alpha_2 = \epsilon_1 + \epsilon_2$, $\alpha_3 = \epsilon_2 - \epsilon_1$, $\alpha_4 = \epsilon_3 - \epsilon_2$, $\ldots$, $\alpha_8 = \epsilon_7 - \epsilon_6$ are a set of primitive roots for E_8. An elementary calculation shows that the root $\epsilon_8 + \epsilon_7 \in D_8$ can be expressed in this basis as

$$\beta_8 = \epsilon_7 + \epsilon_8 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 \in E_8^+.$$

Again, all primitive roots occur in this sum with a factor greater than 1. Hence if $\lambda \in P^+$, then either $\lambda(\beta_8) \notin \{0,1\}$ or $\lambda = 0$. Thus E_8 has no minuscule weights.

E_7: $E_7 = E_8 \cap \mathrm{lin}\{\alpha_1,\ldots,\alpha_7\} = E_8 \cap \{L_7 + L_8 = 0\}$. We have

$$\beta_7 = \epsilon_8 - \epsilon_7 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 \in E_7^+.$$

As above, if $\lambda \in P^+$ is a minuscule, then $\lambda(\alpha_i) = 0$ for $i = 1,\ldots,6$, because the root α_i occurs in the decomposition of β_7 with a factor greater than 1. Since also $\lambda(\alpha_7) \in \{0,1\}$ and $\lambda \neq 0$, the only possible minuscule is $\omega_7\colon \alpha_i \mapsto \delta_{i7}$. We can write $\omega_7 = L_6 + L_8 = L_6 - L_7$ on E_7, and a straightforward calculation shows that ω_7 is indeed a minuscule, and also a fundamental weight.

E_6: $E_6 = E_8 \cap \mathrm{lin}\{\alpha_1,\ldots,\alpha_6\} = E_6 \cap \{L_7 + L_8 = L_6 + L_8 = 0\}$. We have

$$\beta_6 = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \in E_6^+.$$

Thus if $\lambda \in P^+$ is a minuscule for E_6, then necessarily $\lambda(\alpha_i) = 0$ for $i = 2,\ldots,5$. Moreover, $\lambda(\alpha_1), \lambda(\alpha_6) \in \{0,1\}$ and $\lambda(\alpha_1) + \lambda(\alpha_6) = \lambda(\beta_6) \leq 1$, which implies either $\lambda = \omega_1$ or $\lambda = \omega_6$, where ω_i is the fundamental weight $\alpha_j \mapsto \delta_{ij}$. We can write $\omega_6 = L_5 + L_8 = L_5 - L_7 = L_5 - L_6$ and see by a trivial calculation that it is a minuscule. For symmetry reasons, ω_1 must be a minuscule as well because there is a symmetry of the Dynkin diagram E_6 mapping α_1 to α_6.

54. *Show that the simple Lie algebra of type E_8 contains a subalgebra $\mathfrak{sl}_9(\mathbb{C})$ such that the Lie algebra E_8 splits into the sum*

$$\mathfrak{sl}_9(\mathbb{C}) \oplus \Lambda^3(V) \oplus \Lambda^3(V^*)$$

with respect to the adjoint action of $\mathfrak{sl}_9(\mathbb{C})$. Here V is the standard representation of $\mathfrak{sl}_9(\mathbb{C})$. (Hint: Show that the root system of E_8 contains a subsystem A_8.

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