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A Classification of 2-Groups

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Introduction

Just as groups may be used to describe symmetries of objects in a category, crossed modules can describe symmetries of objects in 2-categories. A crossed module may act on X by a homomorphism to $\mathrm{Aut}(X)$, and equivalent crossed modules give equivalent actions [6], [2]. Therefore it is useful to understand crossed modules up to equivalence.

A crossed module is just a particular way to view a strict 2-group [7]. In this thesis, bigroups are classified up to equivalence of bicategories. This is the same classification as Section 8 of John Baez's article [1] written in the language of bicategories. This classification result is then applied to crossed modules.

In Chapter 1, we review all the preliminary notions needed from bicategory theory. In particular, we describe in detail all of the data needed for an equivalence of bicategories.

Chapter 2 begins the classification process using skeletal bigroups. We work with skeleta, because it is in this setting that the classification result is clearest. This result will later be generalised to arbitrary 2-groups. A skeletal bigroup appears to have four pieces of data:

1. the 1-arrows;
2. the 2-arrows;
3. the unitors;
4. the associator.

When we consider skeletal bigroups up to equivalence, this reduces to the following three pieces of data:

1. a group G, describing the 1-arrows;
2. a G-module M, describing the 2-arrows;
3. an element of the third cohomology class $H^3(G, M)$, describing the associator.

In Chapter 3 we show how the results in Chapter 2 for skeletal bigroups can be used in general bigroups. To do this, we show that all bicategories are equivalent to skeletal ones. We prove this using the Whitehead Theorem for Bicategories (Theorem 30), which gives criteria for a pseudofunctor to be an equivalence. Generalising a well known result about 1-categories, it states that a pseudofunctor F is an equivalence if and only if it satisfies all of the following three properties:

1. F is *essentially surjective*;
2. F is *essentially full*;
3. F is *2-fully-faithful.*

These three properties are called the Whitehead criteria and defined in full in Definition 22. The Whitehead Theorem is a powerful result, as it simplifies the task of showing that two bicategories are equivalent. Instead of constructing an equivalence as described in Chapter 1, it is only necessary to check if a pseudofunctor satisfies the above three criteria. We give an elementary proof of this theorem, and then apply it to prove the existence of skeleta. It follows that the classification theorem from Chapter 2 holds for general bigroups.

Chapter 4 discusses our classification result for strict bigroups. The classification of crossed modules up to equivalence of bicategories has already been demonstrated in Chapter 3. However, when discussing crossed modules, we often use a simpler notion of equivalence than "equivalence of bicategories". Equivalences of crossed modules corresponds to a simpler notion, which we call a *strict equivalence* of strict bicategories. In fact, there is no difference between these two notions of equivalence: two strict bicategories are strictly equivalent iff they are equivalent in the sense described in Chapter 1. Theorem 47 is the key result here, as it states that any equivalence can be made strict. Therefore, the classification Theorem in Chapter 2 can be applied to crossed modules. This completes the classification of crossed modules.

1 Preliminaries

1.1 Bicategories

Definition 1. A bicategory $C = (C^0, C, \circ, l, r, a)$ consists of:

- A set of objects C^0.
- For all objects x, y, a category $C(x, y)$, whose objects are 1-arrows $x \to y$ and morphisms are 2-arrows. Composition of 2-arrows in this category is denoted by $\cdot$ and is called vertical composition.
- For all objects x, a unit 1-arrow $1_x : x \to x$.
- For all objects x, y, z, a bifunctor $\circ : C(y, z) \times C(x, y) \to C(x, z)$. This bifunctor provides a product $\circ$ on arrows and a horizontal product $\bullet$ on 2-arrows.
- For all arrows $f : x \to y$, invertible natural transformations $l_f : 1_x \circ f \Rightarrow f$ and $r_f : f \circ 1_y \Rightarrow f$ called left and right unitors, respectively.
- An invertible natural transformation a with components $a_{f,g,h} : (f \circ g) \circ h \Rightarrow f \circ (g \circ h)$ for each composable triple of arrows f, g, h, called an associator.

such that the following diagrams commute for all composable arrows f, g, h, i:

- The *middle triangle identity*

$$\begin{tikzcd}
(f \circ 1) \circ g \arrow[rr, Rightarrow, "a_{f,1,g}"] \arrow[dr, Rightarrow, "r_f \bullet 1_g"'] & & f \circ (1 \circ g) \arrow[dl, Rightarrow, "1_f \bullet l_g"] \\
& f \circ g &
\end{tikzcd} \tag{1}$$

- The *pentagon identity*

$$\begin{tikzcd}
& (f \circ g) \circ (h \circ i) \arrow[dr, Rightarrow, "a_{f,g,h\circ i}"] & \\
((f \circ g) \circ h) \circ i \arrow[ur, Rightarrow, "a_{f\circ g,h,i}"] \arrow[d, Rightarrow, "a_{f,g,h} \bullet 1_i"] & & f \circ (g \circ (h \circ i)) \\
(f \circ (g \circ h)) \circ i \arrow[rr, Rightarrow, "a_{f,g\circ h,i}"] & & f \circ ((g \circ h) \circ i) \arrow[u, Rightarrow, "1_f \bullet a_{g,h,i}"']
\end{tikzcd} \tag{2}$$

Remark 2. The bifunctoriality of the horizontal product is also known as the *interchange law*. In full, it states that for any composable 2-arrows $\alpha, \beta, \gamma, \delta$:

$$(\alpha \bullet \beta) \cdot (\gamma \bullet \delta) = (\alpha \cdot \gamma) \bullet (\beta \cdot \delta)$$

The diagrams in the following lemma were originally included as coherence axioms of a monoidal category, but Max Kelly showed in [4] that they follow from the others.

Lemma 3. Let f, g be composable arrows in a bicategory. Then the following two diagrams of 2-arrows commute:

- The *left triangle identity*

$$
\begin{tikzcd}
(1 \circ f) \circ g \arrow[rr, Rightarrow, "a_{1,f,g}"] \arrow[dr, Rightarrow, "l_f \bullet 1_g"'] & & 1 \circ (f \circ g) \arrow[dl, Rightarrow, "l_{f \circ g}"] \\
& f \circ g &
\end{tikzcd}
\tag{3}
$$

- The *right triangle identity*

$$
\begin{tikzcd}
(f \circ g) \circ 1 \arrow[rr, Rightarrow, "a_{f,g,1}"] \arrow[dr, Rightarrow, "r_{f \circ g}"'] & & f \circ (g \circ 1) \arrow[dl, Rightarrow, "1_f \bullet r_g"] \\
& f \circ g &
\end{tikzcd}
\tag{4}
$$

Proof. Consider the diagram

$$
\begin{tikzcd}
((1 \circ 1) \circ f) \circ g \arrow[rrrr, Rightarrow, "a_{1,1,f} \bullet 1_g"] \arrow[drr, Rightarrow, "(l_1 \bullet 1_f) \bullet 1_g"] \arrow[ddd, Rightarrow, "a_{1 \circ 1, f, g}"'] & & & & (1 \circ (1 \circ f)) \circ g \arrow[dll, Rightarrow, "(1_1 \bullet l_f) \bullet 1_g"'] \arrow[ddd, Rightarrow, "a_{1, 1 \circ f, g}"] \\
& & (1 \circ f) \circ g \arrow[d, Rightarrow, "a_{1,f,g}"] & & \\
& & 1 \circ (f \circ g) & & \\
(1 \circ 1) \circ (f \circ g) \arrow[urr, Rightarrow, "l_1 \bullet 1_{f \circ g}"] \arrow[drr, Rightarrow, "a_{1,1,f \circ g}"'] & & & & 1 \circ ((1 \circ f) \circ g) \arrow[ull, Rightarrow, "1_1 \bullet (l_f \bullet 1_g)"'] \arrow[dll, Rightarrow, "1_1 \bullet a_{1,f,g}"] \\
& & 1 \circ (1 \circ (f \circ g)) \arrow[uu, Rightarrow, "1_1 \bullet l_{f \circ g}"'] & &
\end{tikzcd}
$$

The outer pentagon commutes by the pentagon identity. The upper triangle and lower left triangle commute by the middle triangle identity. The two quadrilaterals commute because a is natural. Therefore, the lower right triangle commutes. We now use the naturality of l:

$$\begin{tikzcd}
1 \circ (1 \circ f) \circ g \arrow[rrrr, Rightarrow, "1_1 \bullet a_{1,f,g}"] \arrow[dr, Rightarrow, "l_{(1\circ f)} \bullet 1_g"] \arrow[dddrr, Rightarrow, "1_1 \bullet (l_f \bullet 1_g)"'] & & & & 1 \circ (1 \circ (f \circ g)) \arrow[dl, Rightarrow, "l_{1\circ(f\circ g)}"'] \arrow[dddll, Rightarrow, "1_1 \bullet l_{f\circ g}"] \\
& (1 \circ f) \circ g \arrow[rr, Rightarrow, "a_{1,f,g}"] \arrow[dr, Rightarrow, "l_f \bullet 1_g"'] & & 1 \circ (f \circ g) \arrow[dl, Rightarrow, "l_{f\circ g}"] & \\
& & f \circ g & & \\
& & 1 \circ (f \circ g) \arrow[u, Rightarrow, "l_{f\circ g}"] & &
\end{tikzcd}$$

The squares of this diagram are naturality squares of l. Therefore the inner triangle, which is the left triangle identity, commutes. The proof that the right triangle identity commutes is similar. $\square$

1.2 Lax Functors, Transformations and Modifications

We now introduce arrows between bicategories, 2-arrows between 1-arrows, and 3-arrows between 2-arrows. Similar to how the category of categories is really a bicategory, bicategories form a certain kind of tricategory. The most general form of an equivalence of bicategories is described using three levels of arrows, and we give the concrete definition of these arrows in this section. We also describe their vertical compositions and units, which are also needed to define equivalences.

Definition 4. Let C, D be two bicategories. A *lax functor* $F = (F^0, F, \mu, \lambda) : C \to D$ consists of:

- A function $F^0 : C^0 \to D^0$ between the objects.
- For all objects x, y, a functor $F : C(x, y) \to D(F^0(x), F^0(y))$.
- For all composable arrows f, g, a natural transformation $\mu_{f,g} : F(f) \circ F(g) \Rightarrow F(f \circ g)$.
- For all objects x, an arrow $\lambda_x : 1_{F^0(x)} \Rightarrow F(1_x)$.

We require that the following diagram commutes for all composable arrows f, g, h:

$$\begin{tikzcd}
(Ff \circ Fg) \circ Fh \arrow[r, Rightarrow, "\mu_{f,g} \bullet 1_{Fh}"] \arrow[d, Rightarrow, "a_{Ff,Fg,Fh}"'] & F(f \circ g) \circ Fh \arrow[r, Rightarrow, "\mu_{f\circ g,h}"] & F((f \circ g) \circ h) \arrow[d, Rightarrow, "F(a_{f,g,h})"] \\
Ff \circ (Fg \circ Fh) \arrow[r, Rightarrow, "1_{Ff} \bullet \mu_{g,h}"] & Ff \circ F(g \circ h) \arrow[r, Rightarrow, "\mu_{f,g\circ h}"] & F(f \circ (g \circ h))
\end{tikzcd} \tag{5}$$

and the following diagrams commute for all $f : x \to y$:

$$
\begin{tikzcd}
1_{F^0 y} \circ Ff \arrow[r, Rightarrow, "\lambda_y \bullet 1_{Ff}"] \arrow[d, Rightarrow, "l_{Ff}"'] & F(1_y) \circ Ff \arrow[d, Rightarrow, "\mu_{1_y, f}"] \\
Ff & F(1_y \circ f) \arrow[l, Rightarrow, "F(l_f)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
Ff \circ 1_{F^0 x} \arrow[r, Rightarrow, "1_{Ff} \bullet \lambda_x"] \arrow[d, Rightarrow, "r_{Ff}"'] & Ff \circ F(1_x) \arrow[d, Rightarrow, "\mu_{f, 1_x}"] \\
Ff & F(f \circ 1_x) \arrow[l, Rightarrow, "F(r_f)"]
\end{tikzcd}
\tag{6}
$$

If the arrows λ and μ are invertible, then we call F a *pseudofunctor*.

Lemma 5 (Composition of lax functors)**.** Let C, D, E be bicategories and let $F : C \to D$, $G : D \to E$ be lax functors (resp. pseudofunctors). Let $F = (F^0, F, \mu^F, \lambda^F)$ and $G = (G^0, G, \mu^G, \lambda^G)$. Then we define the composition $F \circ G$ by [3]:

$$F \circ G = (F^0 \circ G^0, F \circ G, \mu^G_{F\cdot, F\cdot} \circ G(\mu^F), \lambda^G_{F\cdot} \circ G(\lambda^F))$$

This is a lax functor (resp. pseudofunctor).

Proof. We follow the proof of Lemma 4.1.29 of [3]. First we show that $\mu^G_{F\cdot,F\cdot} \circ G(\mu^F)$ is natural. Let f, g be a pair of composable arrows, f', g' be another pair of composable arrows and $\alpha : f \Rightarrow f', \beta : g \Rightarrow g'$ 2-arrows in C. The following naturality diagram commutes:

$$
\begin{tikzcd}
GFf \circ GFg \arrow[r, Rightarrow, "\mu^G_{Ff,Fg}"] \arrow[d, Rightarrow, "GF\alpha \bullet GF\beta"'] & G(Ff \circ Fg) \arrow[r, Rightarrow, "G(\mu^F_{f,g})"] \arrow[d, Rightarrow, "G(F\alpha \bullet F\beta)"] & GF(f \circ g) \arrow[d, Rightarrow, "GF(\alpha \bullet \beta)"] \\
GFf' \circ GFg' \arrow[r, Rightarrow, "\mu^G_{Ff',Fg'}"'] & G(Ff' \circ Fg') \arrow[r, Rightarrow, "G(\mu^F_{f',g'})"'] & GF(f' \circ g')
\end{tikzcd}
$$

To see that diagram (5) commutes for GF, let f, g, h be composable arrows in C and consider the diagram:

$$
\begin{tikzcd}
(GFf \circ GFg) \circ GFh \arrow[rrr, Rightarrow, "a_{GFf,GFg,GFh}"] \arrow[d, Rightarrow, "\mu^G_{Ff,Fg} \bullet 1_{GFh}"'] & & & GFf \circ (GFg \circ GFh) \arrow[d, Rightarrow, "1_{GFf} \bullet \mu^G_{Fg,Fh}"] \\
G(Ff \circ Fg) \circ GFh \arrow[d, Rightarrow, "G(\mu^F_{f,g}) \bullet 1_{GFh}"'] \arrow[dr, Rightarrow, "\mu^G_{Ff \circ Fg, Fh}"] & & & GFf \circ G(Fg \circ Fh) \arrow[d, Rightarrow, "1_{GFf} \bullet G(\mu^F_{g,h})"] \arrow[dl, Rightarrow, "\mu^G_{Ff, Fg \circ Fh}"'] \\
GF(f \circ g) \circ GFh \arrow[d, Rightarrow, "\mu^G_{F(f \circ g), Fh}"'] & G((Ff \circ Fg) \circ Fh) \arrow[dl, Rightarrow, "G(\mu^F_{f,g} \bullet 1_{Fh})"] \arrow[r, Rightarrow, bend right, "G(a_{Ff,Fg,Fh})"'] & G(Ff \circ (Fg \circ Fh)) \arrow[dr, Rightarrow, "G(1_{Ff} \bullet \mu^F_{g,h})"'] & GFf \circ GF(g \circ h) \arrow[d, Rightarrow, "\mu^G_{Ff \circ F(g \circ h)}"] \\
G(F(f \circ g) \circ Fh) \arrow[d, Rightarrow, "G(\mu^F_{f \circ g, h})"'] & & & G(Ff \circ F(g \circ h)) \arrow[d, Rightarrow, "G(\mu^F_{f, g \circ h})"] \\
GF((f \circ g) \circ h) \arrow[rrr, Rightarrow, "GF(a_{f,g,h})"'] & & & GF(f \circ (g \circ h))
\end{tikzcd}
$$

The upper and lower hexagons commute by diagram (5). The "triangles" on the left and right are naturality squares for μ^G. The outside 12-gon is diagram (5) for $F \circ G$.

Finally we show that diagram (6) commutes for $F \circ G$. Let $f : x \to y$ be an arrow in C. Consider the diagram:

$$\begin{tikzcd}
1_{GFy} \circ GFf \arrow[r, Rightarrow, "\lambda^G_{Fy} \bullet 1_{GFf}"] \arrow[dd, Rightarrow, "l_{GFf}"'] & G1_{Fy} \circ GFf \arrow[r, Rightarrow, "G(\lambda^F_y) \bullet 1_{GFf}"] \arrow[d, Rightarrow, "\mu^G_{1_{Fy}, Ff}"'] & GF1_y \circ GFf \arrow[d, Rightarrow, "\mu^G_{F1_y, Ff}"] \\
 & G(1_{Fy} \circ Ff) \arrow[r, Rightarrow, "G(\lambda^F_y \bullet 1_{Ff})"'] \arrow[dl, Rightarrow, "G(l_{Ff})"] & G(F1_y \circ Ff) \arrow[d, Rightarrow, "G(\mu^F_{1_y, f})"] \\
GFf & & GF(1_y \circ f) \arrow[ll, Rightarrow, "GFl_f"]
\end{tikzcd}$$

The top-right square is a naturality square for μ. The trapezoids commute by diagram (6). The outside hexagon is the left half of Diagram (6) for $F \circ G$. The commutativity of the right half is shown in a similar way. $\square$

Definition 6 (Identity Pseudofunctor)**.** Let C be a category. Then $1_C = (1, 1, 1, 1) : C \to C$ is the *identity pseudofunctor*.

The identity pseudofunctor clearly acts as a (strict) unit with respect to vertical composition.

Definition 7. Let C, D be bicategories. Let $(F^0, F, \mu^F, \lambda^F)$ and $(G^0, G, \mu^G, \lambda^G)$ be lax functors $C \to D$. A *lax transformation* $\sigma : F \Rightarrow G$ consists of:

- for all objects $x \in C^0$, an arrow $\sigma_x : F^0 x \to G^0 x$;
- for all arrows $f \in C$, natural 2-arrows $\sigma_f : Gf \circ \sigma_x \Rightarrow \sigma_y \circ Ff$;

such that the following diagram commutes for all composable arrows $f : x \to y$, $g : y \to z$:

$$\begin{tikzcd}
(Gf \circ Gg) \circ \sigma_x \arrow[r, Rightarrow, "a"] \arrow[d, Rightarrow, "\mu^G_{f,g} \bullet 1_{\sigma_x}"'] & Gf \circ (Gg \circ \sigma_x) \arrow[r, Rightarrow, "1_{Gf} \bullet \sigma_g"] & Gf \circ (\sigma_y \circ Fg) \arrow[r, Rightarrow, "a^{-1}"] & (Gf \circ \sigma_y) \circ Fg \arrow[d, Rightarrow, "\sigma_f \bullet 1_{Fg}"] \\
G(f \circ g) \circ \sigma_x \arrow[r, Rightarrow, "\sigma_{f \circ g}"'] & \sigma_z \circ F(f \circ g) & \sigma_z \circ (Ff \circ Fg) \arrow[l, Rightarrow, "1_{\sigma_z} \bullet \mu^F_{f,g}"'] & (\sigma_z \circ Ff) \circ Fg \arrow[l, Rightarrow, "a"']
\end{tikzcd} \tag{7}$$

and the following diagram commutes for all $x \in C^0$:

$$\begin{tikzcd}
1_{G^0 x} \circ \sigma_x \arrow[r, Rightarrow, "l_{\sigma_x}"] \arrow[d, Rightarrow, "\lambda^G_x \bullet 1_{\sigma_x}"'] & \sigma_x \arrow[r, Rightarrow, "r^{-1}_{\sigma_x}"] & \sigma_x \circ 1_{F^0 x} \arrow[d, Rightarrow, "1_{\sigma_x} \bullet \lambda^F_x"] \\
G(1_x) \circ \sigma_x \arrow[rr, Rightarrow, "\sigma_{1_x}"'] & & \sigma_x \circ F(1_x)
\end{tikzcd} \tag{8}$$

If the arrows λ and μ are invertible, then we call σ a *strong transformation*. If the arrows λ and μ are identities, then we call σ a *strict transformation*.

Definition 8 (Vertical composition of pseudonatural transformations)**.** Let F, G, H be parallel pseudofunctors and let $\sigma : F \Rightarrow G$, $\sigma' : G \Rightarrow H$ be two transformations. The (vertical) composition $\sigma'\sigma$ consists of the following data:

- The component on each object x is given by $\sigma'_x \circ \sigma_x$.
- The component on each arrow f is given by the composition

$$\begin{tikzcd}
(\sigma' \circ \sigma) \circ Ff \arrow[r, Rightarrow, dashed, "\sigma'\sigma_f"] \arrow[d, Rightarrow, "a"'] & Ff \circ (\sigma' \circ \sigma_f) \\
\sigma' \circ (\sigma \circ Ff) \arrow[d, Rightarrow, "1\bullet\sigma_f"'] & (Ff \circ \sigma') \circ \sigma_f \arrow[u, Rightarrow, "a"'] \\
\sigma' \circ (Gf \circ \sigma_f) \arrow[r, Rightarrow, "a^{-1}"'] & (\sigma' \circ Gf) \circ \sigma_f \arrow[u, Rightarrow, "\sigma'_f\bullet 1"']
\end{tikzcd}$$

This composition is itself a transformation, as shown in Lemma 4.2.19 of [3].

Definition 9 (Identity transformation)**.** Let C, D be bicategories and let $F : C \to D$ be a lax functor. The *identity transformation* $1_F : F \Rightarrow F$ consists of:

- for all objects x in C, the unit arrow 1_x;
- for all 1-arrows $f : x \to y$ in C, the composite 2-arrow

$$Ff \circ 1 \overset{r}{\Rightarrow} Ff \overset{l^{-1}}{\Longrightarrow} 1 \circ Ff$$

A proof that this is a transformation can be found in Proposition 4.2.12 of [3].

Definition 10. Let C, D be bicategories, let $F, G : C \to D$ be lax functors and let $\sigma, \sigma' : F \Rightarrow G$ be transformations. A *modification* $\Gamma : \sigma \Rrightarrow \sigma'$ consists of a 2-arrow $\Gamma_x : \sigma_x \Rightarrow \sigma'_x$ for every object x, such that the following diagram commutes for all arrows $f \in C(x, y)$:

$$\begin{tikzcd}
Gf \circ \sigma_x \arrow[r, Rightarrow, "1\bullet\Gamma_x"] \arrow[d, Rightarrow, "\sigma_f"'] & Gf \circ \sigma'_x \arrow[d, Rightarrow, "\sigma'_f"] \\
\sigma_y \circ Ff \arrow[r, Rightarrow, "\Gamma_y\bullet 1"'] & \sigma'_y \circ Ff
\end{tikzcd} \tag{9}$$

A modification is *invertible* iff all of its components are invertible. Then Γ^{-1}, defined by taking the inverse of all components of Γ, is a transformation inverse to Γ.

Remark 11. To give meaning to invertible modifications, we can define a (vertical) composition of modifications and an identity modification. Then we find that a modification has an inverse if and only if it is invertible. An invertible modification composed with its inverse gives the identity. This is explained in detail in Section 4.4 of [3].

1.3 Equivalence of Bicategories

We now describe the most general form of an equivalence of bicategories. The idea of this definition becomes clear when one views bicategories as objects belonging to a tricategory. Then this definition is simply the weakest notion of "sameness" of objects in a tricategory.

Definition 12. Let C, D be bicategories. An *equivalence of bicategories* C, D consists of the following data:

- Two pseudofunctors $F : C \to D, \qquad G : D \to C$.
- Four pseudonatural transformations

$$\sigma : GF \Rightarrow 1_C, \quad \sigma^* : 1_C \Rightarrow GF, \quad \tau : FG \Rightarrow 1_D, \quad \tau^* : 1_D \Rightarrow FG$$

- Four invertible modifications

$$\sigma^*\sigma \Rrightarrow 1_{GF}, \quad \sigma\sigma^* \Rrightarrow 1_{1_C}, \quad \tau^*\tau \Rrightarrow 1_{FG}, \quad \tau\tau^* \Rrightarrow 1_{1_D}$$

We also call a pseudofunctor $F : C \to D$ an *equivalence* if it is part of an equivalence of bicategories.

2 The Classification of Skeletal Bigroups

Definition 13. A bigroup is a bicategory with one object c in which all arrows and 2-arrows are invertible.

Definition 14. A skeletal bigroup is a bigroup in which all isomorphic arrows are equal.

A skeletal bigroup is defined with four pieces of data:

1. the 1-arrows;
2. the 2-arrows;
3. the unitors;
4. the associator.

Throughout this chapter, we will consider each of these up to equivalence of bicategories. First, we will see that the 1-arrows correspond to a group G and the 2-arrows correspond to a G-module M. Then, we will show that the unitors add no additional data. Finally, we will see that the associator gives us an element of the third cohomology $H^3(M, G)$. The results are summarised in the following proposition:

Proposition 15. Skeletal bigroups can be classified up to equivalence by the following data:

- a group G;
- a G-module M;
- a 3-cocycle $\omega : M^3 \to G$.

Let $\mathfrak{G}$ and $\mathfrak{H}$ be two skeletal bigroups described in this way by $(G^{\mathfrak{G}}, M^{\mathfrak{G}}, \omega^{\mathfrak{G}})$ and $(G^{\mathfrak{H}}, M^{\mathfrak{H}}, \omega^{\mathfrak{H}})$, respectively. Then $\mathfrak{G}$ and $\mathfrak{H}$ are equivalent as bicategories iff all of the following hold:

- There exists a group isomorphism $\phi : G^{\mathfrak{G}} \to G^{\mathfrak{H}}$.
- There exists a group isomorphism $\psi : M^{\mathfrak{G}} \to M^{\mathfrak{H}}$.
- $F(\omega^{\mathfrak{G}})$ and $\omega^{\mathfrak{H}}$ differ by a coboundary, where $F(\omega^{\mathfrak{G}}) := \psi \circ \omega^{\mathfrak{G}} \circ (\phi^{-1})^3$.

We begin with the 1-arrows of a skeletal bigroup. An immediate consequence of being skeletal is:

Remark 16. All 2-arrows in a skeletal bigroup have the same source and target.

By the above remark, $1_c \circ f = f$ and $(f \circ g) \circ h = f \circ (g \circ h)$ hold for all arrows f, g, h, so the set of 1-arrows together with the operator $\circ$ form a group $G = (G, \circ)$.

Next we will describe the structure of the 2-arrows. Any 2-arrow $\alpha : f \Rightarrow f$ is of the form $1_f \bullet \beta$, where $\beta := 1_{f^{-1}} \bullet \alpha$ is an arrow from 1_c to 1_c. So we can describe any 2-arrow by an element of G and a 2-arrow $1_c \Rightarrow 1_c$. We make use of the Eckmann-Hilton argument to describe the set of 2-arrows $1_c \Rightarrow 1_c$.

Lemma 17 (Eckmann-Hilton)**.** Let M be a set equipped with 2 unital binary operators $\bullet$ and $\cdot$ such that for all $a, b, c, d \in M$:

$$(a \bullet b) \cdot (c \bullet d) = (a \cdot c) \bullet (b \cdot d)$$

Then:

- i The units of the operators are equal.
- ii The operators coincide.
- iii The operators are commutative.
- iv The operators are associative.

Proof.

- i $1_\bullet = 1_\bullet \bullet 1_\bullet = (1_. \cdot 1_\bullet) \bullet (1_\bullet \cdot 1_.) = (1_. \bullet 1_\bullet) \cdot (1_\bullet \bullet 1_.) = 1_. \cdot 1_. = 1_.$
- ii $a \bullet b = (a \cdot 1) \bullet (1 \cdot b) = (a \bullet 1) \cdot (1 \bullet b) = a \cdot b$
- iii $a \cdot b = (1 \bullet a) \cdot (b \bullet 1) = (1 \cdot b) \bullet (a \cdot 1) = b \bullet a = b \cdot a$
- iv $(a \cdot b) \cdot c = (a \bullet b) \cdot (1 \bullet c) = (a \cdot 1) \bullet (b \cdot c) = a \cdot (b \cdot c)$

□

This argument can be applied to the horizontal and vertical products on the set of 2-arrows over an identity arrow in any 2-category. In the setting of bigroups, it means that the set of 2-arrows $1_c \Rightarrow 1_c$ forms an abelian group M.

Lemma 18 (Classification: 1-arrows and 2-arrows)**.** Let $\mathfrak{G}, \mathfrak{H}$ be skeletal bigroups, and let $F : \mathfrak{G} \to \mathfrak{H}$ be an equivalence. Then the corresponding groups of 1-arrows $G^{\mathfrak{G}}, G^{\mathfrak{H}}$ are isomorphic, and the corresponding abelian groups $M^{\mathfrak{G}}, M^{\mathfrak{H}}$ are isomorphic.

Proof. A lax functor F acts as a strict monoid homomorphism on the 1-arrows and the 2-arrows of a skeletal bigroup. The existence of a (weak) inverse pseudofunctor of F, together with $\mathfrak{G}, \mathfrak{H}$ being skeletal, implies that F induces a bijection on 1-arrows and 2-arrows. Therefore $G^{\mathfrak{G}}$ and $G^{\mathfrak{H}}$ are isomorphic, and $M^{\mathfrak{G}}$ and $M^{\mathfrak{H}}$ are isomorphic. □

In a skeletal bigroup the 2-arrows $f \Rightarrow f$ over any arrow f form an abelian group. This is because we can translate a 2-arrow $f \Rightarrow f$ horizontally to a 2-arrow $1_c \Rightarrow 1_c$. This observation gives a new interpretation of the naturality of l, r and a:

Remark 19. Consider the naturality square for l:

$$\begin{tikzcd} 1_c \circ f \arrow[r, Rightarrow, "l_f"] \arrow[d, Rightarrow, "1_{1_c} \bullet \alpha"] & f \arrow[d, Rightarrow, "\alpha"] \\ 1_c \circ g \arrow[r, Rightarrow, "l_g"] & g \end{tikzcd}$$

In a skeletal bigroup, we have $f = g$ and $l_f = l_g$, and by the Eckmann-Hilton argument, the composition of 2-arrows is commutative. Therefore, this diagram simply states that $1_{1_c} \bullet \alpha = \alpha$. If we do the same for the naturality squares of r and a, we find that $\alpha \bullet 1_{1_c} = \alpha$ and $(\alpha \bullet \beta) \bullet \gamma = \alpha \bullet (\beta \bullet \gamma)$ for all 2-arrows α, β, γ.

Next we discuss the unitors. Every choice of unitor yields an equivalent bicategory:

Lemma 20 (Classification: Unitors)**.** Let $\mathfrak{G}, \mathfrak{H}$ be two skeletal bigroups with the same arrows, 2-arrows, horizontal products and associators, but not necessarily the same unitors. Then $\mathfrak{G}$ and $\mathfrak{H}$ are equivalent.

Proof. For any skeletal bigroup, we can rewrite the unitors l_f and r_f for any arrow f in terms of the arrow l_{1_c} and associators. Substituting $f = 1_c$ and $g = f$ into diagram (3) gives us $l_f = a^{-1}_{1,1,f} \cdot (l_{1_c} \bullet 1_f)$. Substituting $f = f$ and $g = 1_c$ into diagram (4) gives us $r_f = a_{f,1,1} \cdot (1_f \bullet r_{1_c})$, and in particular $a_{1,1,1} = 1_{1_c}$. Substituting $f = g = 1_c$ into diagram (1) gives us $l_{1_c} = r_{1_c}$. To summarise:

$$l_f = a^{-1}_{1,1,f} \cdot (l_{1_c} \bullet 1_f) \tag{10}$$

$$r_f = a_{f,1,1} \cdot (1_f \bullet l_{1_c}) \tag{11}$$

Now we construct an equivalence between $\mathfrak{G}$ and $\mathfrak{H}$. Let α be the 2-arrow l_{1_c} in $\mathfrak{G}$ and β the 2-arrow l_{1_c} in $\mathfrak{H}$. We define a homomorphism $F = (F^0, F, \mu^F, \lambda^F) : \mathfrak{G} \to \mathfrak{H}$ which acts as the identity on objects, arrows and 2-arrows, and μ^F is trivial. We have to find a λ^F such that the diagrams in (6) commute. After making the substitutions of equation (10), the left diagram becomes:

$$a^{-1}_{1,1,f} \cdot (\beta \bullet 1_f) = (\lambda_c \bullet 1_f) \cdot 1_f \cdot a^{-1}_{1,1,f} \cdot (\alpha \bullet 1_f)$$

Using the fact that the 2-arrows $f \Rightarrow f$ form an abelian group, this simplifies to

$$(\beta \bullet 1_f) = (\lambda_c \bullet 1_f) \cdot (\alpha \bullet 1_f)$$

Applying the interchange law gives us

$$(\beta \bullet 1_f) = (\lambda_c \cdot \alpha) \bullet (1_f \cdot 1_f)$$

Choosing $\lambda_c := \beta \cdot \alpha^{-1}$ satisfies this equation. A similar calculation shows that this choice of λ_c also makes the right diagram of (6) commute.

We construct $G = (G^0, G, \mu^G, \lambda^G) : \mathfrak{H} \to \mathfrak{G}$ in a similar way, letting $\lambda^G_c := \alpha \cdot \beta^{-1}$. This is a strict inverse of F, so we have an equivalence.

$\square$

The associator of a bigroup can be described in terms of group cohomology. Let G be the group of arrows and M the abelian group of 2-arrows $1_c \Rightarrow 1_c$ belonging to a skeletal bigroup. We can make M into a G-module: for $g \in G$ and $h \in M$ we define the action of g on h by $gh = 1_g \bullet h \bullet 1_{g^{-1}}$.

The associator of a bigroup takes three 1-arrows $f, g, h : c \to c$ and produces a 2-arrow $a_{f,g,h} : f \circ g \circ h \Rightarrow f \circ g \circ h$, subject to the pentagon identity. We can describe the 2-arrow $a_{f,g,h}$ using a 2-arrow $M \ni \omega_{f,g,h} : 1_c \Rightarrow 1_c$ defined by $a_{f,g,h} = \omega_{f,g,h} \bullet 1_{f\circ g\circ h}$. Then the associator corresponds to a certain map $G^3 \to M$. The pentagon identity states that for all $f, g, h, i \in G$:

$$\begin{aligned}(\omega_{f,g,h} \bullet 1_{f\circ g\circ h} \bullet 1_i) \cdot (\omega_{f,g\circ h,i} \bullet 1_{f\circ g\circ h\circ i}) \cdot (1_f \bullet \omega_{g,h,i} \bullet 1_{g\circ h\circ i}) \\ = (\omega_{f\circ g,h,i} \bullet 1_{f\circ g\circ h\circ i}) \cdot (\omega_{f,g,h\circ i} \bullet 1_{f\circ g\circ h\circ i})\end{aligned}$$

This simplifies to

$$\omega_{f,g,h} \cdot \omega^{-1}_{f\circ g,h,i} \cdot \omega_{f,g\circ h,i} \cdot \omega^{-1}_{f,g,h\circ i} \cdot (1_f \bullet \omega_{g,h,i} \bullet 1_f^{-1}) = 0,$$

i.e., ω is a 3-cocycle $G^3 \to M$.

This is the final piece of data from Proposition 15. We now investigate how this behaves under an equivalence of bicategories.

Lemma 21 (Classification: Associators)**.** Let $\mathfrak{G}$, $\mathfrak{H}$ be two skeletal bigroups with the same arrows and 2-arrows. Let $F : \mathfrak{G} \to \mathfrak{H}$ be an equivalence. Let $\omega^{\mathfrak{G}}$ and $\omega^{\mathfrak{H}}$ be the cocycles corresponding to the associators of $\mathfrak{G}$ and $\mathfrak{H}$. Then $F(\omega^{\mathfrak{G}})$ and $\omega^{\mathfrak{H}}$ differ by a coboundary.

Proof. Suppose that $F : \mathfrak{G} \to \mathfrak{H}$ is an equivalence of skeletal bigroups. We write the 2-arrows $\mu_{f,g}$ as $\psi_{f,g} \bullet 1_{f\circ g}$ for $\psi_{f,g} \in M$. Then we can translate diagram (5) to the following statement about 2-arrows $1_c \Rightarrow 1_c$:

$$\omega^{\mathfrak{H}}_{Ff,Fg,Fh} = \psi_{f,g} \cdot \psi_{f\circ g,h} \cdot \psi^{-1}_{f,g\circ h} \cdot (1_f \bullet \psi_{g,h} \bullet 1_f)^{-1} \cdot F(\omega^{\mathfrak{G}}_{f,g,h}) \qquad (12)$$

This means that $F(\omega^{\mathfrak{G}})$ and $\omega^{\mathfrak{H}}$ differ by a coboundary. $\square$

We now compete the proof of Proposition 15 by constructing an equivalence between two skeletal bigroups with matching cohomology classes.

Proof of Proposition 15. Let $\mathfrak{G}$ and $\mathfrak{H}$ be skeletal bigroups with corresponding data $(G^{\mathfrak{G}}, M^{\mathfrak{G}}, \omega^{\mathfrak{G}})$ and $(G^{\mathfrak{H}}, M^{\mathfrak{H}}, \omega^{\mathfrak{H}})$.

If $\mathfrak{G}$ and $\mathfrak{H}$ are equivalent, then Lemma 18 gives isomorphisms $\phi : G^{\mathfrak{G}} \to G^{\mathfrak{H}}$ and $\psi : M^{\mathfrak{G}} \to M^{\mathfrak{H}}$, Lemma 21 show that $\omega^{\mathfrak{H}}$ and $\psi \circ \omega^{\mathfrak{G}} \circ (\phi^{-1})^3$ differ by a coboundary.

For the other direction, assume that there exist group isomorphisms $\phi : G^{\mathfrak{G}} \to G^{\mathfrak{H}}$ and $\psi : M^{\mathfrak{G}} \to M^{\mathfrak{H}}$, and assume that $\phi \circ \omega^{\mathfrak{G}} \circ (\psi^{-1})^3 \cong \omega^{\mathfrak{H}}$. We need to construct an equivalence $F = (F^0, F, \mu, \lambda) : \mathfrak{G} \to \mathfrak{H}$. By Lemma 20 we can assume without loss of generality that $l_1^{\mathfrak{G}} = 1_{1_c}$ and $l_1^{\mathfrak{H}} = 1_{1_c}$. Let F^0 map the object of $\mathfrak{G}$ to the object of $\mathfrak{H}$. Let F be the functor which acts as an isomorphism $G^{\mathfrak{G}} \cong G^{\mathfrak{H}}$ on 1-arrows, and as an isomorphism $H^{\mathfrak{G}} \cong H^{\mathfrak{H}}$ on 2-arrows $c \Rightarrow c$.

The associators of $\mathfrak{G}$ and $\mathfrak{H}$ differ by a coboundary in the way given by (12). Let $\mu_{f,g} := \psi_{f,g} \bullet 1_{f \circ g}$. This choice is natural, and satisfies Coherence Condition (5) by construction. It remains to find a λ that makes the diagrams in (6) commute. We start with the left half of diagram (6), and horizontally translate it to the following equation about 2-arrows $1_c \Rightarrow 1_c$:

$$\lambda \cdot \psi_{1,f} \cdot (F(\omega^{\mathfrak{G}}_{1,1,f}))^{-1} = (\psi_{1,1} \cdot \psi_{1,f} \cdot \psi^{-1}_{1,f} \cdot (1_{1_c} \bullet \psi_{1,f} \bullet 1_{1_c})^{-1} \cdot F(\omega^{\mathfrak{G}}_{1,1,f}))^{-1}$$

which simplifies to

$$\lambda = \psi^{-1}_{1,1} \quad (= \mu^{-1}_{1,1})$$

To show that this choice of λ also makes the right diagram in (6) commute, we follow the same steps but horizontally translate on the left side instead of the right side, i.e., we use $\psi'_{f,g}$ defined by $\mu_{f,g} = 1_{f \circ g} \bullet \psi'_{f,g}$ etc. In this way, we verify that (F^0, F, μ, λ) is coherent, and is really a pseudofunctor.

The pseudofunctor (F^0, F, μ, λ) is strictly invertible, since we can invert all of its data to construct an inverse. Therefore, $\mathfrak{G}$ and $\mathfrak{H}$ are equivalent. $\square$

3 The Whitehead Theorem and Skeleta of Bicategories

A well known result in 1-category theory is that, assuming the Axiom of Choice, every category is equivalent to a skeletal category. A skeletal category is a category in which all isomorphic arrows are equal. This can be proven using another well known result, named the Whitehead Theorem for categories. We follow the language of Chapter 7 in [3]. It states that, assuming the Axiom of Choice, a functor $F : C \to D$ between 1-categories is an equivalence iff it satisfies the following two properties:

1. F is *essentially surjective*: For all objects $x^D \in D$, there exists an object $x^C \in C$ such that Fx^C is isomorphic to x^D.

2. F is *fully faithful*: For any two objects $x^C, y^C \in C$, the map $C(x^C, y^C) \to D(Fx^C, Fy^C)$ is bijective.

In this section, we aim to generalise both of these results to the setting of 2-categories. A *skeletal bicategory* is a bicategory in which all equivalent objects are equal, and all isomorphic 1-arrows are equal. The bicategorical Whitehead criteria are the following:

Definition 22 (Whitehead criteria)**.** Let C, D be two bicategories. The Whitehead criteria for a pseudofunctor $F : C \to D$ are the following:

1. F is *essentially surjective* if it is surjective on objects up to equivalence.

2. F is *essentially full* if for any two objects $x^C, y^C \in C^0$, the functor $F(x^C, y^C)$ is surjective on arrows up to isomorphism.

3. F is *2-fully-faithful* if for any two objects $x^C, y^C \in C^0$ and arrows $f, g : x^C \rightrightarrows y^C$, the functor $F(x^C, y^C)$ gives a bijection of 2-arrows $f \Rightarrow g$ and $Ff \Rightarrow Fg$.

The main result here is the Whitehead Theorem for bicategories (Theorem 30), which states that a pseudofunctor is an equivalence if and only if it satisfies the Whitehead criteria.

Most of the work for this theorem will be done in the proof of Lemma 24, in which we assume that the target bicategory is skeletal. We then prove the existence of skeletal bicategories in Theorem 26. The full Whitehead Theorem follows from these results.

Throughout this chapter we will assume the Axiom of Choice. We also assume that every category is small, so that we can apply it. We do not, however, rely on any deep results in category theory, such as the Yoneda embedding or the coherence theorem for bicategories. The reason for this is that we might want to use the Whitehead Theorem to prove these results. In particular, the coherence theorem can be proven by showing that every bicategory is equivalent to a strict one. It is easier to show this equivalence when we can use the Whitehead Theorem.

We now begin the proof of the Whitehead Theorem with the special case that the target bicategory is skeletal.

Remark 23. Let C, D be bicategories and let $F : C \to D$ be a pseudofunctor that is fully faithful on 2-arrows. Then a diagram of 2-arrows in C commutes iff its image in D commutes. This is true because a commutative diagram of 2-arrows describes an equality of 2-arrows.

Lemma 24. Let C be a bicategory and let D be a skeletal bicategory. A pseudofunctor $F = (F^0, F, \mu^F, \lambda^F) : C \to D$ is an equivalence if all of the following hold:

1. F is essentially surjective on objects;
2. F is essentially full on 1-arrows;
3. F is 2-fully-faithful.

Proof. In the following we use superscript D (like x^D, f^D) to indicate that something belongs to D and superscript C to indicate that something belongs to a chosen pre-image.

The idea of this proof is to choose 2-arrows in C that are pre-images of well-behaved arrows in D. Then, by Remark 23, we can prove that diagrams of 2-arrows in C commute by showing that the images of those diagrams commute in D.

A consequence of D being skeletal is that F must be strictly surjective and strictly full on 1-arrows. This allows us to make the following choices:

- For all objects $x^D \in D^0$, choose an object $x^C \in C^0$ such that $F^0 x^C = x^D$.
- For all objects $x \in C^0$, let $x^D := F^0 x$ and let $x^C \in C^0$ be the chosen object corresponding to x^D. Let $f_x : x \to x^C$ be a 1-arrow in the preimage of $1_{x^D} : F^0 x \to F^0 x^C$. Let $\bar{f}_x : x^C \to x$ be a 1-arrow in the preimage of $1_{x^D} : F^0 x^C \to F^0 x$. Let $\epsilon_x : f_x \circ \bar{f}_x \Rightarrow 1_{x^c}$ be the 2-arrow in the preimage of $l^D_{1_x}$ and let $\eta_x : \bar{f}_x \circ f_x \Rightarrow 1_x$ be the 2-arrow in the preimage of $r^D_{1_x}$. This defines an equivalence $(f_x, \bar{f}_x, \epsilon_x^{-1}, \eta_x)$ between x and x^C.
- For all 1-arrows $f^D \in D(x^D, y^D)$, choose a 1-arrow $f^C \in C(x^C, y^C)$ such that $Ff^C = f^D$.
- For all 1-arrows $f \in C(x, y)$, let $f^D := Ff$ and let $f^C \in C(x^C, y^C)$ be the chosen 1-arrow corresponding to f^D. Let $\alpha_f : (f_y \circ f) \circ \bar{f}_x \Rightarrow f^C$ be a 2-arrow in the preimage of $Fl^C_{f^C} \cdot Fr^C_{1 \circ f^C} : (1 \circ F(f)) \circ 1 \Rightarrow F(f^C)$.
- For all 2-arrows $\alpha^D : f^D \Rightarrow g^D$, choose a 2-arrow $\alpha^C : f^C \Rightarrow g^C$ such that $F\alpha^C = \alpha^D$.

We now construct a pseudofunctor $G = (G^0, G, \mu^G, \lambda^G) : D \to C$:

- The map G^0 sends x^D to x^C.

- The functor G sends the 1-arrows f^D to f^C and the 2-arrows α^D to α^C.
- The natural isomorphism μ^G is defined as the pre-image of $(\mu^F)^{-1} \cdot 1$ in $C(Gf^D \circ Gg^D, G(f^D \circ g^D))$. We can write its components $\mu^G_{f^D,g^D}$ explicitly as the composition:

$$
\begin{aligned}
Gf^D \circ Gg^D &\xRightarrow{\alpha^{-1}\bullet\alpha^{-1}} ((f_z \circ f) \circ \bar{f}_y) \circ ((f_y \circ g) \circ \bar{f}_x) \\
&\xRightarrow{1\bullet a} ((f_z \circ f) \circ \bar{f}_y) \circ (f_y \circ (g \circ \bar{f}_x)) \\
&\xRightarrow{a^{-1}} (((f_z \circ f) \circ \bar{f}_y) \circ f_y) \circ (g \circ \bar{f}_x) \\
&\xRightarrow{a\bullet 1} ((f_z \circ f) \circ (\bar{f}_y \circ f_y)) \circ (g \circ \bar{f}_x) \\
&\xRightarrow{(1\bullet\eta)\bullet 1} ((f_z \circ f) \circ 1) \circ (g \circ \bar{f}_x) \\
&\xRightarrow{r\bullet 1} (f_z \circ f) \circ (g \circ \bar{f}_x) \\
&\xRightarrow{a^{-1}} ((f_z \circ f) \circ g) \circ \bar{f}_x \\
&\xRightarrow{a\bullet 1} (f_z \circ (f \circ g)) \circ \bar{f}_x \\
&\xRightarrow{\alpha} G(f^D \circ g^D) \\
&\xRightarrow{G(\mu^F_{f^D,g^D})^{-1}} G(f^D \circ g^D)
\end{aligned}
\tag{13}
$$

Note that this definition does not depend on a particular choice of f^C, because $F(\mu^G_{f^D,g^D})$ is the same with any choice. The following diagram demonstrates why μ^G is a pre-image of $(\mu^F)^{-1} \cdot 1$:

$$
\begin{tikzcd}
f^D \circ g^D \arrow[dd, Rightarrow, "Fl^{-1}\bullet 1"'] & 1 \circ (f^D \circ g^D) \arrow[l, Rightarrow, "Fl"'] & (1 \circ (f^D \circ g^D)) \circ 1 \arrow[l, Rightarrow, "Fr"'] \\
 & ((1 \circ f^D) \circ g^D) \circ 1 \arrow[dl, Rightarrow, "Fr"'] \arrow[ur, Rightarrow, "Fa\bullet 1"'] & \\
(1 \circ f^D) \circ g^D \arrow[uur, Rightarrow, "Fa"] \arrow[d, Rightarrow, "Fr^{-1}"'] & (1 \circ f^D) \circ (g^D \circ 1) \arrow[l, Rightarrow, "1\bullet Fr"] \arrow[u, Rightarrow, "Fa^{-1}"'] & ((1 \circ f^D) \circ 1) \circ (g^D \circ 1) \arrow[l, Rightarrow, "Fr\bullet 1"] \arrow[dll, Rightarrow, "1\bullet Fr"] \\
((1 \circ f^D) \circ 1) \circ g^D \arrow[dd, Rightarrow, "1\bullet Fl^{-1}"'] & ((1 \circ f^D) \circ (1 \circ 1)) \circ g^D \arrow[l, Rightarrow, "(1\bullet Fr)\bullet 1"] & ((1 \circ f^D) \circ (1 \circ 1)) \circ (g^D \circ 1) \arrow[l, Rightarrow, "1\bullet Fr"] \arrow[u, Rightarrow, "(1\bullet Fr)\bullet 1"'] \\
 & ((1 \circ f^D) \circ 1)) \circ (g^D \circ 1) \arrow[ul, Rightarrow, "1\bullet Fr"] \arrow[ur, Rightarrow, "(1\bullet Fr)\bullet 1"'] & (((1 \circ f^D) \circ 1) \circ 1) \circ (g^D \circ 1) \arrow[l, Rightarrow, "Fr\bullet 1"] \arrow[u, Rightarrow, "Fa\bullet 1"'] \\
((1 \circ f^D) \circ 1) \circ (1 \circ g^D) \arrow[r, Rightarrow, "1\bullet Fr^{-1}"'] & ((1 \circ f^D) \circ 1) \circ ((1 \circ g^D) \circ 1) \arrow[u, Rightarrow, "1\bullet (Fl\bullet 1)"] \arrow[r, Rightarrow, "1\bullet Fa"'] & ((1 \circ f^D) \circ 1) \circ (1 \circ (g^D \circ 1)) \arrow[ul, Rightarrow, "1\bullet Fl"] \arrow[u, Rightarrow, "Fa^{-1}"']
\end{tikzcd}
$$

- The natural isomorphism λ^G is given by the pre-image of $(\lambda^F)^{-1}$.

The naturality and coherence of μ^G and λ^G follow from the naturality and coherence of μ^F and λ^F.

We constructed G such that FG is equal to the identity 1_D. It remains to show that $GF : C \Rightarrow C$ is equivalent to 1_C. We construct transformations $\sigma : 1_C \Rightarrow GF$ and $\sigma^* : GF \Rightarrow 1_C$ with the following data:

- $\sigma_x := f_x$
- σ_g is the composite

$$g^C \circ f_x \xRightarrow{\alpha^{-1}\bullet 1} ((f_y \circ g) \circ \bar{f}_x) \circ f_x \overset{a}{\Rightarrow} (f_y \circ g) \circ (\bar{f}_x \circ f_x) \xRightarrow{1\bullet\eta^{-1}} (f_y \circ g) \circ 1_x \overset{r}{\Rightarrow} f_y \circ g$$

- $\sigma^*_x := \bar{f}_x$
- σ^*_g is the composite

$$g \circ \bar{f}_x \xRightarrow{l^{-1}} 1 \circ (g \circ \bar{f}_x) \xRightarrow{\eta^{-1}\bullet 1} (\bar{f}_y \circ f_y) \circ (g \circ \bar{f}_x) \overset{a}{\Rightarrow} \bar{f}_y \circ (f_y \circ (g \circ \bar{f}_x)) \xRightarrow{1\bullet\alpha} \bar{f}_y \circ g^C$$

These transformations are natural because their images under F are compositions of natural 2-arrows in D. We now check that σ satisfies Coherence Conditions (7) and (8) (the case for σ^* is similar).

The right triangle identity (4) tells us that $r_1 = a_{1,1,1} \cdot r_1$, thus $a_{1,1,1} = 1$ and the middle triangle identity (1) tells us that $l_1 = r_1$. Therefore, the image of Coherence Condition (8) in D commutes:

$$\begin{tikzcd}
1 \circ 1 \arrow[rr, Rightarrow, "Fl"] \arrow[ddrr, Rightarrow, bend right, "Fl^{-1}\bullet 1"'] & & 1 \arrow[rr, Rightarrow, "Fr^{-1}"] & & 1 \circ 1 \\
& 1 \circ (1 \circ 1) \arrow[ul, Rightarrow, "Fl"'] \arrow[r, Rightarrow, "1\bullet Fl"] & 1 \circ 1 \arrow[u, Rightarrow, shift left, "Fl"] \arrow[u, Rightarrow, shift right, "Fr"'] & & \\
& & (1 \circ 1) \circ 1 \arrow[ul, Rightarrow, "a"] \arrow[u, Rightarrow, "Fr\bullet 1"'] \arrow[uurr, Rightarrow, bend right, "Fr"'] \arrow[d, Rightarrow, "Fr^{-1}\bullet 1"'] & & \\
& & ((1 \circ 1) \circ 1) \circ 1 \arrow[r, Rightarrow, "Fa"] & (1 \circ 1) \circ (1 \circ 1) \arrow[ul, Rightarrow, "1\bullet Fr"'] &
\end{tikzcd} \tag{14}$$

Coherence Condition (7) commutes by the following diagram:

Figure 1: Coherence of the transformation σ

$$
\begin{tikzcd}[column sep=small]
(f^C \circ g^C) \circ f_x \arrow[r, Rightarrow, "a"] \arrow[d, Rightarrow, "(1\bullet\alpha^{-1})\bullet 1"'] & f^C \circ (g^C \circ f_x) \arrow[r, Rightarrow, "1\bullet(\alpha^{-1}\bullet 1)"] & f^C \circ (((f_y \circ g) \circ \bar{f}_x) \circ f_x) \arrow[r, Rightarrow, "1\bullet a"] & f^C \circ ((f_y \circ g) \circ (\bar{f}_x \circ f_x)) \arrow[d, Rightarrow, "1\bullet(1\bullet\eta)"] \\
(f^C \circ ((f_y \circ g) \circ \bar{f}_x)) \circ f_x \arrow[urr, Rightarrow, "a"] \arrow[d, Rightarrow, "(\alpha^{-1}\bullet 1)\bullet 1"'] & ((f^C \circ (f_y \circ g)) \circ \bar{f}_x) \circ f_x \arrow[l, Rightarrow, "a\bullet 1"] \arrow[r, Rightarrow, "a"'] \arrow[ddl, Rightarrow, bend left, "((\alpha^{-1})\bullet 1)\bullet 1"] & (f^C \circ (f_y \circ g)) \circ (\bar{f}_x \circ f_x) \arrow[ur, Rightarrow, "a"] \arrow[d, Rightarrow, "1\bullet\eta"] \arrow[ddl, Rightarrow, bend right, "(\alpha^{-1})\bullet 1"'] & f^C \circ ((f_y \circ g) \circ 1) \arrow[d, Rightarrow, "1\bullet r"] \\
(((f_z \circ f) \circ \bar{f}_y) \circ ((f_y \circ g) \circ \bar{f}_x)) \circ f_x \arrow[d, Rightarrow, "a^{-1}\bullet 1"'] & & (f^C \circ (f_y \circ g)) \circ 1 \arrow[ur, Rightarrow, "a"] \arrow[r, Rightarrow, "r"] \arrow[dd, Rightarrow, "(\alpha^{-1}\bullet 1)\bullet 1"'] & f^C \circ (f_y \circ g) \arrow[d, Rightarrow, "a^{-1}"] \arrow[dd, Rightarrow, bend right, "\alpha^{-1}\bullet 1"'] \\
((((f_z \circ f) \circ \bar{f}_y) \circ (f_y \circ g)) \circ \bar{f}_x) \circ f_x \arrow[r, Rightarrow, "a"'] \arrow[d, Rightarrow, "(a^{-1}\bullet 1)\bullet 1"'] & (((f_z \circ f) \circ \bar{f}_y) \circ (f_y \circ g)) \circ (\bar{f}_x \circ f_x) \arrow[d, Rightarrow, "a^{-1}\bullet 1"'] \arrow[dr, Rightarrow, "1\bullet\eta"'] & & (f^C \circ f_y) \circ g \arrow[dd, Rightarrow, bend left, "(\alpha^{-1}\bullet 1)\bullet 1"] \\
(((((f_z \circ f) \circ \bar{f}_y) \circ f_y) \circ g) \circ \bar{f}_x) \circ f_x \arrow[r, Rightarrow, "a"'] \arrow[d, Rightarrow, "((a\bullet 1)\bullet 1)\bullet 1"'] & ((((f_z \circ f) \circ \bar{f}_y) \circ f_y) \circ g) \circ (\bar{f}_x \circ f_x) \arrow[d, Rightarrow, "(a\bullet 1)\bullet 1"'] \arrow[dr, Rightarrow, "1\bullet\eta"'] & (((f_z \circ f) \circ \bar{f}_y) \circ (f_y \circ g)) \circ 1 \arrow[r, Rightarrow, "r"'] \arrow[d, Rightarrow, "a^{-1}\bullet 1"'] & ((f_z \circ f) \circ \bar{f}_y) \circ (f_y \circ g) \arrow[d, Rightarrow, "a^{-1}"] \\
((((f_z \circ f) \circ (\bar{f}_y \circ f_y)) \circ g) \circ \bar{f}_x) \circ f_x \arrow[r, Rightarrow, "a"'] \arrow[d, Rightarrow, "(((1\bullet\eta)\bullet 1)\bullet 1)\bullet 1"'] & (((f_z \circ f) \circ (\bar{f}_y \circ f_y)) \circ g \circ (\bar{f}_x \circ f_x) \arrow[d, Rightarrow, "((1\bullet\eta)\bullet 1)\bullet 1"'] \arrow[dr, Rightarrow, "1\bullet\eta"'] & ((((f_z \circ f) \circ \bar{f}_y) \circ f_y) \circ g) \circ 1 \arrow[r, Rightarrow, "r"'] \arrow[d, Rightarrow, "(a\bullet 1)\bullet 1"'] & (((f_z \circ f) \circ \bar{f}_y) \circ f_y) \circ g \arrow[d, Rightarrow, "a\bullet 1"] \\
((((f_z \circ f) \circ 1) \circ g) \circ \bar{f}_x) \circ f_x \arrow[r, Rightarrow, "a"'] \arrow[d, Rightarrow, "((r\bullet 1)\bullet 1)\bullet 1"'] & (((f_z \circ f) \circ 1) \circ g) \circ (\bar{f}_x \circ f_x) \arrow[d, Rightarrow, "(r\bullet 1)\bullet 1"'] \arrow[dr, Rightarrow, "1\bullet\eta"'] & (((f_z \circ f) \circ (\bar{f}_y \circ f_y)) \circ g \circ 1 \arrow[r, Rightarrow, "r"'] \arrow[d, Rightarrow, "((1\bullet\eta)\bullet 1)\bullet 1"'] & ((f_z \circ f) \circ (\bar{f}_y \circ f_y)) \circ g \arrow[d, Rightarrow, "(1\bullet\eta)\bullet 1"] \\
(((f_z \circ f) \circ g) \circ \bar{f}_x) \circ f_x \arrow[r, Rightarrow, "a"'] \arrow[d, Rightarrow, "(a\bullet 1)\bullet 1"'] & ((f_z \circ f) \circ g) \circ (\bar{f}_x \circ f_x) \arrow[d, Rightarrow, "a\bullet 1"'] \arrow[dr, Rightarrow, "1\bullet\eta"'] & (((f_z \circ f) \circ 1) \circ g) \circ 1 \arrow[r, Rightarrow, "r"'] \arrow[d, Rightarrow, "(r\bullet 1)\bullet 1"'] & ((f_z \circ f) \circ 1) \circ g \arrow[d, Rightarrow, "r\bullet 1"] \\
((f_z \circ (f \circ g)) \circ \bar{f}_x) \circ f_x \arrow[r, Rightarrow, "a"'] \arrow[d, Rightarrow, bend right, "\alpha\bullet 1"'] & (f_z \circ (f \circ g)) \circ (\bar{f}_x \circ f_x) \arrow[dr, Rightarrow, "1\bullet\eta"'] & ((f_z \circ f) \circ g) \circ 1 \arrow[r, Rightarrow, "r"'] \arrow[d, Rightarrow, "a\bullet 1"'] & (f_z \circ f) \circ g \arrow[d, Rightarrow, "a"] \\
((f \circ g)^C \circ f_x \arrow[u, Rightarrow, bend right, "\alpha^{-1}\bullet 1"'] & & (f_z \circ (f \circ g)) \circ 1 \arrow[r, Rightarrow, "r"'] & f_z \circ (f \circ g)
\end{tikzcd}
$$

The final step is to construct invertible modifications $\Gamma : \sigma^* \circ \sigma \Rrightarrow 1_{1_C}$ and $\tilde{\Gamma} : \sigma \circ \sigma^* \Rrightarrow 1_{GF}$. We choose $\Gamma_x := \eta_x$ and $\tilde{\Gamma}_x := \epsilon_x$. We must show that these choices are coherent.

Consider the coherence condition for Γ:

$$
\begin{tikzcd}
f^C \circ 1 \arrow[r, Rightarrow, "r"] \arrow[d, Rightarrow, "1\bullet\epsilon^{-1}"'] & f^C \arrow[r, Rightarrow, "l^{-1}"] & 1 \circ f^C \\
f^C \circ (f_x \circ \bar{f}_x) \arrow[d, Rightarrow, "a^{-1}"'] & & (f_y \circ \bar{f}_y) \circ f^C \arrow[u, Rightarrow, "\epsilon\bullet 1"'] \\
(f^C \circ f_x) \circ \bar{f}_x \arrow[d, Rightarrow, "(\alpha^{-1}\bullet 1)\bullet 1"'] & & f_y \circ (\bar{f}_y \circ f^C) \arrow[u, Rightarrow, "a^{-1}"'] \\
(((f_y \circ f) \circ \bar{f}_x) \circ f_x) \circ \bar{f}_x \arrow[dd, Rightarrow, "a\bullet 1"'] & & f_y \circ (\bar{f}_y \circ ((f_y \circ f) \circ \bar{f}_x)) \arrow[u, Rightarrow, "1\bullet(1\bullet\alpha)"'] \\
& & f_y \circ (\bar{f}_y \circ (f_y \circ (f \circ \bar{f}_x))) \arrow[u, Rightarrow, "1\bullet(1\bullet a^{-1})"'] \\
((f_y \circ f) \circ (\bar{f}_x \circ f_x)) \circ \bar{f}_x \arrow[d, Rightarrow, "(1\bullet\eta)\bullet 1"'] & & f_y \circ ((\bar{f}_y \circ f_y) \circ (f \circ \bar{f}_x)) \arrow[u, Rightarrow, "1\bullet a"'] \\
((f_y \circ f) \circ 1) \circ \bar{f}_x \arrow[d, Rightarrow, "r\bullet 1"'] & & f_y \circ (1 \circ (f \circ \bar{f}_x)) \arrow[u, Rightarrow, "1\bullet(\eta^{-1}\bullet 1)"'] \\
(f_y \circ f) \circ \bar{f}_x \arrow[rr, Rightarrow, "a"'] \arrow[uuuuuuur, Rightarrow, bend right, "\alpha"] & & f_y \circ (f \circ \bar{f}_x) \arrow[u, Rightarrow, "1\bullet l^{-1}"']
\end{tikzcd}
\tag{15}
$$

We have split the above diagram into 2 subdiagrams. The key insight in this step is that the so-called zig-zag identities hold for our η and ϵ:

$$
\begin{tikzcd}
(\bar{f}_x \circ f_x) \circ \bar{f}_x \arrow[rr, Rightarrow, "a"] \arrow[d, Rightarrow, "\eta\bullet 1"'] & & \bar{f}_x \circ (f_x \circ \bar{f}_x) \arrow[d, Rightarrow, "1\bullet\epsilon"] \\
1 \circ \bar{f}_x \arrow[r, Rightarrow, "l"'] & \bar{f}_x & \bar{f}_x \circ 1 \arrow[l, Rightarrow, "r"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(f_x \circ \bar{f}_x) \circ f_x \arrow[rr, Rightarrow, "a"] \arrow[d, Rightarrow, "\epsilon\bullet 1"'] & & f_x \circ (\bar{f}_x \circ f_x) \arrow[d, Rightarrow, "1\bullet\eta"] \\
1 \circ f_x \arrow[r, Rightarrow, "l"'] & f_x & f_x \circ 1 \arrow[l, Rightarrow, "r"]
\end{tikzcd}
$$

When we map these diagrams to D, we get subdiagrams of Diagram (14). Therefore they commute. We use this result to prove that both subdiagrams of Diagram (15) commute:

Figure 2: Commutativity of the left side of Diagram (15)

$$
\begin{tikzcd}
(f^C \circ f_x) \circ \bar{f}_x & f^C \circ (f_x \circ \bar{f}_x) \arrow[l, Rightarrow, "a^{-1}"'] & f^C \circ 1 \arrow[l, Rightarrow, "1\bullet\epsilon^{-1}"'] & f^C \arrow[l, Rightarrow, "r^{-1}"'] \\
(((f_y \circ f) \circ \bar{f}_x) \circ f_x) \circ \bar{f}_x \arrow[u, Rightarrow, "(\alpha\bullet 1)\bullet 1"] \arrow[dd, Rightarrow, "a\bullet 1"'] & ((f_y \circ f) \circ \bar{f}_x) \circ (f_x \circ \bar{f}_x) \arrow[l, Rightarrow, "a^{-1}"'] \arrow[u, Rightarrow, "\alpha\bullet 1"] \arrow[d, Rightarrow, "a"] & ((f_y \circ f) \circ \bar{f}_x) \circ 1 \arrow[l, Rightarrow, "1\bullet\epsilon^{-1}"'] \arrow[u, Rightarrow, "\alpha\bullet 1"] \arrow[d, Rightarrow, "a"] & (f_y \circ f) \circ \bar{f}_x \arrow[l, Rightarrow, "r^{-1}"'] \arrow[u, Rightarrow, "\alpha"'] \\
 & (f_y \circ f) \circ (\bar{f}_x \circ (f_x \circ \bar{f}_x)) \arrow[d, Rightarrow, "1\bullet a"] & (f_y \circ f) \circ (\bar{f}_x \circ 1) \arrow[l, Rightarrow, "1\bullet(1\bullet\epsilon^{-1})"'] \arrow[ur, Rightarrow, "1\bullet r"] & \\
((f_y \circ f) \circ (\bar{f}_x \circ f_x)) \circ \bar{f}_x \arrow[d, Rightarrow, "(1\bullet\eta)\bullet 1"] & (f_y \circ f) \circ ((\bar{f}_x \circ f_x) \circ \bar{f}_x) \arrow[l, Rightarrow, "a^{-1}"'] \arrow[d, Rightarrow, "1\bullet(\eta\bullet 1)"] & & \\
((f_y \circ f) \circ 1) \circ \bar{f}_x \arrow[uuurrr, Rightarrow, bend right=40, "r\bullet 1"'] & (f_y \circ f) \circ (1 \circ \bar{f}_x) \arrow[l, Rightarrow, "a^{-1}"'] \arrow[uuurr, Rightarrow, bend right=30, "1\bullet l"] & &
\end{tikzcd}
$$

Figure 3: Commutativity of the right side of Diagram (15)

$$
\begin{tikzcd}
f^C \arrow[r, Rightarrow, "l^{-1}"] & 1 \circ f^C & (f_y \circ \bar{f}_y) \circ f^C \arrow[l, Rightarrow, "\epsilon\bullet 1"'] & f_y \circ (\bar{f}_y \circ f^C) \arrow[l, Rightarrow, "a^{-1}"'] \\
(f_y \circ f) \circ \bar{f}_x \arrow[u, Rightarrow, "\alpha"] \arrow[r, Rightarrow, "l^{-1}"] \arrow[d, Rightarrow, "a"'] & 1 \circ ((f_y \circ f) \circ \bar{f}_x) \arrow[u, Rightarrow, "1\bullet\alpha"'] \arrow[d, Rightarrow, "1\bullet a"'] & (f_y \circ \bar{f}_y) \circ ((f_y \circ f) \circ \bar{f}_x) \arrow[l, Rightarrow, "\epsilon\bullet 1"'] \arrow[u, Rightarrow, "1\bullet\alpha"'] \arrow[d, Rightarrow, "1\bullet a"'] & f_y \circ (\bar{f}_y \circ ((f_y \circ f) \circ \bar{f}_x)) \arrow[l, Rightarrow, "a^{-1}"'] \arrow[u, Rightarrow, "1\bullet(1\bullet\alpha)"'] \\
f_y \circ (f \circ \bar{f}_x) \arrow[r, Rightarrow, "l^{-1}"] \arrow[ddd, Rightarrow, bend right=60, "1\bullet l^{-1}"'] & 1 \circ (f_y \circ (f \circ \bar{f}_x)) & (f_y \circ \bar{f}_y) \circ (f_y \circ (f \circ \bar{f}_x)) \arrow[l, Rightarrow, "\epsilon\bullet 1"'] & f_y \circ (\bar{f}_y \circ (f_y \circ (f \circ \bar{f}_x))) \arrow[l, Rightarrow, "a^{-1}"'] \arrow[u, Rightarrow, "1\bullet(1\bullet a^{-1})"'] \\
 & (1 \circ f_y) \circ (f \circ \bar{f}_x) \arrow[u, Rightarrow, "a"'] \arrow[ul, Rightarrow, "l\bullet 1"'] & ((f_y \circ \bar{f}_y) \circ f_y) \circ (f \circ \bar{f}_x) \arrow[l, Rightarrow, "(\epsilon\bullet 1)\bullet 1"'] \arrow[u, Rightarrow, "a"'] \arrow[d, Rightarrow, "a\bullet 1"'] & \\
 & & (f_y \circ (\bar{f}_y \circ f_y)) \circ (f \circ \bar{f}_x) \arrow[r, Rightarrow, "a"] \arrow[d, Rightarrow, "1\bullet\eta"'] & f_y \circ ((\bar{f}_y \circ f_y) \circ (f \circ \bar{f}_x)) \arrow[uu, Rightarrow, "1\bullet a"'] \\
 & & (f_y \circ 1) \circ (f \circ \bar{f}_x) \arrow[r, Rightarrow, "a"] \arrow[uuull, Rightarrow, bend left=30, "r\bullet 1"] & f_y \circ (1 \circ (f \circ \bar{f}_x)) \arrow[u, Rightarrow, "1\bullet(\eta^{-1}\bullet 1)"']
\end{tikzcd}
$$

The coherence condition for $\tilde{\Gamma}$ commutes by a similar diagram. Thus we have constructed an equivalence from C to D. $\square$

In order to use this result, we must show that skeletal bicategories really exist, and that every bicategory is equivalent to a skeletal one. The following lemma will help us to construct the skeleton of a bicategory.

Lemma 25. Let $C = (C^0, C, \circ_C, l^C, r^C, a^C)$ be a bicategory. Let

$$D = (D^0, D, \circ_D, l^D, r^D, a^D)$$

be the data for a bicategory, except l, r, a are not known to be natural or coherent. Let $F = (F^0, F, \mu, \lambda) : C \Rightarrow D$ be the data of a pseudofunctor, satisfying Coherence Conditions (5) and (6); in particular, all the 2-arrows λ_x and $\mu_{f,g}$ are invertible. Assume that $1_{F^0(x)}$ belongs to the image of F for all objects x of C.

Then l^D, r^D, a^D restricted to the image of F are natural and satisfy Coherence Conditions (1) and (2).

Proof. We will avoid labelling $\circ, l, r, a$ when it is clear which category they are in. In the diagrams that follow, we will label regions to explain why they commute.

- Regions labelled "nat" commute by naturality.
- Regions labelled "func" commute by functoriality. (The functoriality of $\bullet$ is the interchange law.)
- Regions labelled "square" commute by Coherence Condition (5) for pseudofunctors.
- Regions labelled "hex" commute by Coherence Condition (6) for pseudofunctors.

First, we show that the unitors are natural:

$$\begin{tikzcd}
F(1_x \circ f) \arrow[rrrd, Rightarrow, bend left, "F(l)"] \arrow[ddd, Rightarrow, "F(1\bullet\alpha)"'] & & \text{square} & \\
\text{nat} & F(1_x) \circ Ff \arrow[ul, Rightarrow, "\mu"] \arrow[d, Rightarrow, "1\bullet F(\alpha)"'] \arrow[r, phantom, "\text{func}" near end, shift right=6ex] & 1_{Fx} \circ Ff \arrow[l, Rightarrow, "\lambda\bullet 1"'] \arrow[r, Rightarrow, "l"] \arrow[d, Rightarrow, "1\bullet F(\alpha)"'] & Ff \arrow[d, Rightarrow, "F(\alpha)"] \\
& F(1_x) \circ Fg \arrow[dl, Rightarrow, "\mu"] & 1_{Fx} \circ Fg \arrow[l, Rightarrow, "\lambda\bullet 1"] \arrow[r, Rightarrow, "l"'] & Fg \\
F(1_x \circ f) \arrow[rrru, Rightarrow, bend right, "F(l)"'] & & \text{square} &
\end{tikzcd}$$

The outside square is the naturality condition for l^C and the rightmost square is the naturality condition for l^D. The naturality of r^D is proven similarly.

For Coherence Condition (1), consider the diagram:

$$
\begin{tikzcd}
F((f \circ 1_y) \circ g) \arrow[dddddddd, Rightarrow, "F(a)"'] \arrow[ddddrrrr, Rightarrow, bend left, "F(r \bullet 1)"] & & & & \\
& F(f \circ 1_y) \circ Fg \arrow[ul, Rightarrow, "\mu"] \arrow[dddrr, Rightarrow, bend left, "F(r) \bullet 1"] & \text{nat} & & \\
& (Ff \circ F1_y) \circ Fg \arrow[u, Rightarrow, "\mu \bullet 1"] \arrow[dddd, Rightarrow, "a"'] & \text{square} & & \\
& & (Ff \circ 1_{Fy}) \circ Fg \arrow[ul, Rightarrow, "(1 \bullet \lambda) \bullet 1"'] \arrow[dd, Rightarrow, "a"'] \arrow[dr, Rightarrow, "r \bullet 1"] & & \\
\arrow[r, phantom, "\text{hex}"] & \arrow[r, phantom, "\text{nat}"] & \; & Ff \circ Fg \arrow[r, Rightarrow, "\mu"] & F(f \circ g) \\
& & Ff \circ (1_{Fy} \circ Fg) \arrow[ur, Rightarrow, "1 \bullet l"'] \arrow[dl, Rightarrow, "1 \bullet (\lambda \bullet 1)"] & & \\
& Ff \circ (F1_y \circ Fg) \arrow[d, Rightarrow, "1 \bullet \mu"'] & \text{square} & & \\
& Ff \circ F(1_y \circ g) \arrow[dl, Rightarrow, "\mu"] \arrow[uuurr, Rightarrow, bend right, "1 \bullet F(l)"'] & \text{nat} & & \\
F(f \circ (1_y \circ g)) \arrow[uuuurrrr, Rightarrow, bend right, "F(l)"'] & & & &
\end{tikzcd}
$$

The outermost triangle is Coherence Condition (1) for C and the innermost triangle is Coherence Condition (1) for D.

Now we show that the associator is natural and coherent:

Figure 4: Naturality of the associator

$$
\begin{tikzcd}
F((f\circ g)\circ h) \arrow[rrrrrr, Rightarrow, "F(a)"] \arrow[ddddd, Rightarrow, "F((\alpha\bullet\beta)\bullet\gamma)"'] & & & & & & F(f\circ(g\circ h)) \arrow[ddddd, Rightarrow, "F(\alpha\bullet(\beta\bullet\gamma))"] \\
& F(f\circ g)\circ Fh \arrow[ul, Rightarrow, "\mu"'] \arrow[ddd, Rightarrow, "F(\alpha\bullet\beta)\bullet F\gamma"'] \arrow[rrrr, phantom, "\text{hex}"] & & & & Ff\circ F(g\circ h) \arrow[ur, Rightarrow, "\mu"] \arrow[ddd, Rightarrow, "F\alpha\bullet F(\beta\bullet\gamma)"] & \\
& & (Ff\circ Fg)\circ Fh \arrow[ul, Rightarrow, "\mu\bullet 1"'] \arrow[rr, Rightarrow, "a"] \arrow[d, Rightarrow, "F\alpha\bullet(F\beta\bullet F\gamma)"'] & & Ff\circ(Fg\circ Fh) \arrow[ur, Rightarrow, "1\bullet\mu"] \arrow[d, Rightarrow, "(F\alpha\bullet F\beta)\bullet F\gamma"] & & \\
& & (Ff'\circ Fg')\circ Fh' \arrow[dl, Rightarrow, "\mu\bullet 1"] \arrow[rr, Rightarrow, "a"'] & & Ff'\circ(Fg'\circ Fh') \arrow[dr, Rightarrow, "1\bullet\mu"'] & & \\
& F(f'\circ g')\circ Fh' \arrow[dl, Rightarrow, "\mu"] \arrow[rrrr, phantom, "\text{hex}"] & & & & Ff'\circ F(g'\circ h') \arrow[dr, Rightarrow, "\mu"'] & \\
F((f'\circ g')\circ h') \arrow[rrrrrr, Rightarrow, "F(a)"'] & & & & & & F(f'\circ(g'\circ h'))
\arrow[from=1-1, to=6-2, phantom, "\text{nat}"]
\arrow[from=2-2, to=5-3, phantom, "\text{nat}"]
\arrow[from=2-6, to=5-5, phantom, "\text{nat}"]
\arrow[from=1-7, to=6-6, phantom, "\text{nat}"]
\end{tikzcd}
$$

Figure 5: Coherence of the associator

$$
\begin{tikzcd}[column sep=tiny]
 & & & & & & F((f\circ g)\circ(f\circ i)) \arrow[ddrrrrrr, Rightarrow, "F(a)"] & & & & & & \\
 & & & & & & F(f\circ g)\circ F(h\circ i) \arrow[u, Rightarrow, "\mu"] & & & & & & \\
F(((f\circ g)\circ h)\circ i) \arrow[uurrrrrr, Rightarrow, "F(a)"] \arrow[ddddddd, Rightarrow, "F(a\bullet 1)"'] & & \text{hex} & & F(f\circ g)\circ(Fh\circ Fi) \arrow[urr, Rightarrow, "1\bullet\mu"] & & \text{func} & & (Ff\circ Fg)\circ F(h\circ i) \arrow[ull, Rightarrow, "\mu\bullet 1"'] \arrow[ddr, Rightarrow, "a"] & & \text{hex} & & F(f\circ(g\circ(h\circ i))) \\
 & F((f\circ g)\circ h)\circ Fi \arrow[ul, Rightarrow, "\mu"] \arrow[ddddd, Rightarrow, "F(a)\bullet 1"'] & & & & & & & & & & Ff\circ F(g\circ(h\circ i)) \arrow[ur, Rightarrow, "\mu"'] & \\
 & & & F(f\circ g)\circ Fh)\circ Fi) \arrow[ull, Rightarrow, "\mu\bullet 1"] \arrow[uur, Rightarrow, "a"] & \text{nat} & & (Ff\circ Fg)\circ(Fh\circ Fi) \arrow[uull, Rightarrow, "\mu\bullet 1"] \arrow[uurr, Rightarrow, "1\bullet\mu"'] \arrow[dr, Rightarrow, "a"] & & \text{nat} & Ff\circ(Fg\circ F(h\circ i)) \arrow[urr, Rightarrow, "1\bullet\mu"'] & & & \\
 & \text{nat} & & & & ((Ff\circ Fg)\circ Fh)\circ Fi \arrow[ull, Rightarrow, "(\mu\bullet 1)\bullet 1"] \arrow[ur, Rightarrow, "a"] \arrow[d, Rightarrow, "a\bullet 1"'] & & Ff\circ(Fg\circ(Fh\circ Fi)) \arrow[urr, Rightarrow, "1\bullet a"'] & & & & \text{nat} & \\
 & & \text{hex} & & & (Ff\circ(Fg\circ Fh))\circ Fi \arrow[rr, Rightarrow, "a"'] \arrow[dll, Rightarrow, "(1\bullet\mu)\bullet 1"'] \arrow[drr, phantom, "\text{nat}"] & & Ff\circ((Fg\circ Fh)\circ Fi) \arrow[u, Rightarrow, "1\bullet a"'] \arrow[drr, Rightarrow, "1\bullet(\mu\bullet 1)"] & & & \text{hex} & & \\
 & & & (Ff\circ F(g\circ h))\circ Fi \arrow[rrrrrr, Rightarrow, "a"'] \arrow[dll, Rightarrow, "\mu\bullet 1"'] & & & & & & Ff\circ(F(g\circ h)\circ Fi) \arrow[drr, Rightarrow, "1\bullet\mu"] & & & \\
 & F(f\circ(g\circ h))\circ Fi \arrow[dl, Rightarrow, "\mu"'] & & & & & \text{hex} & & & & & Ff\circ(F(g\circ h)\circ i) \arrow[dr, Rightarrow, "\mu"] \arrow[uuuuu, Rightarrow, "1\bullet F(a)"'] & \\
F((f\circ(g\circ h))\circ i) \arrow[rrrrrrrrrrrr, Rightarrow, "F(a)"'] & & & & & & & & & & & & F(f\circ((g\circ h)\circ i)) \arrow[uuuuuuu, Rightarrow, "F(1\bullet a)"']
\end{tikzcd}
$$

The outermost pentagon is Coherence Condition (2) for C and the innermost pentagon is Coherence Condition (2) for D.

$\square$

Theorem 26. *Every bicategory is equivalent to a skeletal bicategory.*

Proof. Let $C = (C^0, C, \circ_C, l^C, r^C, a^C)$ be a bicategory. For every equivalence class of objects $[x] \in C^0$, we choose a representative x and for all $y \in [x]$ we choose an equivalence $(f_y, \bar{f}_y, \epsilon_y, \eta_y) : y \to x$. Let D^0 be the set of representative objects. For every isomorphism class of 1-arrows $[f]$ in $C(x,y)$, where $x, y \in D^0$, we choose a representative f and an invertible 2-arrow $\alpha_g : g \Rightarrow f$ for all $g \in [f]$. For simplicity, we choose the equivalence $(1_x, 1_x, l_{1_x}, l_{1_x})$ for all $x \in D^0$, and we choose the representative 1-arrow 1_x for all $x \in D^0$.

Having chosen all our representatives, we now construct a bicategory $D = (D^0, D, \circ_D, l^D, r^D, a^D)$ and a pseudofunctor $P = (P^0, P, \mu, \lambda) : C \to D$ with the following data:

- The set D^0 consists of our representative objects.
- For all x, y in D^0, the category $D(x,y)$ is the full subcategory of $C(x,y)$ containing exactly the chosen 1-arrows.
- The map P^0 maps objects y to its chosen representative $x \cong y$.
- The functor P sends 1-arrows $g : x \to y$ to $\alpha(f_y \circ g \circ \bar{f}_x) : P^0(x) \to P^0(y)$ and 2-arrows $\beta : g \Rightarrow h$ to $\alpha(f_y \circ g \circ \bar{f}_x)^{-1} \cdot (1_{f_y} \bullet \beta \bullet 1_{\bar{f}_x}) \cdot \alpha(f_y \circ h \circ \bar{f}_x) : P(g) \Rightarrow P(h)$.
- The horizontal product is defined on 1-arrows by $f \circ_D g := P(f \circ_C g)$ and on 2-arrows by $\beta \bullet_D \gamma := P(\beta \bullet_C \gamma)$.
- The natural isomorphism μ is given by $\mu_{f,g} := ((\alpha_f^{-1} \bullet_C \alpha_g^{-1}) \cdot \alpha_{f \circ_C g}$.
- The natural isomorphism λ is given by $\lambda_y := 1_{1_{P^0(y)}}$
- The natural isomorphisms l^D,r^D,a^D are defined by Coherence Conditions (5) and (6).

The following diagram shows that μ is natural:

$$\begin{tikzcd}
Pg \circ Ph \arrow[r, Rightarrow, "\alpha_g^{-1} \bullet \alpha_h^{-1}"'] \arrow[rr, Rightarrow, bend left, "\mu"] \arrow[d, Rightarrow, "\alpha_g^{-1} \bullet \alpha_h^{-1}"] \arrow[ddd, Rightarrow, bend right, "P(\beta \bullet \gamma)"'] & g \circ h \arrow[r, Rightarrow, "\alpha_{g \circ h}"'] & P(g \circ h) \arrow[d, Rightarrow, "\alpha_{g \circ h}^{-1}"'] \arrow[ddd, Rightarrow, bend left, "P(\beta) \bullet P(\gamma)"] \\
g \circ h \arrow[d, Rightarrow, "\beta \bullet \gamma"] & & g \circ h \arrow[d, Rightarrow, "\beta \bullet \gamma"'] \\
g' \circ h' \arrow[d, Rightarrow, "\alpha_g \bullet \alpha_h"] & & g' \circ h' \arrow[d, Rightarrow, "\alpha_{g' \circ h'}"'] \\
P(g') \circ P(h') \arrow[r, Rightarrow, "\alpha_g^{-1} \bullet \alpha_h^{-1}"] \arrow[rr, Rightarrow, bend right, "\mu"'] & g' \circ h' \arrow[r, Rightarrow, "\alpha_{g' \circ h'}"] & P(g' \circ h')
\end{tikzcd}$$

In Lemma 25, we proved that l^D, r^D, a^D constructed in this way are natural and coherent. Therefore D is a bicategory and P is a pseudofunctor. By construction, D is skeletal. The pseudofunctor P satisfies the Whitehead criteria. Therefore, P is an equivalence. $\square$

We now complete the proof of the Whitehead Theorem for bicategories.

Lemma 27. Let C, D, E be bicategories and let $F : C \to D$, $G : D \to E$ be pseudofunctors satisfying the Whitehead criteria.

Then $GF : C \to E$ satisfies the Whitehead criteria.

Proof. GF is essentially surjective because pseudofunctors preserve equivalences; GF is essentially full because functors preserve isomorphisms; and GF is 2-fully faithful because a composition of bijective maps is bijective. $\square$

Lemma 28. Let C, D be bicategories, and let $F = (F^0, F^1, \mu, \lambda) : C \to D$ be an equivalence of bicategories. Then for any two objects $x, y \in C^0$, the functor $F^1 : C(x, y) \to D(F(x), F(y))$ is an equivalence of categories.

Proof. Let $G : D \to C$ be a weak inverse of F. The composition GF is equivalent to 1_C. For a pair of objects $x, y \in C^0$, this means that $G^1F^1(C(x, y)) \cong 1(C(x, y))$. In the other direction, we get $F^1G^1 \cong 1$. Therefore, F^1 is an equivalence of categories with weak inverse G^1. $\square$

The 2-out-of-3 property is also used to prove the Whitehead Theorem. It holds for general n-categories, and is discussed in a more general setting in [8].

Theorem 29 (2-out-of-3 property for Bicategories)**.** *Equivalences of bicategories satisfy the* 2-out-of-3 property*: If two of the pseudofunctors F, G and GF are equivalences, then so is the third.*

Theorem 30 (The Whitehead Theorem for Bicategories)**.** *Let C, D be bicategories and let $F : C \to D$ be a pseudofunctor.*

The pseudofunctor F is an equivalence if and only if it satisfies the Whitehead criteria.

Proof. First we show the direction that the Whitehead criteria implies equivalence. Assume that F satisfies the Whitehead criteria. Consider the diagram

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[dr, "SF"'] & D \arrow[d, "S"] \\ & \mathrm{Skel}(D) \end{tikzcd}$$

where S is the equivalence constructed in Theorem 26. The pseudofunctor SF is an equivalence by Theorem 24 and Lemma 27. Therefore, by the 2-out-of-3 property, F is an equivalence.

For the other direction, assume that F is an equivalence. Let $G : D \to C$ be a pseudoinverse to F, meaning G satisfies $FG \cong 1_D$ and $GF \cong 1_c$. The equivalence $FG \cong 1_D$ implies that FG is essentially surjective, and it follows that F is essentially surjective.

By Lemma 28, the functors $C(x, y) \to D(F^0x, F^0y)$ are equivalences, and therefore they are essentially surjective and fully faithful. This implies that F is essentially full and 2-fully-faithful. □

We conclude this section by stating the classification of bigroups, which no longer need to be skeletal.

Theorem 31 (Classification of Bigroups)**.** *Bigroups can be classified up to equivalence by the following data:*

- *a group G;*
- *a G-module M;*
- *a 3-cocycle $\omega : M^3 \to G$.*

Proof. The skeleton of a bigroup is a skeletal bigroup. We can see this because the construction of a skeleton as described in Theorem 26 does not introduce any non-invertible arrows. Therefore we can apply the classification of skeletal bigroups (Theorem 15) to general bigroups. □

4 Strictification of Equivalences

4.1 Strict Equivalences

When working with strict bicategories, we might want to avoid using non-strict lax functors, and also avoid transformations and modifications. Using the Whitehead criteria, it is still possible to define an equivalence with these restrictions:

Definition 32. Let C, D be strict bicategories. An *elementary equivalence of strict bicategories* is a strict pseudofunctor $F : C \to D$ that satisfies the Whitehead criteria.

Definition 33. Two strict bicategories are called *strictly equivalent* if they are connected by a zig-zag of strict equivalences.

The "zig-zags" are necessary to turn elementary equivalences, which have a direction, into an equivalence relation. By zig-zag, we mean a path which does not necessarily respect the direction of the elementary equivalence. For example, two strict bicategories C, F are equivalent if they are connected by elementary equivalences in the following way:

$$C \longrightarrow D \longleftarrow E \longrightarrow F$$

The aim of this chapter is to show that the above defined strict equivalence is no different from an equivalence of bicategories. Two strict bicategories are equivalent iff they are strictly equivalent. One direction (strictly equivalent $\implies$ equivalent) is trivial.

As motivation for our definition, we will first discuss equivalences of crossed modules.

Definition 34. A *crossed module* (G, H, τ, α), often written as $H \overset{\tau}{\Rightarrow} G$, consists of:

- two groups G, H;
- a group homomorphism $\tau : H \to G$;
- a G-action on H $\alpha : G \to \mathrm{Aut}(H)$;

such that the following diagrams commute:

$$\begin{tikzcd}
H \times H \arrow[rr, "\tau \times 1"] \arrow[dr, "\text{Adjoint}"'] & & G \times H \arrow[dl, "\alpha"] \\
& H &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times H \arrow[r, "\alpha"] \arrow[d, "1 \times \tau"] & H \arrow[d, "\tau"] \\
G \times G \arrow[r, "\text{Adjoint}"'] & G
\end{tikzcd}$$

Definition 35. Let $(G_1, H_1, \tau_1, \alpha_1), (G_2, H_2, \tau_2, \alpha_2)$ be two crossed modules. A *crossed module homomorphism* (γ, δ) consists of two group homomorphisms $\gamma : G_1 \to G_2$, $\delta : H_1 \to H_2$ such that the following diagrams commute:

$$
\begin{tikzcd}
H_1 \arrow[r, "\delta"] \arrow[d, "\tau_1"] & H_2 \arrow[d, "\tau_2"] \\
G_1 \arrow[r, "\gamma"] & G_2
\end{tikzcd}
\qquad
\begin{tikzcd}
G_1 \times H_1 \arrow[r, "\gamma\times\delta"] \arrow[d, "\alpha_1"] & G_2 \times H_2 \arrow[d, "\alpha_2"] \\
H_1 \arrow[r, "\delta"] & H_2
\end{tikzcd}
$$

Crossed modules are just another way to represent the data of a strict 2-group, and crossed module homomorphisms correspond to strict pseudofunctors. This is discussed in detail in [7]. What is especially interesting to us is the definition of an equivalence of crossed modules.

Definition 36. An *elementary equivalence* of crossed modules $(G_1, H_1, \tau_1, \alpha_1)$, $(G_2, H_2, \tau_2, \alpha_2)$ is a crossed module homomorphism that induces isomorphisms $\ker(\tau_1) \cong \ker(\tau_2)$ and $\mathrm{coker}(\tau_1) \cong \mathrm{coker}(\tau_2)$.

Two crossed modules are *equivalent* if and only if they are connected by a zig-zag of elementary equivalences.

If we view crossed modules as bigroups, then $\ker(\tau)$ corresponds to the 2-arrows $1 \Rightarrow 1$ of the bigroup, and $\mathrm{coker}(\tau)$ corresponds to equivalence classes of 1-arrows. Then this definition corresponds to the definition of strictly equivalent strict bicategories.

Lemma 37. Two crossed modules are equivalent if and only if their corresponding strict 2-groups are strictly equivalent.

Proof. Let F be a pseudofunctor. We will show that the definitions of elementary equivalence coincide.

First, assume that the pseudofunctor F satisfies the Whitehead criteria. The 2-fully-faithfulness condition implies that F is bijective on 2-arrows $1 \Rightarrow 1$. The essential fullness condition implies that F is surjective on equivalence classes of 1-arrows. From 2-fully-faithfulness, it follows that F preserves isomorphisms of 1-arrows. Therefore, for 1-arrows f and g, we have $F(f) \cong F(g)$ implies $f \cong g$. It follows that F is injective on equivalence classes of 1-arrows.

Now assume conversely that F is bijective on 2-arrows $1 \Rightarrow 1$, and F is bijective on equivalence classes of 1-arrows. Then F is 2-fully-faithful and essentially full. The pseudofunctor F is also essentially surjective because strict bigroups contain only 1 object. $\square$

4.2 The Bicategorical Yoneda Embedding

Definition 38. Let C, D be bicategories. Then $[C, D]$ is the bicategory of pseudofunctors, pseudonatural transformations, and modifications from C to D.

Remark 39. Let $\sigma^1, \sigma^2, \sigma^3$ be three composable transformations in $[C, D]$. The associator of $[C, D]$ consists of 2-arrows $(\sigma^1_x \circ \sigma^2_x) \circ \sigma^3_x \Rightarrow \sigma^1_x \circ (\sigma^2_x \circ \sigma^3_x)$. This is given by the associator in D. Similarly, the unitor of $[C, D]$ has components given by the unitor of D.

This means that if D is strict, then $[C, D]$ is also strict.

Definition 40. Let C be a bicategory. C has three opposite bicategories:

- the bicategory C^{op}, in which only the 1-arrows are reversed;
- the bicategory C^{co}, in which only the 2-arrows are reversed;
- the bicategory $C^{\text{co op}}$, in which both the 1-arrows and 2-arrows are reversed.

Definition 41 (Bicategorical Yoneda embedding)**.** Let C be a bicategory. The *Yoneda embedding* $\mathbb{Y} = (\mathbb{Y}^0, \mathbb{Y}, \mu^{\mathbb{Y}}, \lambda^{\mathbb{Y}})$ is a pseudofunctor $C \to [C^{\text{op}}, \text{Cat}]$ given by the following data:

- $\mathbb{Y}^0$ sends an object $a \in C^0$ to the pseudofunctor $\mathbb{Y}^0(a) : C \to \text{Cat}$, consisting of the following data:
  - $\mathbb{Y}^0(a)$ sends an object $x \in C^0$ to the category $C(a, x)$.
  - $\mathbb{Y}^0(a)$ sends a 1-arrow $f \in C$ to the functor $f_* : C(a, x) \to C(a, y)$, which sends a 1-arrow $\phi \in C(a, x)$ to $(f \circ \phi) \in C(a, y)$, and sends a 2-arrow $\alpha \in C(a, x)$ to $(1_f \bullet \alpha) \in C(a, y)$.
  - $\mathbb{Y}^0(a)$ sends a 2-arrow $\alpha \in C$ to the post-composition functor $\alpha_* : C(a, x) \to C(a, y)$.
  - $\mu^{\mathbb{Y}^0(a)}_{f,g} : f_* \circ g_* \to (f \circ g)_*$ sends a composition $f \circ (g \circ \phi)$ to the composition $(f \circ g) \circ \phi$. It is given by the inverse associator of C.
  - Similarly, $\lambda^{\mathbb{Y}^0(a)}_f$ is given by the inverse left unitor of C.
- $\mathbb{Y}$ sends a 1-arrow $\phi \in C(a, b)$ to the pseudonatural transformations $\mathbb{Y}(\phi) : \mathbb{Y}^0(b) \Rightarrow \mathbb{Y}^0(a)$, consisting of the following data:
  - For an object $x \in C^0$, the transformation $\mathbb{Y}(\phi)$ has a component $\mathbb{Y}(\phi)_x : C(b, x) \to C(a, x)$ which sends a 1-arrow $f \in C(b, x)$ to $f \circ \phi \in C(a, x)$, and sends a 2-arrow $\alpha \in C(b, x)$ to $(\alpha \bullet 1_\phi) \in C(a, x)$.
  - For a morphism $f \in C(x, y)$, the transformation $\mathbb{Y}(\phi)$ has a component $\phi_1 \circ (f \circ \phi_2) \Rightarrow (\phi_1 \circ f) \circ \phi_2$ given by the inverse of the associator of C.
- $\mathbb{Y}$ sends a 2-arrow $\alpha \in C(\phi_1, \phi_2)$ to the modification $\mathbb{Y}(\alpha) : \mathbb{Y}(\phi_1) \Rrightarrow \mathbb{Y}(\phi_2)$ that sends f to $1_f \bullet \alpha$.
- Let ϕ, ψ be composable 1-arrows in C. Then $\mu^{\mathbb{Y}}_{\phi,\psi}$ is a modification $\mathbb{Y}(\phi) \circ \mathbb{Y}(\psi) \Rrightarrow \mathbb{Y}(\psi \circ \phi)$. On an object x, this modification is given by 2-arrows $\mathbb{Y}(\psi)(x) \circ \mathbb{Y}(\phi)(x) \Rightarrow \mathbb{Y}(\psi \circ \phi)(x)$, which go from $(h \circ \psi) \circ \phi$ to $h \circ (\psi \circ \phi)$. These are given by the associator of C.
- Let a be an object in C. Then $\lambda^{\mathbb{Y}}_a$ is a modification $1_{\mathbb{Y}^0(a)} \Rrightarrow \mathbb{Y}(1_a)$. Its components at $x \in C^0$ maps $h \in C(a, x)$ to $(h \circ 1_a) \in C(a, x)$. This is given by the inverse of the right unitor of C.

Remark 42 (Strictification of bicategories)**.** The Yoneda embedding is 2-fully-faithful ([6]). When restricted to its image, the Yoneda embedding is essentially surjective, essentially full and 2-fully-faithful. This means that every bicategory is equivalent to a subcategory of Cat. Therefore, every bicategory is equivalent to a strict bicategory.

This can be used to prove the coherence theorem for bicategories, as explained in [5].

Remark 43. It follows from the definition that if C is a strict bicategory, then $\mathbb{Y}$ is a strict pseudofunctor.

4.3 Strictification of Equivalences of Bicategories

Definition 44. Let C be a bicategory and let $D \subseteq C$ be a sub-bicategory. The *repletion* $\overline{D}$ of D is the smallest sub-bicategory of C such that:

- $\overline{D}$ contains D.
- For any equivalence $(f, g, \epsilon, \eta) : c \to d$ in C with $c \in \overline{D}$, we have $c, d, f, g, \epsilon, \eta \in \overline{D}$.
- For any isomorphism $\alpha : f \to g$ in C with $f \in \overline{D}$, we have $f, g, \alpha, \alpha^{-1} \in \overline{D}$.

Let $F : B \to C$ be a lax functor. The *essential image* $\overline{F(B)} \subseteq C$ is the repletion of the image of F.

We will now use the Yoneda embedding to create a strictification procedure for equivalences. Let C, D be strict bicategories and let $F : C \to D$ be an equivalence. Consider the diagram:

$$\begin{tikzcd} C \arrow[r, "\mathbb{Y}"] \arrow[d, "F"'] & \overline{\mathbb{Y}(C)} \\ D \arrow[r, "\mathbb{Y}"'] & \mathbb{Y}(D) \arrow[u, dashed, "F^*"'] \end{tikzcd}$$

Our goal is to construct a strict equivalence $F^* : \mathbb{Y}(D) \to \overline{\mathbb{Y}(C)}$, the existence of which will imply that C and D are strictly equivalent. Note that this diagram will only commute up to equivalence.

Lemma 45. Let C, D be strict bicategories, and let $F : C \to D$ be a pseudofunctor. Then F defines a strict pullback pseudofunctor $F^\times : [D^{\mathrm{op}}, \mathrm{Cat}] \to [C^{\mathrm{op}}, \mathrm{Cat}]$ by pre-composition.

Proof. The pullback $F^\times$ is defined in the following way:

- On objects, a pseudofunctor $G : D^{\mathrm{op}} \to \mathrm{Cat}$ is sent to $GF^{\mathrm{op}} : C^{\mathrm{op}} \to \mathrm{Cat}$.
- On 1-arrows, a transformation $\sigma : G \Rightarrow H$ consists of a 1-arrow and 2-arrows in Cat corresponding to objects and 1-arrows in D. Therefore $F^\times(\sigma) : GF^{\mathrm{op}} \Rightarrow HF^{\mathrm{op}}$ inherits its data from σ. This is automatically strict.

- On 2-arrows, similar to 1-arrows, a modification $F^\times(\Gamma)$ inherits its data from Γ. □

Lemma 46. Let C, D be strict bicategories, and let $F : C \to D$ be an equivalence. Then the pullback pseudofunctor $F^\times : [D^{\mathrm{op}}, \mathrm{Cat}] \to [C^{\mathrm{op}}, \mathrm{Cat}]$ gives a strict equivalence of bicategories $\mathbb{Y}(D)$ and $\overline{\mathbb{Y}(C)}$.

Proof. First we show that $\overline{\mathbb{Y}(C)}$ contains $F^\times(\mathbb{Y}(D))$. We will show that for any object $a \in C^0$, the pseudofunctor $D(Fa, F-)$ is equivalent to $C(a,-)$, and therefore lies in $\overline{\mathbb{Y}(C)}$. We construct a pseudonatural transformation $\sigma : C(a,-) \Rightarrow D(Fa, F-)$ with the following data:

- On objects x, the component $\sigma_x : C(a,x) \Rightarrow D(Fa, Fx)$ is given by F.
- On arrows $f : x \to y$, the component $\sigma_f : Ff_* \circ \sigma_x \Rrightarrow \sigma_y \circ f_*$ is given by $(\mu^F)^{-1}$, shown by the diagram:

$$\begin{tikzcd}
(a \xrightarrow{\psi} x) \arrow[rr, "\sigma_x"] \arrow[d, "f_*"] & & (Fa \xrightarrow{F\psi} Fx) \arrow[d, "Ff_*"] \\
(a \xrightarrow{\psi} x \xrightarrow{f} y) \arrow[r, "\sigma_y"'] & (Fa \xrightarrow{F(f\circ\psi)} Fy) \arrow[r, "\sigma_f=\mu^{-1}"'] & (Fa \xrightarrow{F\psi} Fx \xrightarrow{Ff} Fy)
\end{tikzcd}$$

- After making substitutions, Coherence Condition (7) becomes Coherence Condition (5) for F, and Coherence Condition (8) becomes the middle triangle identity. Therefore, this transformation is coherent.

From the essential fullness and 2-fully-faithfulness of F, it follows that σ is essentially surjective and fully faithful, and therefore an equivalence. Thus we have shown that the objects of $F^\times(\mathbb{Y}(D))$ lie in $\overline{\mathbb{Y}(C)}$. It follows that the arrows and 2-arrows of $F^\times(\mathbb{Y}(D))$ must also lie in $\overline{\mathbb{Y}(C)}$.

Finally we show that $F^\times$ satisfies the Whitehead criteria. Every object in $\overline{\mathbb{Y}(C)}$ is equivalent to one in $\mathbb{Y}(C)$, and by the above paragraph, it is equivalent to an object in $F^\times(\mathbb{Y}(D))$. This is the essential surjectivity property. The essential fullness property holds for similar reasons. □

These results allow us to prove that all equivalences between strict bicategories can be strictified.

Theorem 47. *Any two equivalent strict 2-categories are also strictly equivalent.*

Proof. Let C, D be strict bicategories and $F : C \to D$ an equivalence. Then consider the following zig-zag:

$$C \xrightarrow{\mathbb{Y}} \overline{\mathbb{Y}(C)} \xleftarrow{F^\times} \mathbb{Y}(D) \xleftarrow{\mathbb{Y}} D$$

By Remarks 42 and 43, $\mathbb{Y}$ is a strict equivalence. By Lemma 46, $F^\times$ is a strict equivalence. Therefore C and D are connected by a zig-zag of strict equivalences. □

It follows that our classification of 2-groups also applies to crossed modules, with crossed module equivalences:

Theorem 48 (Classification of crossed modules)**.** *Crossed modules are classified up to equivalence by the following data:*

- *a group G;*
- *a G-module M;*
- *a class in the third cohomology $H^3(G, M)$, given by a 3-cocycle $\omega : G^3 \to M$.*

The crossed modules associated to two such triples (G_i, M_i, ω_i) for $i = 1, 2$ are equivalent if and only if there are isomorphisms $G_1 \cong G_2$ and $M_1 \cong M_2$ such that the induced isomorphism $H^3(G_1^3, M_1) \cong H^3(G_2^3, M_2)$ maps ω_1 to ω_2.

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