Stammeier's C^*-algebras for several injective group endomorphisms as C^*-algebras of diagrams of étale groupoid correspondences

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Contents

1 Introduction

Nicolai Stammeier ([26]) introduces and studies *irreversible algebraic dynamical systems* (G, P, θ), where G is a countable group, P is a countably generated free abelian monoid and θ is an action of P on G by injective endomorphisms fulfilling a certain independence condition. He associates C^*-algebras $\mathcal{O}[G, P, \theta]$.

In his thesis ([1]), Suliman Albandik describes a homomorphism from a bicategory $\mathfrak{Gr}_{\text{prop}}$ of proper locally compact étale groupoids and groupoid correspondences to a bicategory of C^*-algebras and C^*-correspondences introduced by Alcides Buss, Ralf Meyer and Chengchang Zhu in [10]. He constructs bicategorial (co)limits[2] for Ore monoid shaped diagrams of such correspondences. Further, he shows that the homomorphism of bicategories, which maps the diagrams to product systems, maps the limit constructions to the Cuntz-Pimsner algebras of the product systems.

Much of the theory in Albandik's thesis has been extended by Ralf Meyer and his students Celso Antunes and Joanna Ko in [5], [20], [15]. In many cases, those will be our main sources for this part of the theory we use. Especially the construction of the groupoid model is heavily based on [20], in which Meyer provides a construction different from Albandik's.

We make use of the theory above as follows: First, we interpret Stammeier's irreversible algebraic dynamical systems (G, P, θ) as P-shaped diagrams in the category of injective (discrete) group endomorphisms. Then we describe a homomorphism from this category into the bicategory of locally compact étale groupoid correspondences, which contains Albandik's bicategory. In the finite-type case, that is, under the assumption that $\theta_p(G)$ has finite index for all $p \in P$, applying this homomorphism to an irreversible algebraic dynamical system results in a diagram in Albandik's subbicategory. Albandik's theory now gives us a product system, a locally compact étale groupoid, and a C^*-algebra being the Cuntz-Pimsner algebra of the product system and the groupoid C^*-algebra of the groupoid.

In fact, Stammeier, too, reinterprets his C^*-algebra in the finite-type case as the Cuntz-Pimsner algebra of a product system. By comparing the product systems, we conclude that in the finite-type case, Stammeier's C^*-algebra $\mathcal{O}[G, P, \theta]$ coincides with the C^*-algebra obtained through Albandik's and Meyer's theory. Another way to see this, using Meyer's construction of a groupoid model, via two different crossed product constructions for inverse semigroup actions and for semigroup actions, respectively, on C^*-algebras, is roughly sketched.

We study how certain properties of the groupoid model depend on properties of the irreversible algebraic dynamical system and give a sufficient criterion for simplicity of the resulting C^*-algebra. Much of the above is formulated in the more general framework where P is an Ore monoid, and G an arbitrary group.

The possibility to reinterpret Stammeier's irreversible algebraic dynamical systems in terms of groupoid correspondences, as worked out in this master's thesis, is briefly mentioned in [5, Section 4; p. 1338]

In Section 2, we recall the required bicategorical language. In Section 3, we describe Stammeier's irreversible algebraic dynamical systems and some optional properties. In Section 4, we describe the appropriate version of the bicat-

[2]There are different versions of the bicategory of (certain) groupoid correspondences with the arrows going in different directions.

egory of locally compact étale groupoid correspondences, our homomorphism from injective group endomorphisms into this bicategory and thus an encoding of (slightly generalised) irreversible algebraic dynamical systems via groupoid correspondences. In Section 5, we construct a groupoid model for such an encoding (in fact for the more general Ore case), applying a construction by Meyer. In the finite-type case, this will turn out to be a groupoid model for the C^*-algebra of interest. Further, we sketch a way to show this without product systems by comparing Meyer's construction with Stammeier's via crossed product constructions. In Section 6, we apply Albandik's homomorphism of bicategories to obtain a product system and a groupoid C^*-algebra, which in the finite-type case is the Cuntz-Pimsner algebra of the former, and by comparing the product system to a product system provided by Stammeier, conclude that we indeed describe Stammeier's C^*-algebra in the finite type case. In Section 7, we provide sufficient criteria for the groupoid model to be Hausdorff, effective, minimal and locally contracting, respectively, and a sufficient criterion for the associated groupoid C^*-algebra to be simple.

2 Bicategories, homomorphisms of bicategories and diagrams

In order to reinterpret a type of dynamical system and the construction of an associated C^*-algebra by Stammeier (see [26]) in terms of diagrams of groupoid correspondences, groupoid models, and their groupoid C^*-algebras, we need some bicategorial language.

First, we recall what bicategories are. A bicategory has objects like a category. It has also 1-arrows between objects, corresponding roughly to arrows in categories. A major difference is that their composition is only associative up to what is called 2-arrows. A 2-arrow has as domain and codomain parallel 1-arrows. Analogously to associativity, the multiplication with identity 1-arrows only acts trivially up to 2-arrows. The 2-arrows are part of the data of a bicategory. For fixed objects a, b, the 1-arrows $a \to b$ are the objects of a category $\mathrm{Hom}(a, b)$ where the 2-arrows between 1-arrows $a \to b$ are the morphisms. The multiplication of 1-arrows $b \to c$, $a \to b$ extends to a bifunctor $\mathrm{Hom}(b, c) \times \mathrm{Hom}(a, b) \to \mathrm{Hom}(a, c)$.

Definition 2.1 (see [19, Section 1.0] and [1, Definition 2.1])**.** A bicategory $\mathcal{B}$ is given by the data

1. a class of objects $\mathcal{B}^0$,

2. a category $\mathcal{B}(a, b)$ whose objects are called 1-arrows $a \to b$, and whose morphisms are called 2-arrows $f \to g$ between such 1-arrows $f, g \in \mathcal{B}(a, b)$, for all objects $a, b \in \mathcal{B}^0$. For the concatenation of morphisms in the category $\mathcal{B}(a, b)$, which we call vertical composition of 2-arrows in $\mathcal{B}$, we write $\alpha_1 \circ \alpha_2$, for $\alpha_1 \colon f_1 \to f_2$, $\alpha_2 : f_2 \to f_3$, $f_1, f_2, f_3 \in \mathcal{B}^0(a, b)$.

3. a bifunctor $\mathbf{c}_{a,b,c} \colon \mathcal{B}(b, c) \times \mathcal{B}(a, b) \to \mathcal{B}(a, c)$, encoding the concatenation of 1-arrows and the horizontal[3] composition of 2-arrows, for each triple of

[3]For horizontal and vertical composition of 2-arrows, see [23, p. 45], there defined for the 2-category of categories, which is a particular bicategory.

objects $a, b, c \in \mathcal{B}^0$ (we also write $\beta * \alpha := \mathbf{c}_{a,b,c}(\beta, \alpha)$, for $\beta \in \mathcal{B}^1(b, c)$, $\alpha \in \mathcal{B}^1(a, b)$, for the horizontal concatenation of 2-arrows, and $g \circ f := \mathbf{c}_{a,b,c}(g, f)$, for $g \in \mathcal{B}^0(b, c)$, $f \in \mathcal{B}^0(a, b)$, $a, b \in \mathcal{B}^0$, for the concatenation of 1-arrows),

4. a functor $I_a \colon 1 \to \mathcal{B}(a, a)$ (selecting a "unit" with respect to the composition of 1-arrows) for each object $a \in \mathcal{B}^0$, where 1 is the category with exactly one object and one (identity) arrow,

5. a natural isomorphism

$$\mathrm{assoc}_{a,b,c,d} \colon \mathbf{c}_{a,b,d} \circ (\mathbf{c}_{b,c,d} \times \mathrm{id}_{\mathcal{B}(a,b)}) \Rightarrow \mathbf{c}_{a,c,d} \circ (\mathrm{id}_{\mathcal{B}(c,d)} \times \mathbf{c}_{a,b,c}), \tag{1}$$

called the *associator*, for each quadruple of objects $a, b, c, d \in \mathcal{B}^0$, with the 2-arrows

$$\mathrm{assoc}_{h,g,f} \colon (h \circ g) \circ f \to h \circ (g \circ f) \tag{2}$$

for $f \in \mathcal{B}(a, b)^1$, $g \in \mathcal{B}(b, c)^1$, $h \in \mathcal{B}(c, d)^1$, as components,

6. natural isomorphisms $\mathfrak{l}_{a,b} \colon \mathbf{c}_{a,b,b} \circ (I_a \times \mathrm{id}_{\mathcal{B}(a,b)}) \Rightarrow p_2$ and $\mathfrak{r}_{a,b} \colon \mathbf{c}_{a,a,b} \circ (\mathrm{id}_{\mathcal{B}(a,b)} \times I_a) \Rightarrow p_1$, called *unitors*, for each pair of objects $a, b \in \mathcal{B}^0$, with components

$$\mathfrak{l}_f \colon I_b \circ f \to f \tag{3}$$
$$\mathfrak{r}_f \colon f \circ I_a \to f \tag{4}$$

for $f \in \mathcal{B}(a, b)^0$ for $a, b \in \mathcal{B}^0$,

fulfilling the coherence axioms given in [19, Section 1.0]. A bicategory is called small, if its objects, 1-arrows and 2-arrows fit in a set.

Remark 2.2. Bicategories whose 2-arrows are all units correspond one-to-one to categories in the obvious way: The objects are the same in the bicategory and in the category. The 1-arrows in such a bicategory become morphisms in the corresponding category. For the bifunctor $\mathbf{c}_{abc} \colon \mathcal{B}(b, c) \times \mathcal{B}(a, b) \to \mathcal{B}(a, c)$, for objects $a, b, c \in \mathcal{B}$, the map encoding its action on objects is the map $\mathrm{Hom}(b, c) \times \mathrm{Hom}(a, b) \to \mathrm{Hom}(a, c)$ encoding multiplication of arrows $g \colon b \to c$, $f \colon a \to b$ in the category. The data of assoc of a bicategory is choices of certain 2-arrows not involving a choice of domain or codomain, so it becomes trivial given that all 2-arrows are units. The existence of assoc then enforces the associativity of composition of morphisms in the associated purported category. Similarly, given the information that 2-arrows are units, $\mathfrak{r}_a$ and $\mathfrak{l}_a$, for objects $a \in \mathcal{B}^0$, do not contain any information about $\mathcal{B}$, but ensure that the associated purported category has the "units" of $\mathcal{B}$ as actual units and is thus a category. For categories, we use "$\cdot$" rather than "$\circ$" for composition of 1-arrows, that is, morphisms.

Definition 2.3 (compare [19, 1.1])**.** For bicategories $\mathcal{B}'$, $\mathcal{B}$, a homomorphism of bicategories $F \colon \mathcal{B}' \to \mathcal{B}$ has data $(F, F_{a,b}, \phi_{a,b,c}, \phi_a)$ with

- a map $F \colon \mathcal{B}'^0 \to \mathcal{B}^0$ between object classes,
- a functor $F = F_{a,b} \colon \mathcal{B}'(a, b) \to \mathcal{B}(Fa, Fb)$ for all objects $a, b \in \mathcal{B}'^0$ in $\mathcal{B}'$,

- natural isomorphisms $\phi_{a,b,c}\colon \mathbf{c}_{Fa,Fb,Fc} \circ (F_{b,c} \times F_{a,b}) \Rightarrow F_{a,c} \circ \mathbf{c}'_{a,b,c}$, for $a, b, c \in \mathcal{B}'^0$, with components $\phi_{g,f}\colon Fg \circ Ff \to F(g \circ f)$, for $g \in \mathcal{B}'(b,c)^0$, $f \in \mathcal{B}'(a,b)^0$,
- natural isomorphisms $\phi_a\colon I_{Fa} \Rightarrow F_{a,a} \circ I'_a$, for $a \in \mathcal{B}'^0$, each having a single component $\phi_a\colon I_{Fa} \to FI'_a$,

fulfilling the axioms

$$F(\mathrm{assoc}'_{h,g,f}) \circ \phi_{h\circ g,f} \circ (\phi_{h,g} * 1_{Ff}) = \phi_{h,g\circ f} \circ (1_{Fh} * \phi_{g,f}) \circ \mathrm{assoc}_{Fh,Fg,Ff} \tag{5}$$

$$\mathfrak{l}_{Ff} = F(\mathfrak{l}'_f) \circ \phi_{I_b,f} \circ (\phi_b * 1_{Ff}) \tag{6}$$

$$\mathfrak{r}_{Ff} = F(\mathfrak{r}'_f) \circ \phi_{f,I_a} \circ (1_{Ff} * \phi_a), \tag{7}$$

for $h \in \mathcal{B}'(c,d)^0$, $g \in \mathcal{B}'(b,c)^0$, $f \in \mathcal{B}'(a,b)^0$, for $a, b, c, d \in \mathcal{B}'^0$. A homomorphism of bicategories is called *strictly unital* if the components of ϕ_a are unit 2-arrows – implying $I_{Fa} = I'_a$ – for all objects $a \in \mathcal{B}'^0$. The term "strictly unital" is taken from [20, Proposition 3.1].

Remark 2.4. A homomorphism of bicategories of which the domain and codomain bicategory are both categories (in the sense of Remark 2.2) is equivalent to a functor in the most obvious way.

We describe strictly unital homomorphisms from small categories to bicategories, analogously to how diagrams in a particular bicategory of étale groupoid correspondences are described in [20, Proposition 3.1]. In Lemma 4.29, we specialise this description to diagrams with monoid shape in another version $\mathfrak{Gr}$ of the bicategory (following [5]). Furthermore, we use the following description to establish a strictly unital homomorphism of bicategories from the category $\mathfrak{GrpMn}$ to $\mathfrak{Gr}$ in Lemma 4.28.

Lemma 2.5 (compare [20, Proposition 3.1] and [1, Proposition 2.40])**.** *Let $\mathcal{C}$ be a small category and $\mathcal{B}$ a bicategory. A strictly unital homomorphism $\mathcal{C} \to \mathcal{B}$ is described by the data $(A_a, X_f, \phi_{g,f})$ with*

- *an object A_a for each object $a \in \mathcal{C}^0$,*
- *a 1-arrow $X_g \in \mathcal{B}(A_a, A_b)^0$ for each $g \in \mathcal{C}(a,b)$ for each pair $a, b \in \mathcal{C}^1$,*
- *an invertible 2-arrow $\phi_{g,f}\colon X_g \circ X_f \to X_{g\cdot f}$ for each pair of composable morphisms f, g in $\mathcal{C}$*

fulfilling the axioms

$$I_{A_a} = X_{I_a} \tag{8}$$

$$\phi_{hg,f} \circ (\phi_{h,g} * 1_{X_f}) = \phi_{h,gf} \circ (1_{X_h} * \phi_{g,f}) \circ \mathrm{assoc}_{h,g,f} \tag{9}$$

$$\mathfrak{l}_{X_f} = \phi_{I_b,f} \tag{10}$$

$$\mathfrak{r}_{X_f} = \phi_{f,I_a} \tag{11}$$

for all $h \in \mathcal{C}(c,d)$, $g \in \mathcal{C}(b,c)$, $f \in \mathcal{C}(a,b)$, for all $a, b, c, d \in \mathcal{C}^0$. The identity (9) expresses the commutativity of the following diagram, whose arrows represent

2-arrows in $\mathcal{B}$:

$$\begin{tikzcd}
(X_h \circ X_g) \circ X_f \arrow[r, "\phi_{h,g} * 1_{X_f}"] \arrow[d, "\mathrm{assoc}_{hg,f}"'] & X_{hg} \circ X_f \arrow[dr, "\phi_{hg,f}"] & \\
& & X_{hgf} \\
X_h \circ (X_g \circ X_f) \arrow[r, "1_{X_h} * \phi_{g,f}"] & X_h \circ X_{gf} \arrow[ur, "\phi_{h,gf}"'] &
\end{tikzcd}$$

Proof. We demonstrate how the description of a homomorphism $F\colon \mathcal{B}' \to \mathcal{B}$ between general bicategories in Definition 2.3 simplifies under the assumption that F is strictly unital and that the domain bicategory $\mathcal{B}'$ is a category (as in Remark 2.2). We first consider the data.

The object map $F\colon \mathcal{C}^0 \to \mathcal{B}^0$ remains the same.

The Hom-categories of $\mathcal{B}'$ are just sets, that is, categories with only identity-arrows; hence the functors between hom-categories are equivalent to maps (of sets) $\mathcal{B}'^0(a,b)^0 \to \mathcal{B}(A_a, A_b)^0$, for $a, b \in \mathcal{B}'^0$.

For $a, b, c \in \mathcal{B}'^0$, consider the natural isomorphism $\phi_{a,b,c}$ between the functors

$$\mathbf{c}_{Fa,Fb,Fc} \circ (F_{b,c} \times F_{a,b}),\ \ F_{a,c} \circ \mathbf{c}'_{a,b,c}\colon \mathcal{B}'(b,c) \times \mathcal{B}'(a,b) \to \mathcal{B}(a,c).$$

With the domain category of the functors becoming a set, the naturality requirement becomes vacuous. Then the natural isomorphism $\phi_{a,b,c}$ becomes equivalent to the family of its components

$$\phi_{g,f}\colon Fg \circ Ff \to F(g \circ f),\ \text{ for } g \in \mathcal{B}'(b,c)^0, f \in \mathcal{B}'(a,b)^0,$$

for $a, b, c \in \mathcal{B}'^0$.

The natural isomorphisms ϕ_a, for $a \in \mathcal{B}'^0$, become trivial by the assumption of strict unitality. So as data, they vanish; however they survive as an axiom: $I_{Fa} = FI'_a$, for all $a \in \mathcal{B}'^0$.

We now describe how the axioms simplify. To see how (5) simplifies, consider

$$\mathrm{assoc}'_{h,g,f}\colon (h \circ g) \circ f \to h \circ (g \circ f),$$

for $a, b, c, d \in \mathcal{B}'^0$ and $f \in \mathcal{B}'(a,b)^1$, $g \in \mathcal{B}'(b,c)^1$ and $h \in \mathcal{B}'(c,d)^1$. When $\mathcal{B}'$ is a category, then $\mathrm{assoc}'_{h,g,f}$ is the unit 2-arrow $1'_{hgf}$ on the 1-arrow $hgf \in \mathcal{B}'(a,d)^0$, hence its image under the functor $F\colon \mathcal{B}'(a,d) \to \mathcal{B}(Fa, Fd)$ is the identity 2-arrow $F(\mathrm{assoc}'_{hgf}) = 1_{F(hgf)}$ on $F(hgf) \in \mathcal{B}(Fa, Fd)$. Since this is a unit with respect to concatenation of 2-arrows in $\mathcal{B}'$, the term can be left out; this leaves (9).

To see how (6) simplifies, consider first the component $\mathfrak{l}'_f\colon I'_b \circ f \to f$ of the left unitor at $f \in \mathcal{B}'(a,b)$, for objects $a, b \in \mathcal{B}'^0$. When $\mathcal{B}'$ is a category, $\mathfrak{l}'_f$ is the unit 2-arrow on f. Its image under the functor $\phi_{a,b}\colon \mathcal{B}'(a,b) \to \mathcal{B}(a,b)$ is hence a unit 2-arrow, and the term $F(\mathfrak{l}'_f)$ vanishes. Next, consider $\phi_b\colon I_{Fb} \to FI'_a$. When we require strict unitality, then this is the 2-unit on I_{Fb}. Hence the term $(\phi_b * 1_{Ff})$ becomes $(1_{I_{Fb}} * 1_{Ff})$. This is the image of a unit morphism (in $\mathcal{B}'$, a 2-arrow) under the functor $\mathbf{c}_{a,b,b}\colon \mathcal{B}(b,b) \times \mathcal{B}(a,b) \to \mathcal{B}(a,b)$, hence a unit morphism, that is, in $\mathcal{B}$, a unit 2-arrow. Hence the term $(\phi_b * 1_{Ff})$ vanishes as well. This leaves (10). By a similar argument, (7) simplifies to (11). $\square$

Lemma 2.6. *Suppose that $\mathcal{B}''$, $\mathcal{B}'$, $\mathcal{B}$ are bicategories and $F' \colon \mathcal{B}'' \to \mathcal{B}'$, $F \colon \mathcal{B}' \to \mathcal{B}$ are homomorphisms of bicategories. Then the data for a homomorphism $F \circ F' \colon \mathcal{B}'' \to \mathcal{B}$, the composition of F and F', is given by*

- *the concatenation of maps $F \circ F' \colon \mathcal{B}''^0 \to \mathcal{B}^0$ as map between object classes,*
- *the concatenations of functors $F_{F'a,F'b} \circ F'_{a,b}$, for $a, b \in F''^0$, as functors encoding the mapping of 1- and 2-arrows,*
- *the natural isomorphisms*[4]

$$\begin{aligned}&(F_{F'a,F'c} * \phi'_{a,b,c}) \cdot (\phi_{F'a,F'b,F'c} * (F'_{b,c} \times F'_{a,b}))\colon\\ &\mathbf{c}_{FF'a,FF'b,FF'c} \circ (F_{F'b,F'c} \times F_{F'a,F'b}) \circ (F'_{b,c} \times F'_{a,b})\\ &\Rightarrow F_{F'a,F'c} \circ F'_{a,c} \circ \mathbf{c}''_{a,b,c}\end{aligned} \tag{12}$$

  providing the required natural isomorphisms related to multiplication of 1-arrows and horizontal concatenation of 2-arrows,
- *the natural isomorphisms*

$$(F_{F'a,F'a} * \phi'_a) \cdot \phi_{F'a} \colon I_{FF'a} \Rightarrow F_{F'a,F'a} \circ F'_{a,a} \circ I''_a \tag{13}$$

  providing the required natural isomorphisms related to the units of 1-arrows.

Proof. Compare the data to [14, Definition 4.1.24]. Homomorphisms of bicategories are called "lax functors" (see [14, Definition 4.2.1]) there, and the notational conventions are different. Then [14, Lemma 4.1.27] implies that the given data indeed defines a homomorphism of bicategories. The best way to an understanding of this definition might be drawing the diagrams with functors and their domains and the natural isomorphisms between them. □

Definition 2.7 (see [23, p. 5])**.** A *monoid* P is a small category with exactly one object. We write P to denote the set of its morphisms, too, and 1 to denote the unit arrow of the single object.

The term *diagram* can be used to refer to a homomorphism of bicategories where the domain bicategory is small (see [20, first paragraph of Section 3]). We define a monoid shaped diagram with the extra property of strict unitality.

Definition 2.8. A *monoid-shaped diagram* D in a bicategory $\mathcal{B}$ is a monoid P together with a strictly unital homomorphism of bicategories $P \to \mathcal{B}$.

Lemma 2.9 (compare [1, Definition 3.4])**.** *A monoid-shaped diagram in a bicategory $\mathcal{B}$ is of the form $D = (A, P, X_p, \mu_{p,q})$ with data*

- *an object $A \in \mathcal{B}^0$,*
- *a monoid P,*
- *a 1-arrow $X_p \in \mathcal{B}(A, A)^0$ for each $p \in P$,*

[4] Here, "$*$" symbolises whiskering of natural transformations and functors, see [23, Remark 1.7.6].

- *an invertible* 2*-arrow* $\mu_{p,q}\colon X_p \circ X_q \to X_{p\cdot q}$ *for any two* $p, q \in P$.

fulfilling

$$I_A = X_1, \tag{14}$$
$$\mu_{pq,r} \circ (\mu_{p,q} * 1_{X_r}) = \mu_{p,qr} \circ (1_{X_p} * \mu_{q,r}) \circ \mathrm{assoc}_{p,q,r}, \tag{15}$$
$$\mathfrak{l}_{X_p} = \mu_{1,p} \text{ and} \tag{16}$$
$$\mathfrak{r}_{X_p} = \mu_{p,1} \tag{17}$$

for all $p, q, r \in P$.

Proof. This follows immediately from Lemma 2.5. □

Remark 2.10. If the bicategory $\mathcal{B}$ is a category in the sense of Remark 2.2, then a monoid-shaped diagram $D = (A, P, X_p, \mu_{p,q})$ in $\mathcal{B}$ as in Lemma 2.9 simplifies to $D = (A, P, X_p)$ where $X\colon P \to \mathcal{B}$ is a functor and A is the single object in its image.

3 Stammeier's irreversible algebraic dynamical systems

In this section, we recall Stammeier's irreversible algebraic dynamical systems (see [26, Section 1]) and describe how they can be encoded as monoid-shaped diagrams in the category of (discrete) groups and group monomorphisms. We prove a lemma analysing the independence property, a property part of Stammeier's definition for his systems. We recall the definition of C^*-algebras $\mathcal{O}[G, P, \theta]$ associated by Stammeier to his irreversible algebraic dynamical systems. Then we introduce some conventions concerning properties of the monoid shaped diagrams encoding (sometimes a generalisation of) Stammeier's systems.

Definition 3.1. Let P be a free abelian monoid. Then p, q are *relatively prime*, if there are no $x, y \in P$, $z \in P \setminus \{1\}$ such that $p = zx$ and $q = zy$.

Remark 3.2. Stammeier defines what it means for $p, q \in P$ to be relatively prime in the case of a not necessarily free abelian or even commutative, but lattice ordered monoid P in [26, paragraph preceding Definition 1.5]: In this context, p, q are relatively prime, if their greatest common divisor $p \wedge q$ is 1. The canonical lattice order given on a free abelian monoid P is given by

$$p \leq q \text{ if and only if there exists } x \in P \text{ such that } q = px.$$

Then $p \wedge q$ is the largest monoid element such that there exist $y, z \in P$ such that $p = (p \wedge q)y$ and $q = (p \wedge q)z$. Then $p \wedge q = 1$ is equivalent to there being no $x, y \in P$, $z \in P \setminus \{1\}$ such that $p = za$ and $q = zb$, since in the free abelian P, 1 has no divisors except for itself. This shows that Definition 3.1 is compatible with the usage of the term "relatively prime" in [26].

Definition 3.3 (see [26, Proposition 1.1 and Definition 1.3])**.** Suppose that θ_1, θ_2 are commuting injective group endomorphisms of a (discrete) group G. Then θ_1 and θ_2 are *independent* if

$$\theta_1(G) \cap \theta_2(G) = \theta_1\theta_2(G). \tag{18}$$

We cite Stammeier's definition of an irreversible algebraic dynamical system:

Definition 3.4 ([26, Definition 1.5]). An *irreversible algebraic dynamical system* (G, P, θ) is

(A) a countably infinite, discrete group G with unit 1_G,

(B) a countably generated, free abelian monoid P with unit 1_P, and

(C) a P-action θ on G by injective group endomorphisms for which θ_p and θ_q are independent if and only if p and q are relatively prime.

An irreversible algebraic dynamical system (G, P, θ) is said to be

- *minimal*, if $\bigcap_{p\in P} \theta_p(G) = 1_G$,
- *commutative*, if G is commutative,
- *of finite type*, if $[G : \theta_p(G)]$ is finite for all $p \in P$, and
- *of infinite type*, if $[G : \theta_p(G)]$ is infinite for all $p \neq 1_P$.

Lemma 3.5. *Let $\theta_1, \dots, \theta_m$, $\theta'_1, \dots, \theta'_{m'}$ be commuting group endomorphisms of a group G, where $m, m' \in \mathbb{N}_{\geq 0}$. Then $\prod_{k=1}^{m} \theta_k$ and $\prod_{k'=1}^{m'} \theta'_{k'}$ are independent if and only if $\theta_k, \theta'_{k'}$ are independent for all $k \in \{1, \dots, m\}$, $k' \in \{1, \dots, m'\}$.*

Proof. [26, Lemma 1.4] states that for three commuting injective group endomorphisms $\theta_1, \theta_2, \theta_3$ of G, $\theta_1\theta_2$ and θ_3 are independent if and only if θ_1 and θ_3 are independent and θ_2 and θ_3 are independent. Applying this result inductively proves the statement of the lemma. □

Lemma 3.6. *An injective endomorphism θ in G is independent of itself if and only if it is surjective.*

Proof. If θ is surjective, then $\theta(G) \cap \theta(G) = \theta^2(G)$, thus θ is independent of itself. Conversely, if θ is independent of itself, that is, $\theta(G) = \theta^2(G)$, then, for every $g \in G$, there exists $h \in G$ such that $\theta(g) = \theta^2(h)$, and, since θ is injective, $g = \theta(h)$. Hence $G \subseteq \theta(G)$, that is, G is surjective. This concludes the proof. □

Lemma 3.7. *Let P be the free abelian monoid on a generating countable set B. Let $\theta\colon P \to \mathrm{Mono}(G)$ be a monoid homomorphism into the monoid $\mathrm{Mono}(G)$ of injective group endomorphisms of G. Then Stammeier's "independence condition" (C) for θ is equivalent to requiring independence for $\theta_{p_1}, \theta_{p_2}$ for any two distinct elements p_1, p_2 of B. Furthermore, if "independence" of θ in this sense holds, then θ is injective.*

Proof. Suppose, "independence" in the sense of (C) holds, that is, (18) holds for $\theta_p, \theta_{p'}$, $p, p' \in P$, if and only if p and p' are relatively prime. Two distinct $b, b' \in B$ are relatively prime, so $\theta_b, \theta_{b'}$ are independent.

Conversely, suppose that for any two distinct elements b, b' of B, $\theta_b, \theta_{b'}$ fulfil (18). We show that any $p, p' \in P$ are relatively prime if and only if $\theta_p, \theta_{p'}$ are independent. Let $p, p' \in P$. If $p = 1$ or $p' = 1$, then $\theta_p, \theta_{p'}$ are independent, and p, p' are relatively prime. So we now consider the case when $p \neq 1$ and $q \neq 1$.

Write $p = \prod_{k=1}^{m} b_k$, $p' = \prod_{k=1}^{m'} b'_k$ for $b_1, \dots, b_m, b'_1, \dots, b'_{m'} \in B$, $m, m' \in \mathbb{N}_{\geq 0}$. Then p, p' are relatively prime if and only if $b_k \neq b_{k'}$ for all $k \in \{1, \dots, m\}$, $k' \in \{1, \dots, m'\}$. The latter is equivalent to $\theta_{b_k}, \theta_{b_{k'}}$ being independent for all k, k', by assumption. This, in turn, is equivalent to θ_p, $\theta_{p'}$ being independent, by Lemma 3.5.

Suppose that the two equivalent criteria for "independence" of θ hold. Let $p, p' \in P$ such that $\theta_p = \theta_{p'}$. First suppose that θ_p is surjective. Then θ_p is independent of itself by Lemma 3.6. So p is relatively prime with itself and hence $p = 1$. Similarly, $q = 1$. (Compare [26, Remark 1.6].) Now suppose that the identical $\theta_p, \theta_{p'}$ are not surjective. Then they are not independent, by Lemma 3.6, and hence p, p', are not relatively prime. Then they can be written $p = za$, $p' = za'$, for $a, a', z \in P$, $z \neq 1$. Since P is finitely generated, we can require without loss of generality that a, a' are relatively prime. Then $\theta_z \theta_a = \theta_p = \theta_{p'} = \theta_z \theta_{a'}$. Since θ_z is injective, $\theta_a = \theta_{a'}$. Then $\theta_a, \theta_{a'}$ must be surjective, because they are independent, by Lemma 3.6. According to the case treated above, $a = a' = 1$, and hence $p = z = p'$. $\square$

Stammeier associates C^*-algebras to irreversible algebraic dynamical systems:

Definition 3.8 ([26, Definition 3.1])**.** $\mathcal{O}[G, P, \theta_p]$ is the universal C^*-algebra generated by a unitary representation $(u_g)_{g \in G}$ of the group G and a representation $(s_p)_{p \in P}$ of the monoid P by isometries subject to the (additional) relations

$$s_p u_g = u_{\theta_p(g)} s_p \tag{19}$$

$$s_p^* u_g s_q = \begin{cases} u_{g_1} s_{(p \wedge q)^{-1} q} s^*_{(p \wedge q)^{-1} p} u_{g_2} & \text{if } g = \theta_p(g_1)\theta_q(g_2), \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

$$1 = \sum_{[g] \in G/\theta_p(G)} e_{g,p} \quad \text{if } [G : \theta_p(G)] < \infty, \tag{21}$$

where $e_{g,p} = u_g s_p s_p^* u_g^*$.

We are going to show that if an irreversible algebraic dynamical system (G, P, θ) is of finite type, that is, if $\theta_p(G) \leq G$ has finite index for all $p \in P$, then Stammeier's C^*-algebra $\mathcal{O}[G, P, \theta]$ can be obtained as a groupoid C^*-algebra by applying theory from [1] and [20].

We interpret Stammeier's irreversible algebraic dynamical systems (see Definition 3.4) as a special case of a monoid-shaped diagram of group monomorphisms.

Definition 3.9. Let $\mathfrak{GrpMn}$ be the category of (discrete) groups and group monomorphisms.

Lemma 3.10. *A monoid-shaped diagram in $\mathfrak{GrpMn}$ has the form $D_{\mathfrak{GrpMn}} = (G, P, \theta_p)$ for a (discrete) group G, a monoid P and an injective group endomorphism $\theta_p \colon G \to G$ for each $p \in P$ such that $p \mapsto \theta_p$ is a monoid homomorphism $P \to \mathrm{Mono}(G)$, where $\mathrm{Mono}(G)$ is the monoid of injective group endomorphisms of G.*

Proof. This follows from Remark 2.10 and the observation that a functor $P \to \mathfrak{GrpMn}$ with G as the single group in its image is equivalent to a homomorphism from P to $\mathrm{Mono}(G)$. $\square$

An irreversible algebraic dynamical system is a monoid-shaped diagram $D_{\mathfrak{GrpMn}} = (G, P, \theta_p)$ in $\mathfrak{GrpMn}$ with a countably infinite group G, and a countably generated free abelian monoid P, fulfilling the independence conditions (C).

The above motivates the following conventions for this work:

Standing Assumption 3.11. Throughout the present work, let G be a group, P a monoid and $\theta\colon P \to \mathrm{Mono}(G)$ an action of P on G by injective group endomorphisms.

Definition 3.12. Despite Standing Assumption 3.11, we will occasionally call monoid shaped diagrams in $\mathfrak{GrpMn}$ *dynamical systems (in $\mathfrak{GrpMn}$)* and denote them by $D_{\mathfrak{GrpMn}}$.

Such a "dynamical system" induces monoid shaped diagrams in other bicategories, namely $\mathfrak{Gr}$ (Section 4) and $\mathfrak{Corr}$ (Section 6), which we call *dynamical systems (in $\mathfrak{Gr}$ or $\mathfrak{Corr}$, respectively).*

Remark 3.13. We say that *Stammeier's conditions hold* or call it *Stammeier's case*, if

- G is countably infinite,
- P is a countably generated free abelian monoid and
- Stammeier's independence condition (C) holds for θ.

In accordance with Definition 3.4, we say that

- *Stammeier's finite-type condition* holds, if $\theta_p G \leq G$ has finite index for all $p \in P$, and
- *Stammeier's "minimality" condition* holds, if $\bigcap_p \theta_p(G) = \{1_G\}$.

We may combine the above in expressions like *Stammeier's finite-type, "minimal" case.*

Beware that in Section 7, there is the notion of *minimality* of a topological groupoid, which (in this work) has no interesting relation to Stammeier's "minimality" of irreversible algebraic dynamical systems.

At various places in this work, we will say which version of Stammeier's set of conditions fulfils requirements of results, especially in Section 7, where properties of a groupoid model are discussed. To this end, we state an observation:

Remark 3.14. By Lemma 3.7, in Stammeier's case, $\theta\colon P \to \mathrm{Mono}(G)$ is injective.

See also Remark 5.16.

4 Translation to groupoid correspondences

In this section we recall the definition of a suitable bicategory $\mathfrak{Gr}$ of groupoid correspondences (Section 4.1) and describe a homomorphism from the category of group monomorphisms to $\mathfrak{Gr}$ (Section 4.2). This enables us to encode Stammeier's irreversible algebraic dynamical systems as monoid shaped diagrams in $\mathfrak{Gr}$ (Section 4.3).

4.1 The bicategory $\mathfrak{Gr}$ of locally compact étale topological groupoids and locally compact étale groupoid correspondences

We recall the definition of the bicategory $\mathfrak{Gr}$ of locally compact étale (topological) groupoids and locally compact étale groupoid correspondences with injective, biequivariant, continuous maps as 2-arrows described in [5]. See Remark 4.24 for the particular choice of groupoids and groupoid correspondences out of different versions occurring in [5] and [20].

Definition 4.1 (see [5, Definition 2.1])**.** A *topological groupoid* is a groupoid $\mathcal{G}$ with topologies on the arrow and object spaces $\mathcal{G}$ and $\mathcal{G}^0$ such that the range and source maps $\mathbf{r}, \mathbf{s}\colon \mathcal{G} \to \mathcal{G}^0$ and the multiplication and inverse maps are continuous. A topological groupoid is an *étale (topological) groupoid* if $\mathbf{r}$ and $\mathbf{s}$ are local homeomorphisms. An étale groupoid is *locally compact* if the object space $\mathcal{G}^0$ is Hausdorff and locally compact.

In the following, let $\mathcal{G}$ be a topological groupoid, if nothing else is stated. We do not only denote the groupoid itself by $\mathcal{G}$, but also its arrow space, and $\mathcal{G}^0$ is regarded as a subset of $\mathcal{G}$.

The objects in the bicategory $\mathfrak{Gr}$ we are interested will be *locally compact* étale groupoids; we follow [5], see also Remark 4.24.

Definition 4.2 ([5, Definition 2.3])**.** A *right $\mathcal{G}$-space* is a topological space $\mathcal{X}$ with a continuous map $\mathbf{s}\colon \mathcal{X} \to \mathcal{G}^0$, the *anchor map*, and a continuous map

$$\mathrm{mult}\colon \mathcal{X} \times_{\mathbf{s},\mathcal{G}^0,\mathbf{r}} \mathcal{G} \to \mathcal{X}, \quad \mathcal{X} \times_{\mathbf{s},\mathcal{G}^0,\mathbf{r}} \mathcal{G} := \{(x,g) \in \mathcal{X} \times \mathcal{G} : \mathbf{s}(x) = \mathbf{r}(g)\},$$

which we denote multiplicatively as $\cdot$, such that

(1) $\mathbf{s}(x \cdot g) = \mathbf{s}(g)$ for $x \in \mathcal{X}$, $g \in \mathcal{G}$ with $\mathbf{s}(x) = \mathbf{r}(g)$;

(2) $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 \cdot g_2)$ for $x \in \mathcal{X}$, $g_1, g_2 \in \mathcal{G}$ with $\mathbf{s}(x) = \mathbf{r}(g_1)$, $\mathbf{s}(g_1) = \mathbf{r}(g_2)$;

(3) $x \cdot \mathbf{s}(x) = x$ for all $x \in \mathcal{X}$.

Left $\mathcal{G}$-spaces are defined accordingly, and their anchor maps are denoted by $\mathbf{r}$ (for *range*) rather than $\mathbf{s}$ (for *source*), see [5, paragraph after Definition 2.4]; see also Definition 5.1.

Definition 4.3 ([5, Definition 2.4])**.** The *orbit space* $\mathcal{X}/\mathcal{G}$ is the quotient $\mathcal{X}/\sim_{\mathcal{G}}$ with the quotient topology, where $x \sim_{\mathcal{G}} y$ if there is an element $g \in \mathcal{G}$ with $\mathbf{s}(x) = \mathbf{r}(g)$ and $x \cdot g = y$. We write $\mathrm{p}\colon \mathcal{X} \to \mathcal{X}/\mathcal{G}$ for the orbit space projection.

Definition 4.4 ([5, Definition 2.5])**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be right $\mathcal{G}$-spaces. A continuous map $f\colon \mathcal{X} \to \mathcal{Y}$ is *$\mathcal{G}$-equivariant* if $\mathbf{s}(f(x)) = \mathbf{s}(x)$ for all $x \in \mathcal{X}$ and $f(x \cdot g) = f(x) \cdot g$ for all $x \in \mathcal{X}$, $g \in \mathcal{G}$ with $\mathbf{s}(x) = \mathbf{r}(g)$.

Definition 4.5 ([5, see Definition 2.7 and 2.12])**.** A continuous map f is *proper* if the map $f \times id_Z$ is closed for any topological space Z. A right $\mathcal{G}$-space is *proper* if the map

$$\mathcal{X} \times_{\mathbf{s},\mathcal{G}^0,\mathbf{r}} \mathcal{G} \to \mathcal{X} \times \mathcal{X}, \quad (x,g) \mapsto (x \cdot g, x) \tag{22}$$

is proper. A right $\mathcal{G}$-space is *basic*, if the map in (22) is a homeomorphism onto its image with the subspace topology from $\mathcal{X} \times \mathcal{X}$.

Lemma 4.6 (see [6, §10, Théorème 1 (TG I.75)][5]). *A continuous map $f\colon X \to Y$ between two topological spaces X and Y is proper if and only if f is closed and $f^{-1}(y)$ is compact for all $y \in Y$.*

Lemma 4.7 ([5, "Definition and Lemma" 3.4]). *Let $\mathcal{X}$ be a space with a basic right $\mathcal{G}$-action. Let $\mathrm{p}\colon \mathcal{X} \to \mathcal{X}/\mathcal{G}$ be the orbit space projection. The image of the map in (22) is the subset $\mathcal{X} \times_{\mathcal{X}/\mathcal{G}} \mathcal{X} = \mathcal{X} \times_{\mathrm{p},\mathcal{G}^0,\mathrm{p}} \mathcal{X}$ of all $(x_1, x_2) \in \mathcal{X} \times \mathcal{X}$ with $\mathrm{p}(x_1) = \mathrm{p}(x_2)$. The inverse to the map in (22) induces a continuous map*

$$\mathcal{X} \times_{\mathcal{X}/\mathcal{G}} \mathcal{X} \to \mathcal{X} \times_{\mathbf{s},\mathcal{G}^0,\mathbf{r}} \mathcal{G} \to \mathcal{G}, \quad (x_1, x_2) \mapsto \langle x_2 \,|\, x_1 \rangle \,. \tag{23}$$

That is, $\langle x_1 \,|\, x_2 \rangle$ is defined for $x_1, x_2 \in \mathcal{X}$ with $\mathrm{p}(x_1) = \mathrm{p}(x_2)$ in $\mathcal{X}/\mathcal{G}$, and it is the unique $g \in \mathcal{G}$ with $s(x_1) = r(g)$ and $x_2 = x_1 g$. Conversely, if $g \in \mathcal{G}$ with $x_2 = x_1 g$ for $x_1, x_2 \in \mathcal{X}$ with $\mathrm{p}(x_1) = \mathrm{p}(x_2)$ is unique and depends continuously on $(x_1, x_2) \in \mathcal{X} \times_{\mathcal{X}/\mathcal{G}} \mathcal{X}$, then the right $\mathcal{G}$-action on $\mathcal{X}$ is basic.

Lemma 4.8 ([5, Proposition 2.16]). *Let $\mathcal{G}$ be a locally compact étale groupoid and $\mathcal{X}$ a right $\mathcal{G}$-space. The following are equivalent:*

1. *the action of $\mathcal{G}$ on $\mathcal{X}$ is basic and the orbit space $\mathcal{X}/\mathcal{G}$ is Hausdorff;*
2. *the action of $\mathcal{G}$ on $\mathcal{X}$ is free and proper.*

We now give the definition of locally compact étale groupoid correspondences, the 1-arrows of $\mathfrak{Gr}$:

Definition 4.9 (see [5, Definition 3.1]). Let $\mathcal{H}$ and $\mathcal{G}$ be locally compact étale groupoids. A *locally compact étale groupoid correspondence* from $\mathcal{G}$ to $\mathcal{H}$, denoted $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$, is a space $\mathcal{X}$ with commuting actions of $\mathcal{H}$ on the left and $\mathcal{G}$ on the right, such that the right anchor map $\mathbf{s}\colon \mathcal{X} \to \mathcal{G}^0$ is a local homeomorphism and the right $\mathcal{G}$-action is free and proper.

Remark 4.10. By Lemma 4.8, in Definition 4.9, rather than requiring that the right $\mathcal{G}$-action be free and proper, one can equivalently require that it be basic and that the orbit space $\mathcal{X}/\mathcal{G}$ be Hausdorff.

Remark 4.11 (see [5, Remark 3.2]). The underlying space $\mathcal{X}$ of a locally compact étale groupoid correspondence is locally compact and it is not necessarily Hausdorff.

Definition 4.12 ([5, Definition 3.3]). A locally compact étale groupoid correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ is *proper* if the map $\mathbf{r}_*\colon \mathcal{X}/\mathcal{G} \to \mathcal{H}^0$ induced by $\mathbf{r}$ is proper. It is *tight* if $\mathbf{r}_*$ is a homeomorphism.

Example 4.13 (see [5, Example 4.2]). Requiring the two groupoids in Definition 4.9 to be discrete groups $\mathcal{G} = G$ and $\mathcal{H} = H$, a locally compact étale groupoid correspondence $\mathcal{X}\colon H \leftarrow G$ is a discrete topological space $\mathcal{X}$ with commuting group actions of H on the left and G on the right, such that right action is free: $\mathcal{X}$ being discrete is equivalent to the right anchor map being a local homeomorphism, and the right action being basic is equivalent to it being free; the orbit space is automatically Hausdorff because it is the quotient space of a discrete space (and hence discrete). The correspondence $\mathcal{X}$ is proper if and

[5] Beware that there, "quasi-compact" means what is usually called *compact* (see [6, §9, Définition 1 (TG I.59)]).

only if the map $\mathbf{r}_*\colon \mathcal{X}/G \to H^0 = \{*\}$ is proper, where $\mathcal{X}/G$ is the quotient set with respect to the right G-action; hence by Lemma 4.6, $\mathcal{X}$ is proper if and only if $\mathcal{X}/G$ is finite.

Example 4.14. Let $\mathcal{G}$ be a locally compact étale groupoid. This groupoid itself gives rise to a locally compact étale groupoid correspondence $\mathcal{G}\colon \mathcal{G} \leftarrow \mathcal{G}$ consisting of $\mathcal{G}$ as a topological space, $\mathbf{r}, \mathbf{s}\colon \mathcal{G} \to \mathcal{G}^0$ as the anchor maps, and having the regular left and right $\mathcal{G}$-actions on itself as left and right $\mathcal{G}$-actions as in Definition 4.9. We denote this correspondence by $\mathcal{G}$, just as the groupoid it comes from. Such correspondences will be needed as the "units" with respect to the concatenation of 1-arrows in the bicategory of locally compact étale groupoid correspondences. See Lemma 4.21 and Definition 2.1, and [5, paragraph before Lemma 6.3]. The correspondence $\mathcal{G}$ is tight and thus proper:

For $g, h \in \mathcal{G}$, there is $x \in \mathcal{G}$ with $gx = h$ if and only if $\mathbf{r}(g) = \mathbf{r}(h)$. Hence the surjective continuous $\mathbf{r}\colon \mathcal{G} \to \mathcal{G}^0$ descends to a continuous bijection $\mathbf{r}_*\colon \mathcal{G}/\mathcal{G} \to \mathcal{G}^0$ along the quotient map $\mathrm{p}\colon \mathcal{G} \to \mathcal{G}/\mathcal{G}$. The inclusion map of the object space $\iota\colon \mathcal{G}^0 \to \mathcal{G}$ is a (continuous) section for $\mathbf{r}$, that is, $\mathbf{r} \circ \iota = \mathrm{id}_{\mathcal{G}^0}$. Since $\mathrm{p} \circ \mathbf{r}_* = \mathbf{r}$, this implies $\mathbf{r}_* \circ \mathrm{p} \circ \iota = \mathrm{id}_{\mathcal{G}^0}$. So $\mathbf{r}_*$ is a continuous bijection with a continuous section, hence a homeomorphism. Thus $\mathcal{G}$ is tight.

The following is the category whose objects are the 1-arrows between two fixed objects $\mathcal{G}$, $\mathcal{H}$ of $\mathfrak{Gr}$, and whose morphisms are the 2-arrows in between such 1-arrows:

Lemma 4.15 (see [5, around Lemma 6.1])**.** *For locally compact étale groupoids $\mathcal{H}$, $\mathcal{G}$, a category $\mathfrak{Gr}(\mathcal{G}, \mathcal{H})$ is obtained by taking the locally compact étale groupoid correspondences $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ as objects and the injective, $\mathcal{H}, \mathcal{G}$-equivariant, continuous maps in between with the usual composition of maps as morphisms.*

Lemma 4.16 (see [5, Lemma 6.1])**.** *The morphisms of $\mathfrak{Gr}(\mathcal{G}, \mathcal{H})$ are homeomorphisms onto open subsets of their respective codomain correspondences.*

We now state how 1-arrows of $\mathfrak{Gr}$, that is, locally compact étale groupoid correspondences, are composed, see [5, Section 5; p.1341]. Consider locally compact étale groupoids $\mathcal{H}, \mathcal{G}, \mathcal{K}$ and locally compact étale groupoid correspondences $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$, $\mathcal{Y}\colon \mathcal{G} \leftarrow \mathcal{K}$. Let

$$\mathcal{X} \times_{\mathcal{G}^0} \mathcal{Y} := \mathcal{X} \times_{\mathbf{s}, \mathcal{G}^0, \mathbf{r}} \mathcal{Y} := \{(x, y) \in \mathcal{X} \times \mathcal{Y} : \mathbf{s}(x) = \mathbf{r}(y)\} \tag{24}$$

and consider the orbit space $\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y}$ of $\mathcal{X} \times_{\mathcal{G}^0} \mathcal{Y}$ with respect to the left $\mathcal{G}$-action given by

$$g \cdot (x, y) := (x \cdot g^{-1}, g \cdot y) \tag{25}$$

for each $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $g \in \mathcal{G}$ with $\mathbf{s}(x) = \mathbf{r}(g)$ and $\mathbf{s}(g) = \mathbf{r}(y)$. We denote the image of $(x, y) \in \mathcal{X} \times_{\mathcal{G}} \mathcal{Y}$ in the quotient by $[x, y]$. We equip $\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y}$ with the left $\mathcal{H}$-action $h \cdot [x, y] = [h \cdot x, y]$ and the right $\mathcal{K}$-action $[x, y] \cdot k = [x, y \cdot k]$. Well-definedness of those actions is to be established.

Remark 4.17 (see [5, paragraph before Lemma 5.1])**.** Note that $\mathcal{X} \circ_{\mathcal{G}} Y$ can be defined as the orbit space of a diagonal $\mathcal{G}$-action similar to (25) even when Y in place of $\mathcal{Y}$ is merely "a correspondence short of the right action of any étale groupoid", that is, just a topological space with a left $\mathcal{G}$-action. $\mathcal{X} \circ_{\mathcal{G}} Y$ can be equipped with a left $\mathcal{H}$-action just as above, turning it into a topological space with a left $\mathcal{H}$-action. This will occur when actions of diagrams of correspondences on topological spaces are defined, see Definition 5.2 and Lemma 5.3.

We now describe how 2-arrows in $\mathfrak{Gr}$, that is, injective, bi-equivariant, continuous maps between correspondences, are composed horizontally, see [5, between Remark 6.2 and Lemma 6.3]. Let $\mathcal{X}_1, \mathcal{X}_2 \colon \mathcal{H} \leftarrow \mathcal{G}$ and $\mathcal{Y}_1, \mathcal{Y}_2 \colon \mathcal{G} \leftarrow \mathcal{K}$ be locally compact étale groupoid correspondences and let $\alpha \colon \mathcal{X}_1 \to \mathcal{X}_2$ and $\beta \colon \mathcal{Y}_1 \to \mathcal{Y}_2$ be 2-arrows. Define

$$\alpha \circ_{\mathcal{G}} \beta \colon \mathcal{X}_1 \circ_{\mathcal{G}} \mathcal{Y}_1 \to \mathcal{X}_2 \circ_{\mathcal{G}} \mathcal{Y}_2 \colon \qquad [(x,y)] \mapsto [(\alpha(x), \beta(y)]. \tag{26}$$

Lemma 4.18 ([5])**.** *The above construction describes a bifunctor*

$$\begin{aligned} \circ_{\mathcal{G}} \colon \mathfrak{Gr}(\mathcal{G},\mathcal{H}) \times \mathfrak{Gr}(\mathcal{K},\mathcal{G}) &\to \mathfrak{Gr}(\mathcal{K},\mathcal{H}), && (27)\\ (\mathcal{X},\mathcal{Y}) &\mapsto \mathcal{X} \circ_{\mathcal{G}} \mathcal{Y} && (28)\\ (\alpha,\beta) &\mapsto \alpha \circ_{\mathcal{G}} \beta && (29) \end{aligned}$$

If $\mathcal{X}, \mathcal{Y}$ are both proper, or tight, respectively, so is the composition $\mathcal{X} \circ \mathcal{Y}$.

Proof. That the result of the composition of two locally compact étale groupoid correspondences is again such a correspondence (including well-definedness of the left and right action), and the preservation property with respect to tightness and properness, is the content of [5, Proposition 5.7]. That the composition is functorial is stated in [5, between Remark 6.2 and Lemma 6.3]. $\square$

Remark 4.19. Note the use of $\alpha \circ_{\mathcal{G}} \beta$ to denote the horizontal concatenation of 2-arrows α, β as above in place of the notation $\alpha * \beta$, used in the case of generic bicategories. The former will be our standard notation in any bicategories involving correspondences.

The associators for the composition of 1-arrows of $\mathfrak{Gr}$ are given by the following lemma.

Lemma 4.20 (see [5, Lemma 6.4][6])**.** *Let $\mathcal{G}_i$ for $1 \le i \le 4$ be locally compact étale groupoids. Let $\mathcal{X}_i \colon \mathcal{G}_i \leftarrow \mathcal{G}_{i+1}$ for $1 \le i \le 3$ be locally compact étale groupoid correspondences. The map*

$$\text{assoc} \colon (\mathcal{X}_1 \circ_{\mathcal{G}_2} \mathcal{X}_2) \circ_{\mathcal{G}_3} \mathcal{X}_3 \to \mathcal{X}_1 \circ_{\mathcal{G}_2} (\mathcal{X}_2 \circ_{\mathcal{G}_3} \mathcal{X}_3), \qquad [[x_1,x_2],x_3] \mapsto [x_1,[x_2,x_3]], \tag{30}$$

is a $\mathcal{G}_1, \mathcal{G}_4$-equivariant homeomorphism, which is natural with respect to $\mathcal{G}_i, \mathcal{G}_{i+1}$-equivariant, continuous maps $\alpha_i \colon \mathcal{X}_i \to \mathcal{X}_i'$ for $1 \le i \le 3$.

Recall the "unit" correspondence $\mathcal{G}$ for an étale groupoid $\mathcal{G}$ from Example 4.14. The unitors are given by the following maps:

Lemma 4.21 ([5, Lemma 6.3])**.** *Let $\mathcal{X} \colon \mathcal{H} \leftarrow \mathcal{G}$ be a locally compact étale groupoid correspondence. The maps*

$$\mathfrak{l} \colon \mathcal{H} \circ_{\mathcal{H}} \mathcal{X} \to \mathcal{X}, \quad [h,x] \mapsto h \cdot x, \tag{31}$$

$$\mathfrak{r} \colon \mathcal{X} \circ_{\mathcal{G}} \mathcal{G} \to \mathcal{X}, \quad [x,g] \mapsto x \cdot g, \tag{32}$$

[6] Deviating form the cited source, we have turned around the direction of the associators, replacing the original homeomorphisms by their inverses, in order to achieve compatibility with Leinster's ([19]) convention regarding the direction of associators.

are $\mathcal{H}, \mathcal{G}$-equivariant homeomorphisms, which are natural for $\mathcal{H}, \mathcal{G}$-equivariant, continuous maps $\mathcal{X} \to \mathcal{X}'$.

Finally, we have recalled all the ingredients for the *bicategory of locally compact étale groupoids and locally compact étale groupoid correspondences.*

Lemma 4.22 ([5, Proposition 6.5])**.** *The following defines a bicategory $\mathfrak{Gr}$:*

1. *The objects be* locally compact étale groupoids *as in Definition 4.1.*
2. *For two objects $\mathcal{H}, \mathcal{G}$, the category of 1-arrows from $\mathcal{G}$ to $\mathcal{H}$ and of 2-arrows in between be given by the category $\mathfrak{Gr}(\mathcal{G}, \mathcal{H})$ in Lemma 4.15 of* locally compact étale groupoid correspondences *from $\mathcal{G}$ to $\mathcal{H}$ and* injective, $\mathcal{H}, \mathcal{G}$-equivariant, continuous maps *between them.*
3. *The bifunctor encoding the composition of 1-arrows and horizontal composition of 2-arrows be the one from Lemma 4.18 sending correspondences $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ and $\mathcal{Y}\colon \mathcal{G} \leftarrow \mathcal{K}$ to $\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y}$.*
4. *The "units" with respect to composition of correspondences (the images of the functors $I_{\mathcal{G}}$) be the correspondences $\mathcal{G}\colon \mathcal{G} \leftarrow \mathcal{G}$ for groupoids $\mathcal{G}$ from Example 4.14.*
5. *The components of the natural isomorphisms $\mathrm{assoc}_{a,b,c,d}$ (for $a, b, c, d \in \mathcal{B}^0$) – that is, of the associators for composition of 1-arrows – be the maps from Lemma 4.20.*[7]
6. *The components of the natural isomorphisms $\mathfrak{l}_{a,b}, \mathfrak{r}_{a,b}$ (for $a, b \in \mathcal{B}^0$) – that is, of the unitors for composition of 1-arrows – be the maps from Lemma 4.21.*

Lemma 4.23 (compare [5, Last paragraph of Section 6] and [1, Section 2.4] and [20, paragraph following Proposition 2.19])**.** *Restricting the class of 1-arrows in $\mathfrak{Gr}$ to proper or tight correspondences (see Definition 4.12), and restricting the Hom-categories $\mathfrak{Gr}(\mathcal{G}, \mathcal{H})$, for $\mathcal{G}, \mathcal{H} \in \mathfrak{Gr}^0$, to the full subcategories generated by such correspondences, results in bicategories $\mathfrak{Gr}_{\mathrm{prop}}$ and $\mathfrak{Gr}_{\mathrm{tight}}$, respectively, where $\mathfrak{Gr}_{\mathrm{tight}} \subseteq \mathfrak{Gr}_{\mathrm{prop}}$*

Proof. Tight and proper correspondences each define subbicategories of $\mathfrak{Gr}$ by Lemma 4.18 and the observation that the unit correspondences are tight and *a fortiori* proper (see Example 4.14). Every tight correspondence is proper because every homeomorphism is a proper map (consider this in the context of Definition 4.12). □

Remark 4.24. In [5], the bicategory $\mathfrak{Gr}$ of *locally compact* étale groupoids and *locally compact* étale groupoid correspondences with *injective*, biequivariant, continuous maps as 2-arrows, as in Lemma 4.22, is considered. The original bicategory described by Albandik in his thesis [1] is the same except that (1) only invertible 2-arrows (that is, biequivariant homeomorphisms) are allowed (see [1, second paragraph of Section 2.4]), and (2) what we call a locally compact groupoid correspondence $\mathcal{G} \leftarrow \mathcal{H}$ from $\mathcal{H}$ to $\mathcal{G}$, for locally compact étale groupoids $\mathcal{G}, \mathcal{H}$, he calls a groupoid correspondence $\mathcal{G} \to \mathcal{H}$ from $\mathcal{G}$ to $\mathcal{H}$;

[7] Recall that in [5], the associators go in the opposite direction.

names and symbols for the anchor maps (source, $\mathbf{s}$, range, $\mathbf{r}$) of the left and right groupoid actions are swapped accordingly; what we consider a 1-arrow from $\mathcal{H}$ to $\mathcal{G}$, he considers a 1-arrow from $\mathcal{G}$ to $\mathcal{H}$. Also, for locally compact étale groupoids $\mathcal{G}, \mathcal{H}$ he calls *locally compact* étale groupoid correspondences $\mathcal{G} \leftarrow \mathcal{H}$ just "groupoid correspondences" (see [1, Definition 2.18], take into account [1, Standing assumption 2.9]).

In [20], the bicategory of étale groupoids and étale groupoid correspondences with *not necessarily injective*, continuous, biequivariant maps as 2-arrows is considered and denoted by $\mathfrak{Gr}$, see [20, Section 2, especially paragraphs before Definition 2.1, Definition 2.7 and paragraph after 2.19]. For étale groupoids $\mathcal{G}, \mathcal{H}$, [20, Definition 2.7] defines an étale groupoid correspondence $\mathcal{X} \colon \mathcal{H} \leftarrow \mathcal{G}$ to be a space with commuting actions of $\mathcal{H}$ on the left and $\mathcal{G}$ on the right such that the right anchor map $\mathbf{s} \colon \mathcal{X} \to \mathcal{G}^0$ is a local homeomorphism and the right $\mathcal{G}$-action is basic. If $\mathcal{G}$ and $\mathcal{H}$ are locally compact, then such an étale groupoid correspondence $\mathcal{X} \colon \mathcal{H} \leftarrow \mathcal{G}$ is a locally compact étale groupoid correspondence as in Definition 4.9 (see also Remark 4.10), if and only if the orbit space $\mathcal{X}/\mathcal{G}$ is Hausdorff. See also [20, Definition 2.9].

The category $\mathfrak{Gr}$ of locally compact étale groupoids and locally compact étale groupoid correspondences with injective, biequivariant, continuous 2-arrows used in the present work and in [5] is denoted by $\mathfrak{Gr}_{\mathrm{lc}}$ in [20] (see [20, first paragraph on p. 3, and between Proposition 2.19 and Theorem 2.22]).

The stricter variant is used here and in [5] in order to allow for the homomorphism from $\mathfrak{Gr}$ into a bicategory of C^*-correspondences (see Section 6.2), see [5, Remark 2.2] and [20, from Definition 2.9 to Example 2.10].

4.2 The homomorphism from the category of group monomorphisms to $\mathfrak{Gr}$

In this section, we describe a homomorphism of bicategories from $\mathfrak{GrpMn}$ to $\mathfrak{Gr}$. The interesting component of this homomorphism is the construction of a groupoid correspondence from an injective group endomorphism. It is mentioned in [5, last paragraph of Example 4.2], where it is noted that such groupoid correspondences are implicitly used by Stammeier ([26]).

Let G and H be (discrete) groups. Let $\theta \colon G \to H$ be an injective group homomorphism. For this data, define $\mathcal{X} := H$ as a set and equip it with the regular left action by H, given by

$$h.k := h \cdot k \in \mathcal{X} \qquad \text{for all } k \in \mathcal{X} = H, h \in H. \tag{33}$$

and the right $\mathcal{G}$-action given by

$$k.g := k \cdot \theta(g) \in \mathcal{X} \qquad \text{for all } k \in \mathcal{X} = H, g \in G. \tag{34}$$

Lemma 4.25 (see also [5, Example 4.2, last paragraph])**.** *Let G, H be (discrete) groups and $\theta \colon G \to H$ an injective group homomorphism. The construction above associates to θ a locally compact étale groupoid correspondence $\mathcal{X}_\theta \colon H \leftarrow G$. It is proper if and only if $\theta(G)$ has finite index in H.*

Proof. We have $\mathcal{X}_\theta = H$ as a discrete topological space. The left and right actions by the group H and G, respectively, commute because $(h \cdot x) \cdot \theta(g) = h \cdot (x \cdot \theta(g))$ for $h \in H$, $x \in \mathcal{X}_\theta = H$, $g \in G$. The right action is free, because

$x \cdot \theta(g_1) = x \cdot \theta(g_2)$ implies $g_1 = g_2$, for $x \in \mathcal{X}_\theta = H$, $g_1, g_2 \in G$, since θ is injective. By Example 4.13, the given data thus constitute a locally compact étale groupoid correspondence $\mathcal{X}_\theta \colon H \leftarrow G$ which is proper if and only if $\mathcal{X}/G = H/\theta(G)$ is finite. $\square$

Remark 4.26. In 4.1, left and right actions of groupoids on groupoid correspondences are indicated by "$\cdot$". We now consider concrete correspondences, which have as underlying spaces the groups acting on them from the left. Later, we will exclusively consider the case where the group acting on the right will be identical to the underlying group, as well. Since in this case the correspondence and the group acting on it from both sides have the same underlying set, it is crucial to distinguish the right group action from the product within the group; the results are typically different. Thus some care has to be taken when it comes to the notation. In the context of correspondences coming from group monomorphisms, we will use "$\cdot$" for the multiplication within a group whenever this interpretation is possible; terms explicitly involving the application of a left or right action will be avoided – for example by replacing them with the result of the application – or the symbol "." will be used.

Let H, G, K be (discrete) groups and $\theta_1 \colon G \to H$, $\theta_2 \colon K \to G$ be injective group homomorphisms. Consider the map

$$\mathcal{X}_{\theta_1} \times \mathcal{X}_{\theta_2} \to \mathcal{X}_{\theta_1\theta_2} \colon (h, g) \mapsto h \cdot \theta_1(g). \tag{35}$$

The underlying sets of $\mathcal{X}_{\theta_1}$, $\mathcal{X}_{\theta_2}$ and $\mathcal{X}_{\theta_1\theta_2}$ are H, G and H, respectively.

Lemma 4.27. *The map in (35) descends to an H-K-equivariant homeomorphism*

$$\phi_{\theta_1,\theta_2} \colon \mathcal{X}_{\theta_1} \circ \mathcal{X}_{\theta_2} \to \mathcal{X}_{\theta_1\theta_2} \colon [y, x] \mapsto y \cdot \theta_1(x). \tag{36}$$

Proof. The relation

$$(y, x) \sim (y \cdot \theta_1(g), g^{-1} \cdot x)$$

for $y \in \mathcal{X}_{\theta_1}$, $x \in \mathcal{X}_{\theta_2}$ and $g \in G$ defines an equivalence relation on $\mathcal{X}_{\theta_1} \times \mathcal{X}_{\theta_2}$. The set $\mathcal{X}_{\theta_1} \circ \mathcal{X}_{\theta_2}$ is the resulting set of equivalence classes, see (25), (33) and (34).

We first show that the map in (36) is well defined and injective. To this end, let $y, y' \in \mathcal{X}_{\theta_1}$, $x, x' \in \mathcal{X}_{\theta_2}$. Then

$$[y, x] = [y \cdot \theta_1(x), x^{-1} \cdot x] = [y \cdot \theta_1(x), 1_G] \tag{37}$$

$$[y', x'] = [y' \cdot \theta_1(x'), x'^{-1} \cdot x'] = [y' \cdot \theta_1(x'), 1_G]; \tag{38}$$

the right hand sides are equal if and only if $y \cdot \theta_1(x) = y' \cdot \theta_1(x')$, which shows that ϕ_{θ_1,θ_2} is well defined by (36) and injective.

Biequivariance of ϕ_{θ_1,θ_2} is witnessed by commutativity of the following diagram for all $h \in H$, $k \in K$, $y \in \mathcal{X}_{\theta_1}$ and $x \in \mathcal{X}_{\theta_2}$:

$$\begin{tikzcd}
{[y, x\theta_2(k)]} \arrow[d, mapsto, "\phi_{\theta_1,\theta_2}"] & {[y, x]} \arrow[l, mapsto, "- \cdot k"'] \arrow[r, mapsto, "h \cdot -"] \arrow[d, mapsto, "\phi_{\theta_1,\theta_2}"] & {[hy, x]} \arrow[d, mapsto, "\phi_{\theta_1,\theta_2}"] \\
\begin{array}{c} y\theta_1(x\theta_2(k)) \\ = y\theta_1(x)\theta_1 \circ \theta_2(k) \end{array} & y\theta_1(x) \arrow[l, mapsto, "- \cdot k"'] \arrow[r, mapsto, "h \cdot -"] & hy\theta_1(x).
\end{tikzcd}$$

Additionally to being injective, as shown above, ϕ_{θ_1,θ_2} is surjective, because $\phi_{\theta_1,\theta_2}([h, 1_G]) = h$ for all $h \in \mathcal{X}_{\theta_1} = H$, and, as a set, $\mathcal{X}_{\theta_1\theta_2} = H$. Thus ϕ_{θ_1,θ_2} is a homeomorphism between discrete spaces. $\square$

Lemma 4.28. *The construction above defines a strictly unital homomorphism* $\mathfrak{GrpMn} \to \mathfrak{Gr}$ *with the data* $(\{G\}_G, \mathcal{X}_\theta, \phi_{\theta_1,\theta_2})$ *in terms of Lemma 2.5.*

Proof. The required map between the classes of objects is just the reinterpretation of a discrete group as a locally compact étale groupoid, that is, in terms of Lemma 2.5, $A_G := G$, as groupoids, for each discrete group G. For two given groups G, H, to an injective group homomorphism $\theta\colon G \to H$ is assigned the locally compact groupoid correspondence $\mathcal{X}_\theta\colon H \leftarrow G$, that is, in terms of Lemma 2.5, $X_\theta := \mathcal{X}_\theta \in \mathfrak{Gr}^0(G, H)$ for $\theta \in \mathfrak{GrpMn}(G, H)$. The invertible 2-arrows in $\mathfrak{Gr}$ are biequivariant homeomorphisms between correspondences with equal domains and codomains. For two composable injective group homomorphisms θ_1, θ_2, the required invertible 2-arrow $\phi_{\theta_1,\theta_2}\colon \mathcal{X}_{\theta_1} \circ \mathcal{X}_{\theta_2} \to \mathcal{X}_{\theta_1\cdot\theta_2}$ is given by (36).

It remains to show that the axioms (8)–(11) are fulfilled. In the present situation, (8) amounts to $G = \mathcal{X}_{\mathrm{id}_G}$ for all groups G, where the left hand side denotes the “unit” groupoid correspondence on G consisting of G as a set and being equipped with the left and right regular G-actions, see Example 4.14. This is fulfilled, since the groupoid correspondence on the right hand side is exactly the same by (33) and (34); specifically, the right G-action on $\mathcal{X}_{\mathrm{id}_G}$ is the regular G-action.

We show that identity (9) holds by demonstrating that the equivalent diagram in Lemma 2.5 commutes. Taking into account the definitions of horizontal concatenation of 2-arrows (see (26)) and the associator (see Lemma 4.20) in $\mathfrak{Gr}$ and the maps ϕ_{θ_1,θ_2} (see Lemma 4.27), this is shown thus: The diagram

$$\begin{tikzcd}
{[[x,y],z]} \arrow[r, mapsto, "\phi_{h,g} \circ_K \mathrm{id}_{\mathcal{X}_k}"] \arrow[d, mapsto, "\mathrm{assoc}_{hg,k}"'] & {[x\theta_h(y), z]} \arrow[dr, mapsto, "\phi_{hg,k}"] & \\
 & & {x\theta_h(y)\theta_{hg}(z) = x\theta_h(y\theta_g(z))} \\
{[x,[y,z]]} \arrow[r, mapsto, "\mathrm{id}_{\mathcal{X}_h} \circ_G \phi_{g,k}"] & {[x, y\theta_g(z)]} \arrow[ur, mapsto, "\phi_{h,gk}"'] &
\end{tikzcd}$$

commutes for groups J, K, G, H, group monomorphisms $k\colon J \to K$, $g\colon K \to G$, $h\colon G \to H$ and $x \in \mathcal{X}_h$, $y \in \mathcal{X}_g$, $z \in \mathcal{X}_k$.

We verify (10) and (11). Let K,G,H be groups and $f\colon K \to G$ and $h\colon G \to H$ group monomorphisms. We have to show that $\mathfrak{l}_{\mathcal{X}_f} = \phi_{\mathrm{id}_G,f}$ and $\mathfrak{r}_{\mathcal{X}_h} = \phi_{h,\mathrm{id}_G}$. To this end, let $g \in G = \mathcal{X}_{\mathrm{id}_G}$, $x \in \mathcal{X}_h$ and $y \in \mathcal{X}_f$. Then, indeed,

$$\begin{aligned}
&\phi_{\mathrm{id}_G,f}([g, y]) = g \cdot \mathrm{id}_G(y) = g \cdot y = g.y = \mathfrak{l}_{\mathcal{X}_f}([g, y]) \text{ and} \\
&\phi_{h,\mathrm{id}_G}([x, g]) = x \cdot \theta_h(g) = x.g = \mathfrak{r}_{\mathcal{X}_h}([x, g]),
\end{aligned}$$

where “.” denotes the left G-action on $\mathcal{X}_f$, and the right G-action on $\mathcal{X}_h$, respectively; see (31) and (32) for the definitions of the unitors $\mathfrak{l}_{\mathcal{X}_f}$ and $\mathfrak{r}_{\mathcal{X}_h}$. $\square$

4.3 Dynamical systems in $\mathfrak{Gr}$

In this section, we describe monoid shaped diagrams (see Definition 2.8) in $\mathfrak{Gr}$ in general and the monoid-shaped diagrams arising from composing dynamical systems in $\mathfrak{GrpMn}$ (see Definition 3.12) with the homomorphism of bicategories $\mathfrak{GrpMn} \to \mathfrak{Gr}$ described in Section 4.2. The latter will be called dynamical systems in $\mathfrak{Gr}$.

Lemma 4.29 (compare [5, first paragraph after Theorem 7.13] and [1, Definition 3.4]). *A monoid-shaped diagram in $\mathfrak{Gr}$ is of the form $D = (\mathcal{G}, P, \mathcal{X}_p, \mu_{p,q})$ with data*

- *a locally compact étale groupoid $\mathcal{G}$,*
- *a monoid P,*
- *a locally compact étale groupoid correspondence $\mathcal{X}_p \colon \mathcal{G} \leftarrow \mathcal{G}$ for each $p \in P$,*
- *a $\mathcal{G}$-biequivariant homeomorphism $\mu_{p,q} \colon \mathcal{X}_p \circ \mathcal{X}_q \to \mathcal{X}_{p \cdot q}$ for each pair $p, q \in P$,*

fulfilling

$$\mathcal{X}_1 = \mathcal{G}, \tag{39}$$
$$\mu_{pq,r} \circ (\mu_{p,q} * \mathrm{id}_{\mathcal{X}_r}) = \mu_{p,qr} \circ (\mathrm{id}_{\mathcal{X}_p} * \mu_{q,r}) \circ \mathrm{assoc}_{p,q,r}, \tag{40}$$
$$\mathfrak{l}_{\mathcal{X}_p} = \mu_{1,p} \text{ and} \tag{41}$$
$$\mathfrak{r}_{\mathcal{X}_p} = \mu_{p,1} \tag{42}$$

for all $p, q, r \in P$.

Proof. This directly follows from 2.9 and the definition of $\mathfrak{Gr}$ in Section 4.1. □

Lemma 4.30 (compare[8] [5, second paragraph after Theorem 7.13]). *Applying the homomorphism of bicategories $\mathfrak{GrpMn} \to \mathfrak{Gr}$ from Lemma 4.28 to a dynamical system in $\mathfrak{GrpMn}$ (that is, a monoid-shaped diagram in $\mathfrak{GrpMn}$) $D_{\mathfrak{GrpMn}} = (G, P, \theta_p)$ as in Lemma 3.10 results in the monoid-shaped diagram $D_{\mathfrak{Gr}} = (G, P, \mathcal{X}_p, \mu_{p,q})$ in $\mathfrak{Gr}$ where*

- *$\mathcal{X}_p \colon G \leftarrow G$ is the locally compact groupoid correspondence with underlying discrete space $\mathcal{X}_p = G$ and the left and right G-actions given by*
$$g.x := g \cdot x \tag{43}$$
$$x.g := x \cdot \theta_p(g) \tag{44}$$
for $x \in \mathcal{X}_p$, $g \in G$, for each $p \in P$,
- *$\mu_{p,q}$ is the G-biequivariant homeomorphism*
$$\mathcal{X}_p \circ \mathcal{X}_q \to \mathcal{X}_{pq}, \tag{45}$$
$$[g, h] \mapsto g\theta_p(h), \tag{46}$$
for each pair $p, q \in P$.

We omit the rather technical proof.

[8]The relationships between the present and the last lemma is not analogous to the relationship between the given sources with which they should be compared.

Remark 4.31. Let $D_{\mathfrak{Gr}}$ be as in Lemma 4.30 and $p \in P$. Then for $x_1, x_2 \in \mathcal{X}_p$ with $x_1\theta_p(G) = x_2\theta_p(G)$, the "bracket product"

$$\langle x_2 \,|\, x_1 \rangle_p := \langle x_2 \,|\, x_1 \rangle$$

defined by the continuous map in (23) is the group element in G such that $x_1 = x_2\theta_p(\langle x_2 \,|\, x_1 \rangle_p)$.

Definition 4.32. A monoid shaped diagram $D_{\mathfrak{Gr}}$ in $\mathfrak{Gr}$ coming from a dynamical system $D_{\mathfrak{GrpMn}}$ in $\mathfrak{GrpMn}$ as in Lemma 4.30 will be called a *dynamical system (in $\mathfrak{Gr}$)* in the present work.

Remark 4.33. In [1], Albandik describes, put in terms of Lemma 2.5, strictly unital homomorphisms from categories into his bicategory of groupoid correspondences ([1, Proposition 2.40]). He calls them "functors".[9] Despite the fact that in his bicategory "$\mathfrak{Gr}$", he considers only *invertible* biequivariant continuous maps as 2-arrows, his "functors $\mathcal{C} \to \mathfrak{Gr}$", for categories $\mathcal{C}$, are exactly our strictly unital homomorphisms $\mathcal{C}^{\mathrm{op}} \to \mathfrak{Gr}$, because all 2-arrows being part of the data of a homomorphism from a category to a bicategory are invertible. The domain $\mathcal{C}^{\mathrm{op}}$ instead of $\mathcal{C}$ accounts for the fact that Albandik's 1-arrows go in the other direction (see Remark 4.24).

In [1, Definition 3.4 and preceding paragraph], an *action of P in "$\mathfrak{Gr}$"* (or an *action of P on $\mathcal{G}$*, for a locally compact étale groupoid $\mathcal{G}$), for an Ore monoid P, is defined as a "functor $P^{\mathrm{op}} \to \mathfrak{Gr}$" (with $\mathcal{G}$ in its image) in the sense above, that is, a P-shaped diagram in $\mathfrak{Gr}$ (with $\mathcal{G}$ in its image) in the sense of Lemma 4.29. Compare the concrete description in [1, Definition 3.4].[10]

5 Construction of a groupoid model

In this section, for the case that P fulfils the right Ore conditions, we construct a groupoid model encoding $D_{\mathfrak{Gr}}$, applying a construction by Meyer ([20, Section 4.2]).

The construction involves a universal action of the diagram, an inverse semigroup associated to the diagram together with an action of it by partial homeomorphisms, and the transformation groupoid of this action, which is the groupoid model.

If $\theta_p(G) \leq G$ has finite index for all $p \in P$, then $D_{\mathfrak{Gr}}$ consists only of proper correspondences, and by [15, Theorem 6.3], this implies that the groupoid model is locally compact. Also, as pointed out by Meyer, then it coincides with a construction by Albandik [1, Section 3]. This will be used to show that the C^*-algebra associated to the groupoid model is the Cuntz-Pimsner algebra associated to a certain product system in Section 6.

In particular, in Stammeier's finite-type case, Stammeier's C^*-algebra is the groupoid C^*-algebra of the groupoid model $\mathcal{L}$. In remarks in the appropriate places, we draft an alternative way to show this by comparing Meyer's construction to Stammeier's construction via different crossed product constructions for inverse semigroup actions and semigroup actions on C^*-algebras.

[9] If the domain is a bicategory, strict unitality is not required, see [1, Definition 2.2]; but it is encoded in the description in [1, Proposition 2.40] of "functors" with a category as domain.

[10] There, what is called "an action of an Ore monoid P in the bicategory $\mathfrak{Gr}$" just above the cited definition, is called "an action of P^{op} on $\mathcal{G}$ by correspondences". Referring to Albandik, we will call it an action of P on $\mathcal{G}$ (or in $\mathfrak{Gr}$).

5.1 Actions and groupoid models for dynamical systems in $\mathfrak{Gr}$

We describe what an action of a dynamical system in $\mathfrak{Gr}$ on a topological space is (see [20, Section 4.1]). A groupoid model for a dynamical system is defined by requiring a natural bijection between such actions and actions of the groupoid model, see Definition 5.4.

Definition 5.1 (compare Definitions 4.2 and 4.4 (or, equivalently, [5, Definitions 2.3 and 2.5]))**.** Let $\mathcal{G}$ be a topological groupoid. A (left) action of $\mathcal{G}$ on a topological space Y is a left $\mathcal{G}$-space Y. The data is thus a continuous anchor map $\mathbf{r}\colon Y \to \mathcal{G}^0$ and a continuous map

$$\text{mult}\colon \mathcal{G} \times_{\mathbf{s},\mathcal{G}^0,\mathbf{r}} Y \to Y, \quad \mathcal{G} \times_{\mathbf{s},\mathcal{G}^0,\mathbf{r}} Y := \{(g, y) \in \mathcal{G} \times Y | \, \mathbf{s}(g) = \mathbf{r}(y)\} \tag{47}$$

which we denote multiplicatively as $\cdot$, which fulfil the conditions

(1) $\mathbf{r}(g \cdot y) = \mathbf{r}(g)$ for $g \in \mathcal{G}$, $y \in Y$, with $\mathbf{s}(g) = \mathbf{r}(y)$;

(2) $g_1 \cdot (g_2 \cdot y) = (g_1 \cdot g_2) \cdot y$ for $g_1, g_2 \in \mathcal{G}$, $y \in Y$ with $\mathbf{s}(g_1) = \mathbf{r}(g_2)$, $\mathbf{s}(g_2) = \mathbf{r}(y)$;

(3) $\mathbf{r}(y) \cdot y = y$ for all $y \in Y$.

Let Y, Y' be $\mathcal{G}$-actions. A continuous map $f\colon Y \to Y'$ is called $\mathcal{G}$-equivariant, if $\mathbf{r}(f(y)) = \mathbf{r}(y)$ for all $y \in Y$ and $f(g \cdot y) = g \cdot f(y)$ for all $y \in Y$, $g \in \mathcal{G}$ with $\mathbf{r}(y) = \mathbf{s}(g)$.

We specialise the definition in [20, Definition 4.5] of actions of (strictly unital, category-shaped) diagrams (as the term is used in the cited source) of étale groupoid correspondences on topological spaces: We are only interested in the case where the diagram is a dynamical system $D_{\mathfrak{Gr}}$ of locally compact étale groupoid correspondences as in Definition 4.32.

Definition 5.2 (compare [20, Definitions 4.5 and 4.8])**.** Let $D_{\mathfrak{Gr}} = (G, P, \mathcal{X}_p, \mu_{p,q})$ be a dynamical system in $\mathfrak{Gr}$. A *$D_{\mathfrak{Gr}}$-action* $((\alpha_p)_{p\in P}, Y)$ consists of a topological space Y and open, continuous, surjective maps $\alpha_p\colon \mathcal{X}_p \times Y \to Y$ for $p \in P$, denoted multiplicatively as $\alpha_p(x, y) = x \cdot_p y$, such that

(i) $x_1 \cdot_{p_1} (x_2 \cdot_{p_2} y) = x_1\theta_{p_1}(x_2) \cdot_{p_1p_2} y$ for $p_i \in P$, $x_i \in \mathcal{X}_{p_i}$, $i = 1, 2$, and $y \in Y$ and

(ii) if $x \cdot_p y = x' \cdot_p y'$ for $x, x' \in \mathcal{X}_p$, $y, y' \in Y$, there is $\eta \in G$ with $x' = x \cdot \theta_p(\eta)$ and $y = \eta \cdot_1 y'$; equivalently, $\mathrm{p}(x) = \mathrm{p}(x')$ for the orbit space projection $\mathrm{p}\colon \mathcal{X}_p \to \mathcal{X}_p/G$ and $y = \langle x \,|\, x'\rangle_p \cdot_1 y'$.

A continuous map $f\colon Y \to Y'$ between two spaces with $D_{\mathfrak{Gr}}$-actions is called $D_{\mathfrak{Gr}}$-equivariant if $f(x \cdot_p y) = x \cdot_p f(y)$ for all $p \in P$, $x \in \mathcal{X}_p$ and $y \in Y$.

Lemma 5.3 (see [20, Lemma 4.7])**.** *The multiplication map α_p descends to a $\mathcal{G}$-equivariant homeomorphism $\dot{\alpha}\colon \mathcal{X}_p \circ_{\mathcal{G}} Y \to Y$.*

Proof. A direct comparison immediately shows that Definition 5.2 is a special case of [20, Definition 4.5], namely, a restriction from general category-shaped strictly unital diagrams F in a larger bicategory to dynamical systems $D_{\mathfrak{Gr}}$ in $\mathfrak{Gr}$, a subbicategory of that larger bicategory considered in [20] (see Remark

4.24). Since the domain category P of $D_{\mathfrak{Gr}}$ has only one object, the partition of Y and the anchor maps r for the action as in [20, Definition 4.5] are here unnecessary. There are technical simplifications because here, it is at no point necessary to require composability of arrows in the domain category. Hence, the lemma follows from [20, Lemma 4.7]. $\square$

Definition 5.4 (compare [20, Definition 4.13])**.** A groupoid model for $D_{\mathfrak{Gr}}$-actions is an étale topological groupoid $\mathcal{L}$ with natural bijections between the sets of $\mathcal{L}$-actions and $D_{\mathfrak{Gr}}$-actions on the same space in the sense that a continuous map $Y \to Y'$ is $\mathcal{L}$-equivariant if and only if it is $D_{\mathfrak{Gr}}$-equivariant.

Lemma 5.5 (compare [20, Proposition 4.16])**.** *Let $\mathcal{L}$ and $\mathcal{L}'$ be two groupoid models for $D_{\mathfrak{Gr}}$-actions. There is a unique isomorphism of topological groupoids $\mathcal{L} \cong \mathcal{L}'$ that is compatible with the equivalence between actions of $\mathcal{L}$, $\mathcal{L}'$ and $D_{\mathfrak{Gr}}$.*

5.2 The universal action

In [20, Section 8], a universal action for diagrams[11] of groupoid correspondences of Ore shape is constructed. In this section, we apply this construction to the dynamical system $D_{\mathfrak{Gr}}$ in $\mathfrak{Gr}$ (see Lemma 4.30). This necessitates the assumption that P fulfils the right Ore conditions (see Definition 5.15). We thus construct a universal action for $D_{\mathfrak{Gr}}$ under this assumption.

Definition 5.6 (compare [20, Definition 4.12])**.** An action $((\alpha_p)_{p\in P}, Y)$ of a dynamical system $D_{\mathfrak{Gr}}$ is called universal if for every action $((\tilde{\alpha}_p)_{p\in P}, \tilde{Y})$, there is a unique $D_{\mathfrak{Gr}}$-equivariant continuous map $f\colon \tilde{Y} \to Y$.

Let $D_{\mathfrak{Gr}} = (G, P, \mathcal{X}_p, \mu_{p,q})$ be a dynamical system in $\mathfrak{Gr}$. The space upon which $D_{\mathfrak{Gr}}$ acts universally is defined as a limit of a certain diagram in the category **Top** of topological spaces. The shape category $\mathcal{D}$ for this diagram is defined as the comma category $P \downarrow \bullet$, where $\bullet$ is the only object of P. It has as objects arrows of P and as morphisms pairs (p, q), with domain pq and codomain p, where $p, q \in P$. The composition of such morphisms is given by $(p, q)(pq, r) := (p, qr)$. This is illustrated by

$$\begin{tikzcd} & \bullet \arrow[dl, "p"'] \\ \bullet & \bullet \arrow[l, "pq"'] \arrow[u, "\mathbf{q}"] \\ & \bullet \arrow[ul, "pqr"] \arrow[u, "\mathbf{r}"] \arrow[uu, bend right, "\mathbf{qr}"'] \end{tikzcd} \quad (\text{in } P) \qquad \begin{tikzcd} \mathrm{p} \\ \mathrm{pq} \arrow[u, "{(p,\mathbf{q})}"] \\ \mathrm{pqr} \arrow[u, "{(pq,\mathbf{r})}"] \arrow[uu, bend right, "{(p,\mathbf{qr})}"'] \end{tikzcd} \quad (\text{in } \mathcal{D}). \tag{48}$$

For comma categories, see [23, Exercises 1.3.vi–vii]. Compare the category $\mathcal{D}$ to the categories $\mathcal{D}_x$ in [20, paragraph following Lemma 8.2].

The data of the diagram $\mathcal{D} \to$ **Top** is given as follows: The image of $p \in \mathcal{D}^0$ is the quotient space of $\mathcal{X}_p$ with respect to the right G-action, namely

$$\mathcal{X}_p/G = G/\theta_p(G). \tag{49}$$

[11] strictly unital homomorphisms of bicategories with a category as domain

For $p, q \in P$, the image of $(p, q) \in \mathcal{D}^1$ is the map

$$\pi_{p,q}\colon G/\theta_{pq}(G) \to G/\theta_p(G) \tag{50}$$
$$g\theta_{pq}(G) \mapsto g\theta_p(G). \tag{51}$$

Then $(G/\theta_p(G), \pi_{p,q})$ is a diagram in **Top** with domain category $\mathcal{D}$, see [20, paragraph following Lemma 8.2].

Remark 5.7. We denote by $[g]_p$ the coset $g\theta_p(G) \in G/\theta_p(G)$, for $g \in G$ and $p \in P$.

The limit of $(G/\theta_p(G), \pi_{p,q})$ will serve as the space on which the universal action of the dynamical system $D_{\mathfrak{G}\mathfrak{r}}$ will be defined (compare [20, paragraph following Lemma 8.2]).

Lemma 5.8. *A limit of $(G/\theta_p(G), \pi_{p,q})$ is given by the set*

$$\Omega := \left\{ ([g_p]_p)_{p\in P} \in \prod_{p\in P} G/\theta_p(G) \,\middle|\, [g_{pq}]_p = [g_p]_p \text{ for all } p, q \in P \right\} \tag{52}$$

equipped with the topology on Ω generated by the cylinder sets

$$Z^{\Omega}_{(h_1,p_1),\dots,(h_m,p_m)} := \left\{ ([g_p])_{p\in P} \in \Omega \,\middle|\, [g_{p_i}]_{p_i} = [h_i]_{p_i},\ \text{for } i = 1, \dots, m \right\} \tag{53}$$

for $p_j \in P$, $h_j \in \mathcal{X}_{p_j} = G$, for $j = 1, \dots, m$, for $m \in \mathbb{N}$, which are closed in this topology, and the projections

$$\mathfrak{p}_p\colon \Omega \to \prod_{p'\in P} G/\theta_{p'}(G) \to G_p/\theta_p(G)\colon ([g_{p'}]_{p'})_{p'\in P} \mapsto [g_p]_p \tag{54}$$

as legs of the limit cone.

Proof. By [23, Proposition 3.5.2], **Top** is complete; hence the limit exists; and in the proof of the proposition, it is explained that since the forgetful functor $U\colon$ **Set** $\to$ **Top** is represented by the space with one point, by [23, Proposition 3.4.5], U preserves limits. In fact, a limit object in **Top** can be obtained from the limit of the underlying set diagram by equipping the limit set with the coarsest topology such that the legs of the limit cone are continuous (see [23, Example 3.5.3]). By [23, Theorem 3.4.12], a limit object for $(G/\theta_p(G), \pi_{p,q})$ is alternatively given by the limit object, or, more specifically, equaliser, for the diagram

$$\prod_{p\in\mathcal{D}^0=P} G/\theta_p(G) \ \underset{d}{\overset{c}{\rightrightarrows}}\ \prod_{(p,q)\in\mathcal{D}^1} G/\theta_p(G) \tag{55}$$

in **Top**, where

$$c(([g_p]_p)_{p\in P}) := ([g_p]_p)_{(p,q)\in\mathcal{D}^1}$$

and

$$d(([g_p]_p)_{p\in P}) := ([\pi_{p,q}([g_{pq}]_{pq})]_p)_{(p,q)\in\mathcal{D}^1} = ([g_{pq}]_p)_{(p,q)\in\mathcal{D}^1}.$$

The underlying set of this equaliser is the equaliser of the underlying diagram of sets (see above). Thus, the definition of Ω in (52) is such that the set Ω is this underlying set.

What is the coarsest topology on Ω, such that c and d are continuous? A subbasis of $\prod_{(p,q)\in\mathcal{D}^1} G/\theta_p(G)$ is given by the family of closed cylinder sets

$$\tilde{Z}_{h,(p,q)} \tag{56}$$

$$:= \left\{ ([g_{(r,s)}]_r)_{(r,s)\in\mathcal{D}^1} \in \prod_{(r,s)\in\mathcal{D}^1} G/\theta_r(G) \,\middle|\, [g_{(p,q)}]_p = [h]_p \text{ for all } q \in P \right\}. \tag{57}$$

The right topology on Ω is thus the one generated by the sets

$$\Omega \cap c^{-1}(\tilde{Z}_{h,(p,q)}) = \left\{([g_r])_{r\in P} \in \Omega \,\middle|\, [g_p]_p = [h]_p\right\} = Z^{\Omega}_{(h,p)}$$

and

$$\Omega \cap d^{-1}(\tilde{Z}_{h,(p,q)}) = \left\{([g_r])_{r\in P} \in \Omega \,\middle|\, [g_{pq}]_p = [h]_p \text{ for all } q \in P\right\} = Z^{\Omega}_{(h,p)}$$

for $h \in G$, $(p,q) \in \mathcal{D}^1$; and those sets are closed in this topology. The last equality is due to the coherence relations in the definition of Ω. Such sets are all of the form as the ones in (53) and the ones in (53) are exactly the intersections of positive finite numbers of such sets. Thus the sets in (53) form a basis of Ω and are closed. Inspecting the theory [23, Theorem 3.4.14] is based upon reveals that the maps in (54) are the legs of the (original) limit cone. $\square$

Remark 5.9. Albandik constructs a diagram of which $(G/\theta_p(G), \pi_{p,q})$ is the special case obtained by replacing a general locally compact étale groupoid by the group G (see [1, Section 3.3.1]). He considers the limit $\mathcal{H}^0$ (see [1, (3.29)]) of it. Thus, in our special case, $\Omega = \mathcal{H}^0$.

Lemma 5.10. *The space Ω is compact if and only if $\theta_p(G) \leq G$ has finite index for all $p \in P$.*

Proof. Suppose that $\theta_p(G) \leq G$ has finite index for all $p \in P$. Then $G/\theta_p(G)$ is finite for all $p \in P$. By Tychonoff's Theorem, this implies that $\prod_{p\in P} G/\theta_p(G)$ with the product topology is compact. The topology of $\Omega \subseteq \prod_{p\in P} G/\theta_p(G)$ is the relative topology with respect to the product topology (see Lemma 5.8). Since Ω is closed in the product topology, it is compact. Conversely, suppose that there is $p \in P$ such that $\theta_p(G) \leq G$ has infinite index. Then $\bigsqcup_{[x]\in G/\theta_p(G)} Z^{\Omega}_{(x,p)}$ is an infinite partition of Ω consisting of open sets, which implies that Ω is not compact. $\square$

Definition 5.11 ([26, Definition 3.6 and Lemma 3.9])**.** Let the diagonal of $\mathcal{O}[G, P, \theta_p]$, denoted $\mathcal{D}$, be its sub-C^*-algebra generated by the projections $e_{g,p}$ for $g \in G$ and $p \in P$. Let G_θ be its spectrum.

Remark 5.12. In Stammeier's finite-type case, the map $\Omega \to G_\theta$ by $\omega = ([g_p])_{p\in P} \mapsto \chi_\omega$, where

$$\chi_\omega(e_{g,p}) = \mathbb{1}_{[g]_p = [g_p]_p}, \tag{58}$$

for $g \in G$, $p \in P$, where $\mathbb{1}$ is the indicator function, seems likely to be a homeomorphism, such that $C(\Omega)$ is isomorphic to $\mathcal{D}$. (Ω is compact by Lemma 5.10) Compare the description of the topology of G_θ from [26, Lemma 3.9] to Ω. This is not so in the non-finite-type case: By [26, Lemma 3.9], G_θ is compact. But if there exists $p \in P$ such that $\theta_p(G) \leq G$ has infinite index, then, by Lemma 5.10, Ω is not compact.

Now we specify how $D_{\mathfrak{Gr}}$ acts on Ω. This is a special case of the construction in [20, around Lemma 8.4].

Lemma 5.13 (compare [20, second to last paragraph before Lemma 8.4])**.** *The space $\Omega \subseteq \prod_{p\in P} G/\theta_p(G)$ carries a left G-action coming from the left regular G-action on G, given by*

$$g.([\omega_p]_p)_{p\in P} := ([g\omega_p]_p)_{p\in P} \tag{59}$$

for all $g \in G$, $([\omega_p]_p)_{p\in P} \in \Omega$.

Proof. For each $p \in P$, G acts on the discrete space $\mathcal{X}_p/G = G/\theta_p(G)$ by

$$g.[x]_p := [gx]_p,$$

for all $g \in G$, $x \in \mathcal{X}_p = G$. Those actions consist of maps

$$a_p\colon G \times (G/\theta_p(G)) \to G/\theta_p(G),$$

for $p \in P$, which are continuous maps between discrete spaces. For $p, q \in P$, the diagram of continuous maps

$$\begin{tikzcd} G \times (G/\theta_p(G)) \arrow[r, "a_p"] & G/\theta_p(G) \\ G \times (G/\theta_{pq}(G)) \arrow[u, "\mathrm{id}_G \times \pi_{p,q}"] \arrow[r, "a_{pq}"] & G/\theta_{pq}(G), \arrow[u, "\pi_{p,q}"'] \end{tikzcd} \qquad \begin{tikzcd} (g, [x]_p) \arrow[r, mapsto] & {[gx]_p} \\ (g, [x]_{pq}) \arrow[u, mapsto] \arrow[r, mapsto] & {[gx]_{pq}} \arrow[u, mapsto] \end{tikzcd}$$

commutes, thus witnessing that the maps a_p, $p \in P$, form a natural transformation between the $\mathcal{D}$-shaped diagrams $(G \times (G/\theta_p(G)), \mathrm{id}_G \times \pi_{p,q})$ and $(G/\theta_p(G), \pi_{p,q})$. The former diagram has the limit $G \times \Omega$ where $(\mathrm{id}_G \times \mathfrak{p}_p)$, $p \in P$, form the limit cone. Now the maps

$$(a_p \circ (\mathrm{id}_G \times \mathfrak{p}_p))\colon G \times \Omega \to G/\theta_p(G)\colon \qquad (g, ([\omega_p]_p)_{p\in P}) \mapsto [g\omega_p]_p \tag{60}$$

form a cone over $(G/\theta_p(G), \pi_{p,q})$. This cone induces a continuous map $a\colon G \times \Omega \to \Omega$, for which the diagram

$$\begin{tikzcd} G \times (G/\theta_p(G)) \arrow[r, "a_p"] & G/\theta_p(G) \\ G \times \Omega \arrow[u, "\mathrm{id}_G \times \mathfrak{p}_p"] \arrow[r, "a"] & \Omega, \arrow[u, "\mathfrak{p}_p"'] \end{tikzcd}$$

commutes for all $p \in P$. Thus, since $\mathfrak{p}(([g_p]_p)_{p\in P}) = [g_p]_p$, a is determined by (60) as

$$a\colon G \times \Omega \to \Omega\colon \qquad (g, ([\omega_p]_p)_{p\in P}) \mapsto ([g\omega_p]_p)_{p\in P}. \tag{61}$$

Being a continuous map, a witnesses that (59) defines an action of G on Ω. $\square$

Specialising [20], we describe how, given that P fulfils the Ore conditions (see 5.15), the left G-action on Ω from Lemma 5.13 extends to an action of $D_{\mathfrak{Gr}}$ on Ω. Compare the following to [20, paragraph preceding Lemma 8.4].

The definition of Ω suggests that the maps

$$\mathcal{X}_p \times \mathcal{X}_q \to \mathcal{X}_{pq}\colon \qquad (x, y) \mapsto \mu_{p,q}([x, y]) = [x\theta_p(y)] \tag{62}$$

could somehow be used in order to define an action of $\mathcal{X}_p$ on Ω. Indeed, if we define

$$\Omega' := \lim_{q,(q,s)} (G/\theta_{pq}(G), \pi_{pq,s}), \tag{63}$$

$$= \left\{ ([g_{pq}])_{q\in P} \in \prod_{q\in P} G/\theta_{pq}(G) \,\middle|\, [g_{pqs}]_{pq} = [g_{pq}]_{pq} \text{ for all } q, s \in P \right\} \tag{64}$$

the maps in (62) induce a map

$$\mathcal{X}_p \times \Omega \to \Omega' \qquad \big(x, ([g_q])_q\big) \mapsto ([x\theta_p(g_q)]_{pq})_q. \tag{65}$$

There is also a canonical map

$$\Omega' \leftarrow \Omega, \qquad ([g_{pq}]_{pq})_q \leftarrow\!\shortmid ([g_r])_r. \tag{66}$$

Remark 5.14. While an element of Ω' can be seen as a "partial specification" of an element of Ω – a relation given by the map in (66) – beware that the q-component $[g_{pq}]_{pq}$ of $([g_{pq}]_{pq})_{q\in P} \in \Omega'$ corresponds to the pq-component $[g_{pq}]_{pq}$ of $([g_{\tilde p}]_{\tilde p})_{\tilde p \in P} \in \Omega$. So when elements of Ω' and Ω are seen as dependent functions $(q \in P) \to G/\theta_{pq}(G)$, and $(\tilde p \in P) \to G/\theta_{\tilde p}(G)$, respectively, the elements of the former do not come from those of the latter merely by restriction of the domain.

The data of an inverse of the map in (66) would consist of a way to extend a coherent choice of an element of $\mathcal{X}_{pq}/\mathcal{G}$ for each $q \in P$ to a coherent choice of elements of $\mathcal{X}_r/\mathcal{G}$ for *all* $r \in P$. In order to obtain such an extension in a canonical way, we require the monoid P to fulfil the right[12] Ore conditions:

Definition 5.15. [see [1, paragraph after Definition 3.1 and Definition 3.2] and [2, Definition 3.7], compare [20, Lemma 8.4]] A monoid P fulfils the *right Ore conditions* if

- for any $p_1, p_2 \in P$ there exist $q_1, q_2 \in P$ such that $p_1 q_1 = p_2 q_2$ and
- for any $p_1, p_2, r \in P$ for which $rp_1 = rp_2$ there exists $q \in P$ such that $p_1 q = p_2 q$.

Remark 5.16. In Stammeier's case, P is commutative. It is easy to check that then, P fulfils the right Ore conditions.

Let again $p \in P$. If P fulfils the right Ore conditions, then for each $r \in P$, there are $q_{p,r}, s_{p,r} \in P$, such that $pq_{p,r} = rs_{p,r}$. Hence, given an element $([g_{pq}]_{pq})_q$ of Ω', we can now specify an arbitrary r-component $[g_r]_r$ of an element $([g_r]_r)_r$ of Ω by

$$[g_r]_r := \pi_{r,s_{p,r}}([g_{pq_{p,r}}]_{rs_{p,r}}) = [g_{pq_{p,r}}]_r, \tag{67}$$

where for each $r \in P$, $q_{p,r} \in P$ is such that $pq_{p,r} = rs$ for some $s \in P$.

Lemma 5.17 (compare [20, Lemma 8.5])**.** *Suppose that P fulfils the right Ore conditions. Then* (67) *defines an inverse for the map in* (66)*. Composing it with the maps in* (65)*, we obtain the maps*

$$\begin{aligned} \alpha_p \colon \mathcal{X}_p \times \Omega \to \Omega' \to \Omega \\ (x, ([g_q])_q) \mapsto ([x\theta_p(g_{q_{p,r}})]_r)_r, \end{aligned} \tag{68}$$

[12] as opposed to left, not to wrong

where for each $r \in P$, $q_{r,s} \in P$ is such that $pq_{p,r} = rs$, for $p \in P$, which descend to homeomorphisms

$$\mathcal{X}_p \circ_G \Omega \xrightarrow{\sim} \Omega, \tag{69}$$

and thus define a $D_{\mathfrak{G}\mathfrak{r}}$-action $(\alpha_p)_{p\in P}$ on Ω. This action is universal.

Proof. Inspecting the constructions in [20, Section 8], considering the references above and the proof of [20, Lemma 8.5], one can see that our constructions are a special case thereof. By [20, Lemma 8.5], the maps presented as

$$\mathcal{X}_g \times_{s,\mathcal{G}^0_{s(g)},r} \Omega_s \to \varprojlim_{\mathcal{D}_{s(g)}} (\mathcal{X}_{gh}/\mathcal{G}_{s(h)}, \pi_{gh,k}) \cong \Omega_{r(g)}, \tag{70}$$

for "$g \in \mathcal{C}$", descend to homeomorphisms "$\mathcal{X}_g \circ \Omega_{s(g)} \to \Omega_{r(g)}$", which define an "$F$-action" on "$\Omega$".

Under our specialisation, "F" corresponds to $D_{\mathfrak{G}\mathfrak{r}}$, "$\Omega$" and "$\Omega_{r(g)}$", for "$g \in \mathcal{C}$", both correspond to Ω at the same time, "$\mathcal{X}_g$", for "$g \in \mathcal{C}$", correspond to $\mathcal{X}_p$, for $p \in P$;

$$\varprojlim_{\mathcal{D}_{s(g)}} (\mathcal{X}_{gh}/\mathcal{G}_{s(h)}, \pi_{gh,k}) \tag{71}$$

– in the proof denoted as "Ω'" – corresponds to Ω', and the first parts of the maps in (70), for $g \in \mathcal{C}$, correspond to the maps in (65), for $p \in P$. Inspecting the proof of [20, Lemma 8.5], one can see that the map from right to left in the isomorphism on the right in (70) corresponds to (66), and that the inverse of that map specialises to a map $\Omega' \to \Omega$ which is indeed described by (67). This shows that (67) is well defined, defines an inverse to (66), that (68) is well defined and that the picture in (70) corresponds to the picture in (68). Then indeed [20, Lemma 8.5] implies that the maps in (68) descend to homeomorphisms

$$\mathcal{X}_p \circ_G \Omega \xrightarrow{\sim} \Omega,$$

$p \in P$ which define a $D_{\mathfrak{G}\mathfrak{r}}$-action on Ω. By [20, Theorem 8.7], this action is universal. □

Remark 5.18. In Remark 5.14 we mentioned that the elements of Ω' can be seen as partial specifications of elements of Ω, a relation given by the map in (66).

Lemma 5.17 shows that, given that P fulfils the right Ore conditions, such a partial specification determines an element of Ω, which can be obtained by the formula in (67). Specifically, an element $([g_{\tilde{p}}]_{\tilde{p}})_{\tilde{p}\in P} \in \Omega$ is determined by $[g_{pq}]_{pq}$, for $q \in P$.

5.3 Slice-wise action of $D_{\mathfrak{G}\mathfrak{r}}$ by partial homeomorphisms

In [20, Section 5], starting from a diagram[13] F containing étale groupoid correspondences $\mathcal{X}_g$, $g \in \mathcal{C}$, and an action $(\vartheta_g \colon \mathcal{X}_g \circ_{\mathcal{G}_{\mathbf{s}(g)}} Y_{\mathbf{s}(g)} \to Y_{\mathbf{r}(g)})_{g\in\mathcal{C}}$ of this diagram on a space Y, Meyer constructs an inverse semigroup together with an action by partial homeomorphisms on $Y = \bigsqcup_{x\in\mathcal{C}^0} Y_x$.

The first step of this construction is to specify a type of open subsets (slices) $\mathcal{U}$ (see [20, Section 5.1]) of the correspondences $\mathcal{X}_g$, $g \in \mathcal{C}$, which are "small

[13] strictly unital homomorphisms with a category as domain

enough” such that the quotient map $\mathfrak{q}_g\colon \mathcal{X}_g \times_{\mathbf{s},\mathcal{G}_{\mathbf{s}(g)},\mathbf{r}} Y_{\mathbf{s}(g)} \to \mathcal{X}_g \circ_{\mathcal{G}_{\mathbf{s}(g)}} Y_{\mathbf{s}(g)}$ with respect to the diagonal action by $\mathcal{G}_{\mathbf{s}(g)}$ together with such $\mathcal{U}$ gives rise to a homeomorphism

$$\begin{aligned} Y_{\mathbf{s}(g)} \supseteq \mathbf{r}^{-1}(\mathbf{s}(\mathcal{U})) &\cong \mathcal{U} \times_{\mathbf{s},\mathcal{G}_{\mathbf{s}(g)},\mathbf{r}} \mathbf{r}^{-1}(\mathbf{s}(\mathcal{U})) \\ &\to \mathfrak{q}_g(\mathcal{U} \times_{\mathbf{s},\mathcal{G}_{\mathbf{s}(g)},\mathbf{r}} \mathbf{r}^{-1}(\mathbf{s}(\mathcal{U}))) \subseteq \mathcal{X}_g \circ_{\mathcal{G}_{\mathbf{s}(g)}} Y_{\mathbf{s}(g)}, \end{aligned} \tag{72}$$

for $g \in \mathcal{C}$, see [20, Lemma 5.1]. Post-composing the homeomorphisms ϑ_g encoding the action of F on Y results in partial homeomorphisms $\vartheta(\mathcal{U})$ from $Y_{\mathbf{s}(g)}$ to $Y_{\mathbf{r}(g)}$, for $g \in \mathcal{C}$.

With $\mathcal{S}(\mathcal{X})$ being the set of slices of a correspondence $\mathcal{X}$,

$$\bigsqcup_{g\in\mathcal{C}} \mathcal{S}(\mathcal{X}_g) \Big/ \{\text{empty slices of } \mathcal{X}_g, \text{ for all } g \in \mathcal{C}, \text{ are equivalent}\}$$

inherits a multiplication from the diagram F, see [20, Section 5.1]. The action of F on Y induces an action on Y by the partial homeomorphisms $\vartheta(\mathcal{U})$, for $\mathcal{U} \in \mathcal{S}(\mathcal{X}_g)$, $g \in \mathcal{C}$, which respects this multiplication, see [20, Section 5.2].

This first step will be specialised to the dynamical system $D_{\mathfrak{Gr}}$ in $\mathfrak{Gr}$ (see Lemma 4.30) and its universal action $((\alpha_p)_{p\in P}, \Omega)$ (see Lemma 5.17) constructed in Section 5.1. The construction of this universal action used that P fulfils the right Ore conditions (see Definition 5.15), a requirement we will therefore implicitly assume to be fulfilled throughout this section.

Definition 5.19 ([5, Definition 7.2] or [20, paragraph preceding Definition 2.1 and Definition 2.12])**.** A *slice* of an étale groupoid $\mathcal{G}$ is an open subset $\mathcal{U} \subseteq \mathcal{G}$ such that $\mathbf{s}|_{\mathcal{U}}$ and $\mathbf{r}|_{\mathcal{U}}$ are injective. Let $\mathcal{G}$ and $\mathcal{H}$ be locally compact étale groupoids. A *slice* of a locally compact étale groupoid correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ is an open subset $\mathcal{U} \subseteq \mathcal{X}$ such that $\mathbf{s}|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{G}^0$ and $\mathrm{p}|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{X}/\mathcal{G}$ are injective.

Remark 5.20 (see [5, second paragraph after Definition 7.2])**.** The slices of $\mathcal{G}$ seen as a correspondence $\mathcal{G} \leftarrow \mathcal{G}$ (see Example 4.14) are the same as its slices as a groupoid.

Remark 5.21. In other texts ([1],[2],[4],[25]), slices (of groupoids or of correspondences) are called *bisections*. The articles [5],[20] and [15] use the term *slices*, as we do, following [11].

Let $\mathcal{U}$ and $\mathcal{V}$ be slices of $\mathcal{X}_p$ and $\mathcal{X}_q$, respectively. Then

$$\mathcal{U}\mathcal{V} := \{\mu_{p,q}([u,v]) \in \mathcal{X}_{pq} \,|\, u \in \mathcal{U}, v \in \mathcal{V}, \mathbf{s}(u) = \mathbf{r}(v)\} \tag{73}$$

defines a slice of $\mathcal{X}_{pq}$ – inheriting the range map from $\mathcal{U}$ and the source map from $\mathcal{V}$ – by [5, Lemma 7.11] and the fact that $\mu_{p,q}$ is a biequivariant homeomorphism. For two slices $\mathcal{U}_1$ and $\mathcal{U}_2$ of $\mathcal{X}_p$,

$$\langle \mathcal{U}_1 \,|\, \mathcal{U}_2 \rangle_p := \left\{ \langle u \,|\, v \rangle_p \in \mathcal{G} \,\middle|\, x \in \mathcal{U}_1, y \in \mathcal{U}_2, \mathrm{p}(x) = \mathrm{p}(y) \right\} \tag{74}$$

defines a slice of $\mathcal{G}$ – as a correspondence, or, equivalently, as a groupoid. Compare [20, Section 5.1].

Lemma 5.22. *For a groupoid correspondence between two discrete groups as in Example 4.13, and specifically for the $\mathcal{X}_p$, $p \in P$, in our dynamical system $D_{\mathfrak{Gr}}$, the slices are just the empty set, and the sets containing exactly one element of the correspondence.*

Proof. Let $\mathcal{X}: H \leftarrow G$ be a locally compact étale groupoid correspondence between two discrete groups H and G. Since the anchor map $\mathbf{s}: \mathcal{X} \to G^0 =: \{*\}$ is a local homeomorphism, $\mathcal{X}$ carries the discrete topology. If $\mathcal{U}$ is a slice of $\mathcal{X}$, then, since $\mathbf{s}|_{\mathcal{U}}: \mathcal{U} \to G^0 = \{*\}$ is injective, $\mathcal{U}$ can have at most one element. Conversely, if $\mathcal{U} \subseteq \mathcal{X}$ is the empty set or a singleton, it is open, and the maps $\mathbf{s}|_{\mathcal{U}}: \mathcal{U} \to \mathcal{G}^0$ and $\mathrm{p}|_{\mathcal{U}}: \mathcal{U} \to \mathcal{X}/\mathcal{G}$ are injective, since their domain contains at most one element; hence $\mathcal{U}$ is a slice of $\mathcal{X}$. Compare the proof of [20, Lemma 9.2]. $\square$

We describe a semigroup of (labelled) slices which will later be extended to an inverse semigroup $I(D_{\mathfrak{Gr}})$ described in Section 5.4.

Lemma 5.23 (compare [20, Section 5.1])**.** *The dynamical system $D_{\mathfrak{Gr}}$ together with the multiplication of slices given by (73) gives rise to a semigroup $\mathcal{S}(D_{\mathfrak{Gr}})$ with underlying set*

$$\big\{(x, p) \,\big|\, p \in P, x \in \mathcal{X}_p\big\} \sqcup \{0\} \tag{75}$$

with multiplication given by

$$(x_1, p_1) \cdot (x_2, p_2) := (\mu_{p_1,p_2}([x_1, x_2]), p_1p_2) = (x_1 \cdot \theta_{p_1}(x_2), p_1p_2), \tag{76}$$

for $x_i \in \mathcal{X}_{p_i}$, $p_i \in P$, $i = 1, 2$, and the rule that 0 is an absorbing element.

Proof. The multiplication of slices given by (73) induces a multiplication on the set

$$\bigsqcup_{p\in P} \mathcal{S}(\mathcal{X}_p).$$

This multiplication is associative due to (40). The set $\{(p, \emptyset) \,|\, p \in P\}$ of empty slices for different $p \in P$ is an absorbing subset. Hence identifying its elements is a semigroup congruence relation. The non-empty slices of $\mathcal{X}_p$ are of the form $(p, \{x\})$, $x \in \mathcal{X}_p$, $p \in P$ by Lemma 5.22. By replacing $\{x\}$ by x and swapping the components of the pairs, taking the quotient with respect to the congruence relation described above, and calling the unified empty slice, which is an absorbing element in the quotient, 0, we obtain a semigroup matching the description in the lemma. $\square$

Lemma 5.24. *Define the semigroup $\mathcal{S}^\times(D_{\mathfrak{Gr}}) := \mathcal{S}(D_{\mathfrak{Gr}})\backslash\{0\}$. Then $\mathcal{S}^\times(D_{\mathfrak{Gr}}) \cong G \rtimes_\theta P$, where $G \rtimes_\theta P$ is the semidirect product of discrete semigroups as in* [26, second paragraph after Definition A.2] *(see also* [26, Proposition 3.18]*).*

Proof. From the definition of $\mathcal{S}(D_{\mathfrak{Gr}})$ in Lemma 5.23 it is easy to see that $\mathcal{S}^\times(D_{\mathfrak{Gr}}) = \mathcal{S}(D_{\mathfrak{Gr}}) \setminus \{0\}$ is a subsemigroup of $\mathcal{S}(D_{\mathfrak{Gr}})$ with the multiplication given by (76) and that it coincides as a semigroup with $G \rtimes_\theta P$ as described in [26, second paragraph following Definition A.2]. $\square$

Remark 5.25. In Stammeier's case, $G \rtimes_\theta P$ is identical to the semigroup "$G \rtimes_\theta P$" occurring in [26, Proposition 3.18].

Following [20], we will describe an encoding of the action in (68) by an action of $\mathcal{S}(D_{\mathfrak{G}\mathfrak{r}})$ by partial homeomorphisms on Ω. To this end, we define partial homeomorphisms of a set and their composition.

Definition and Lemma 5.26 (see [11, Definition 4.2] and [20, Example 5.6])**.** *Let Y be a topological space. Let f and g be two partial homeomorphisms on Y, that is, homeomorphisms between open subsets of Y. Then a partial homeomorphism gf, the* concatenation *of g and f, is defined by*

$$gf\colon f^{-1}(\mathrm{dom}(g)\cap \mathrm{im}(f)) \to g(\mathrm{dom}(g)\cap \mathrm{im}(f))\colon x\mapsto g(f(x)). \tag{77}$$

This concatenation, together with the empty map $\emptyset\colon \emptyset\to\emptyset$, which is absorbing with respect to the concatenation, turns the set $\mathcal{I}(Y)$ of partial homeomorphisms on Y into a semigroup with zero. When $f\colon U\to V$ is a partial homeomorphism on Y with open domain $U\subseteq Y$ and open codomain $V\subseteq Y$, then we write $f^*\colon V\to U$ for its inverse. See Lemma 5.35 for the motivation for this notation.

The partial homeomorphisms on Ω associated to $(x,p)\in\mathcal{S}(D_{\mathfrak{G}\mathfrak{r}})$, $p\in P$, $x\in\mathcal{X}_p$, are described in the next lemma.

Lemma 5.27 (compare [20, Lemma 5.1 and paragraph preceding Lemma 5.2])**.** *For $p\in P$, the map (68) induces a homeomorphism*

$$\vartheta_{(x,p)}\colon \Omega\to Z^{\Omega}_{(x,p)}\subseteq\Omega, \tag{78}$$

for each $x\in\mathcal{X}_p$, described by

$$\vartheta_{(x,p)}(([g_q])_q) = ([x\theta_p(g_{q_{p,r}})]_r)_r \text{ for } ([g_q]_q)_q\in\Omega, \tag{79}$$

where $q_{p,r}\in P$ is such that there exists $s_{p,r}\in P$ with $pq_{p,r}=rs_{p,r}$, for all $r\in P$.

Proof. Fix $p\in P$ and $x\in\mathcal{X}_p$. By Lemma 5.22, $\{x\}$ is a slice of $\mathcal{X}_p$. Then, by [20, Lemma 5.1],

$$\Omega\to\mathfrak{q}(\{x\}\times\Omega)\subseteq\mathcal{X}_p\circ\Omega\colon \omega\mapsto\mathfrak{q}(x,\omega),$$

is a homeomorphism. Here, $\mathfrak{q}\colon\mathcal{X}_p\times\Omega\to\mathcal{X}_p\circ\Omega$ is the quotient map with respect to the diagonal action of G. Post-composing the homeomorphism $\mathcal{X}_p\circ\Omega\to\Omega$ in (69) gives a partial homeomorphism which is described by (79), compare [20, paragraph preceding Lemma 5.2]. It remains to show that its image is $Z^{\Omega}_{(x,p)}$. For any $([g_p]_p)_{p\in P}\in\Omega$, $(\vartheta_{(x,p)}(([g_q]_q)_{q\in P}))_p = [x\theta_p(g_1)]_p = [x]_p$, and hence $\vartheta_{(x,p)}(([g_q]_q)_{q\in P})\in Z^{\Omega}_{(x,p)}$. Conversely, suppose that $([\tilde g_q]_q)_{q\in P}\in\Omega$ with $[\tilde g_p]_p=[x]_p$. Then, for each $q\in P$, $[\tilde g_{pq}]_p=[x]_p$, and hence, there exists $g_q\in G$ such that $\tilde g_{pq}=x\theta_p(g_q)$. For $q,r\in P$, since $[\tilde g_{pqr}]_{pq}=[\tilde g_{pq}]_{pq}$, there is $h\in G$ such that $\tilde g_{pqr}=\tilde g_{pq}\theta_{pq}(h)$; hence

$$x\theta_p(g_{qr}) = \tilde g_{pqr} = \tilde g_{pq}\theta_{pq}(h) = x\theta_p(g_q)\theta_{pq}(h) = x\theta_p(g_q\theta_q(h)),$$

which implies $g_{qr}=g_q\theta_q(h)$ by injectivity of θ_p, and hence $[g_{qr}]_q=[g_q]_q$; thus, $([g_q]_q)_{q\in P}$ is an element of Ω. Then

$$\big(\vartheta_{(x,p)}(([g_q]_q)_{q\in P})\big)_{pq} = [x\theta_p(g_q)]_{pq} = [\tilde g_{pq}]_{pq}.$$

By Remark 5.18, this implies that $\vartheta_{(x,p)}(([g_q]_q)_{q\in P}) = ([\tilde g_{\tilde p}]_{\tilde p})_{\tilde p\in P}$. We have thus shown that the image of $\vartheta_{(x,p)}$ is $Z^{\Omega}_{(x,p)}$, which completes the proof. □

Lemma 5.28 (compare [20, Lemma 5.2])**.** *Mapping $0 \in \mathcal{S}(D_{\mathfrak{Gr}})$ to the empty partial homeomorphism $\emptyset \in \mathcal{I}(\Omega)$ and the other elements of $D_{\mathfrak{Gr}}$, that is, the pairs (x, p), for $p \in P$, $x \in \mathcal{X}_p$, to the partial homeomorphisms (78), defines a homomorphism $\mathcal{S}(D_{\mathfrak{Gr}}) \to \mathcal{I}(\Omega)$ of semigroups with zero.*

Proof. Since $0 \in \mathcal{S}(D_{\mathfrak{Gr}})$ and $\emptyset \in \mathcal{I}(\Omega)$ are the zeros in their respective semigroups with zero, it suffices to show that

$$\vartheta_{(x_1,p_1)(x_2,p_2)} = \vartheta_{(x_1,p_1)}\vartheta_{(x_2,p_2)} \tag{80}$$

for all $p_i \in P$, $x_i \in \mathcal{X}_p$, $i = 1, 2$. The pairs (x_i, p_i) come from slices $\{x_i\}$. Their multiplication is based on the product of their associated slices as in (73) and hence in [20, Section 5.1], see the proof of Lemma 5.23. The association of partial homeomorphisms to pairs (x_i, p_i) thus corresponds to the association of partial homeomorphisms to slices in [20, paragraph preceding Lemma 5.2]. Hence, [20, Lemma 5.2] implies (80). This concludes the proof. □

Remark 5.29. Suppose Stammeier's conditions hold and Stammeier's finite-type condition holds. Then the action ϑ of $\mathcal{S}^\times(D_{\mathfrak{Gr}})$ on Ω by partial homeomorphisms induces an action ϑ_* of $\mathcal{S}^\times(D_{\mathfrak{Gr}})$ on $C(\Omega)$ by isometries. When we identify $\mathcal{S}^\times(D_{\mathfrak{Gr}})$ with $G \rtimes_\theta P$ as in Lemma 5.24, the presumed isomorphism $C(\Omega) \to \mathcal{D}$ in Remark 5.12 would be likely to be equivariant for ϑ_* and the action of $G \rtimes_\theta P$ on $\mathcal{D}$ by isometries described in [26, Proposition 3.18].

Remark 5.30. In the next proposition, we use Laca's and Raeburn's semigroup crossed product construction for actions of semigroups on unital C^*-algebras (see [16, Section 2]).

Remark 5.31. Suppose that Stammeier's conditions hold and that Stammeier's finite-type condition holds. Suppose that we can identify $C(\Omega)$ with Stammeier's $\mathcal{D}$ as in Remark 5.12 and that this identification is equivariant for the actions of $\mathcal{S}^\times(D_{\mathfrak{Gr}}) \cong G \rtimes_\theta P$ on $C(\Omega)$ and $\mathcal{D}$ (see Remark 5.29). Then, by [26, Proposition 3.18] and Remark 5.29, the semigroup crossed product $C(\Omega) \rtimes_{\vartheta_*} \mathcal{S}^\times(D_{\mathfrak{Gr}})$ is isomorphic to $\mathcal{O}[G, P, \theta]$.

5.4 Extension of the action to an inverse semigroup

We are halfway through the application of Meyer's construction of an inverse semigroup with an action by partial homeomorphisms, starting with an action of an Ore-category shaped diagram of correspondences (see [20, Section 5]), to the universal action of $D_{\mathfrak{Gr}}$ in the case that P fulfils the right Ore conditions.

In Section 5.3, we specialised the first step of the construction, thus obtaining a semigroup $\mathcal{S}(D_{\mathfrak{Gr}})$ of (P-indexed) slices and an action of it by partial homeomorphisms on Ω. In this section, specialising the remainder of Meyer's construction (see [20, Sections 5.2 and 5.3]), we extend this action to an inverse semigroup. Roughly speaking, this can be seen as adding the partial inverses which will eventually be turned into inverses in a groupoid in Section 5.5.

Lemma 5.32 (compare [20, Lemma 5.2])**.** *For $p \in P$ and $x, y \in \mathcal{X}_p$,*

$$\vartheta^*_{(x,p)}\vartheta_{(y,p)} = \begin{cases} \vartheta_{(\langle x \,|\, y\rangle_p, 1)}, & \text{if } [x]_p = [y]_p, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that for $x, y \in \mathcal{X}_p$ with $[x]_p = [y]_p$, $\langle x \mid y \rangle_p$ is the element $g \in G$ such that $y = x\theta_p(g)$ (see Remark 4.31).

Proof. This follows from [20, Lemma 5.2] analogously to how (80) does; see the proof of Lemma 5.28. Recall that the pairs (x, p), (y, q) come from the slices $\{x\}, \{y\} \in \mathcal{S}(\mathcal{X}_p)$, see the proof of Lemma 5.28. It has to be observed that $(\langle x \mid y \rangle_p, 1)$, for $x, y \in \mathcal{X}_p$ with $[x]_p = [y]_p$, $p \in P$, come from the slice $\langle \{x\} \mid \{y\} \rangle$ as defined in (74). □

Definition 5.33 ([20, second and third paragraph of Section 5.3])**.** An *inverse semigroup* is a semigroup S with the property that for each element $a \in S$, there exists a unique element a^* such that $aa^*a = a$ and $a^*aa^* = a^*$.

An element $e \in S$ is *idempotent* if $e^2 = e$. The set of idempotents of S is denoted by $E(S)$. The idempotent elements form a commutative subsemigroup, and $ses^* \in E(S)$ for all $s \in S$, $e \in E(S)$.

Observe that $s^*s, ss^* \in E(S)$ for any $s \in S$ in an inverse semigroup S.

Lemma 5.34. *Let S be an inverse semigroup. Let $e, f \in E(S)$ be idempotents and let $s, u \in S$.*

1. *If $s^*ue = e$, then $ue = se$.*
2. *$ef = fe$.*
3. *$es = ss^*es$.*

Proof. In order to prove the first assertion, assume $s^*ue = e$ and consider

$$es^*(ue)es^* = e(s^*ue)s^* = es^*, \tag{81}$$
$$ue(es^*)ue = ue(s^*ue) = ue. \tag{82}$$

Then $ue = se$ follows from the uniqueness condition of the "inverses" in an inverse semigroup (Definition 5.33).

For the second assertion, see [18, Theorem 3].

For the third assertion, observe that $ess^* = ss^*e$ by (2), because s^*s is idempotent. Multiplying both sides with s on the right yields $es = ss^*es$. □

Lemma 5.35 (see [11, Definition 4.2] and [20, Example 5.6])**.** *Let Y be a topological space. Then the semigroup with zero $\mathcal{I}(Y)$ of homeomorphisms $U \to V$, where U, V are open subsets of Y, with multiplication given by (77), as in Definition and Lemma 5.26, extends to an inverse semigroup with zero with inverses f^*, for $f\colon U \to V$, given by the inverse partial homeomorphism of f defined on V.*

Definition 5.36 (compare [11, Definition 4.3])**.** An action of an inverse semigroup S on a topological space Y by local homeomorphisms is a semigroup homomorphism $\vartheta\colon S \to \mathcal{I}(Y)$ such that $Y = \bigcup_{s \in S} \mathrm{dom}(\vartheta_s)$.

We denote by D_e the domain of ϑ_e for an idempotent $e \in E(S)$. Hence, D_{s^*s} is the domain of ϑ_s, for an arbitrary $s \in S$ (see [11, Notation 4.4]).

Lemma 5.37 (see [11, neighbourhood of Notation 4.4 and Proposition 4.5])**.** *Suppose that $e, f \in E(S)$ are idempotents and $s \in S$. Then $\vartheta_e = \mathrm{id}_{D_e}$, $D_{ef} = D_e \cap D_f$ and $\vartheta_{ef} = \mathrm{id}_{D_e \cap D_f}$. Furthermore, ses^* is idempotent and $D_{ses^*} = \vartheta_s(D_e \cap D_{s^*s})$.*

Proof. Let $x \in D_e$. Since ϑ_e is idempotent, $\vartheta_e(x) \in D_e$ and $\vartheta_e(\vartheta_e(x)) = \vartheta_e(x)$. Since ϑ_e is injective, $\vartheta_e(x) = x$. Thus, ϑ_e is the identity map on its domain D_e.

It follows easily from the definition of multiplication in $\mathcal{I}(Y)$ that $\vartheta_e\vartheta_f = \mathrm{id}_{D_e}\mathrm{id}_{D_f}$ has $D_e \cap D_f$ as domain and codomain and is the identity on it.

Since idempotents commute (see Lemma 5.34), $ses^*ses^* = ss^*sees^* = ses^*$; hence ses^* is idempotent. For the last statement, see [11, Proposition 4.5]. □

Lemmata 5.23, 5.28 and 5.32 justify the following definition:

Definition 5.38 (compare [20, Definition 5.7])**.** Let $I(D_{\mathfrak{Gr}})$ be the universal inverse semigroup with zero 0 for generators (x, p), for $p \in P$, $x \in \mathcal{X}_p$, and relations

$$(x_1, p_1)(x_2, p_2) \sim (x_1\theta_{p_1}(x_2), p_1p_2) \tag{83}$$

$$(x, p)^*(y, p) \sim \begin{cases} (\langle x \,|\, y\rangle_p, 1), & \text{if } [x]_p = [y]_p, \\ 0, & \text{otherwise,} \end{cases} \tag{84}$$

for $p, p_1, p_2 \in P$ and $x, y \in \mathcal{X}_p$, $x_i \in \mathcal{X}_i$, $i = 1, 2$.

Lemma 5.39 (compare [20, paragraph following Definition 5.8])**.** *The family of maps $\vartheta_{(x,p)}$, $p \in P$, $x \in \mathcal{X}_p$, extends to an action of the inverse semigroup $I(D_{\mathfrak{Gr}})$ on Ω.*

Proof. The assertion follows from Lemmata 5.23, 5.28 and 5.32. □

5.5 The groupoid model as transformation groupoid

In Sections 5.3 and 5.4, Meyer's construction of an inverse semigroup with an action by partial homeomorphisms starting from an Ore-category shaped diagram F acting on a space (see [20, Section 5]) has been applied to the universal action of $D_{\mathfrak{Gr}}$ on Ω in the case that P fulfils the right Ore conditions.

In [20, Section 5], Meyer shows that when the action is universal for F, then the transformation groupoid of the constructed action of the inverse semigroup is a groupoid model for F. For the case that P fulfils the right Ore conditions, applying this to our specialisation, with $D_{\mathfrak{Gr}}$ and its universal action on Ω underlying the construction, in this section, we obtain a groupoid model for $D_{\mathfrak{Gr}}$.

Definition 5.40 ([20, Definition 5.10], see also [11, Section 4])**.** Let S be an inverse semigroup and let Y be a topological space with an action $\vartheta\colon S \to \mathcal{I}(Y)$. The *transformation groupoid* $S \ltimes Y$ is defined as follows. Its object space is Y, and its set of arrows is the set of equivalence classes of pairs (t, x) for $t \in S$ and $x \in \mathrm{dom}(\vartheta_t)$, where $(t, x) \sim (u, x')$ if $x = x'$ and there is an idempotent element $e \in E(S)$ such that ϑ_e is defined at $x = x'$ and $te = ue$. We define the range and source maps by $\mathbf{s}(t, x) := x$, $\mathbf{r}(t, x) = \vartheta_t(x)$ and $(u, \vartheta_t(x)) \cdot (t, x) = (ut, x)$ whenever this is defined. There is a unique topology on the arrow space that makes $S \ltimes Y$ an étale groupoid.

Remark 5.41. Unlike Exel ([11]), we neither require Y to be locally compact nor Hausdorff. Exel calls transformation groupoids *groupoids of germs* (see [11, Proposition 4.17]). The term *groupoid of germs* might be used differently by other authors.

Lemma 5.42 (see [11])**.** *A basis for the topology on $S \ltimes Y$ is given by the slices*

$$\Theta(s, \mathcal{U}) := \{[s, x]_\sim \mid x \in \mathcal{U}\}, \tag{85}$$

for $s \in S$ and $\mathcal{U} \subseteq D_{s^*s}$ open in Y. (Here the sets $\mathcal{U}$ may be restricted to belong to a given basis of the topology on Y.)

Proof. The definition of $\Theta(s, \mathcal{U})$ in (85) coincides with [11, (4.12)]. By [11, paragraph preceding Proposition 4.14], those sets form the basis for a topology on $S \ltimes Y$. By [11, Proposition 4.17], with this topology, $S \ltimes Y$ becomes an étale topological groupoid. In Definition 5.40 (or equivalently [20, Definition 5.10]), $S \ltimes Y$ is equipped with the *unique* topology making $S \ltimes Y$ an étale groupoid. Thus, this must be the one generated by the sets in (85). By [11, paragraph preceding Proposition 4.18], the basic sets in (85) are slices.

In [11], the space Y acted upon is assumed to be locally compact and Hausdorff (see also Remark 5.41). This is however not required for the statements cited in this proof. $\square$

Let $\mathcal{L} := I(D_{\mathfrak{Gr}}) \ltimes \Omega$ be the transformation groupoid for the action

$$\vartheta \colon I(D_{\mathfrak{Gr}}) \to \mathcal{I}(\Omega)$$

described in Section 5.4.

Proposition 5.43 (compare [20, Corollary 8.8])**.** *Suppose that P fulfils the right Ore conditions. The étale groupoid $\mathcal{L}$ is a groupoid model for $D_{\mathfrak{Gr}}$ (see Definition 5.4).*

Proof. For the proof, we apply [20, Proposition 5.12]. This proposition occurs in [20, Section 5], where for a diagram (a strictly unital homomorphism of bicategories with a category as domain) "F" and a universal action of "F" on a space "Y", an inverse semigroup "$I(F)$", an action "$I(F) \to \mathcal{I}(Y)$" and the transformation groupoid "$I(F) \ltimes Y$" are constructed. The cited proposition states that the transformation groupoid "$I(F) \ltimes Y$" is a groupoid model for "F", if the action of "F" on "Y" is universal. Why can we apply [20, Proposition 5.12] to our situation to yield our proposition? Under the "special case relation" already present in the proof of Lemma 5.17, our dynamical system $D_{\mathfrak{Gr}}$ and the space Ω correspond to the diagram "F" and the space "Ω" from [20, Section 8], respectively. Now "Ω" can be plugged in for "Y" in [20, Section 5]. This "composed special case relation" can be extended by viewing the universal action $((\alpha_p)_{p \in P}, \Omega)$ from Section 5.2 as a specialisation of the one of "F" on "Y" in [20, Section 5]. The whole construction hence leading up to $I(D_{\mathfrak{Gr}}) \ltimes \Omega$ corresponds to the construction leading up to "$I(F) \ltimes Y$" in [20, Section 5]; see the references along the way. Now, since $((\alpha_p)_{p \in P}, \Omega)$ is universal by Lemma 5.17, [20, Proposition 5.12] implies that $\mathcal{L} = I(D_{\mathfrak{Gr}}) \ltimes \Omega$ is a groupoid model for $D_{\mathfrak{Gr}}$. $\square$

Proposition 5.44. *Suppose that P fulfils the right Ore conditions. If $\theta_p(G) \leq G$ has finite index for all $p \in P$, then the groupoid model, the étale groupoid $\mathcal{L}$, is locally compact, that is, its object space $\mathcal{L}^0 = \Omega$ is Hausdorff and locally compact.*

Proof. Suppose that $\theta_p(G)$ has finite index in G for all $p \in P$. Then by Lemma 4.25, $D_{\mathfrak{Gr}}$ is a diagram of proper, locally compact groupoid correspondences. Then by [15, Theorem 6.3], the space Ω underlying the universal action is Hausdorff and locally compact, and (hence) the groupoid model $\mathcal{L}$ is locally compact. Implicitly required for the cited Theorem and its application is the fact that the groupoid model is unique up to isomorphism, see Lemma 5.5. $\square$

Remark 5.45. Suppose that P fulfils the right Ore conditions and that $\theta_p(G) \leq G$ has finite index for all $p \in P$. Since the dynamical system $D_{\mathfrak{Gr}}$ is a P-shaped diagram in $\mathfrak{Gr}$ (see Lemma 4.29), by Remark 4.33, and because the correspondences $\mathcal{X}_p$, $p \in P$, are proper by Lemma 4.25, it is "an action of P on G by proper correspondences" as in [1, first paragraph of Section 3.3.1].

In [1, Section 3.3.1], from such an action, another action by *tight* correspondences $\tilde{\mathcal{X}}_p$ over a locally compact étale groupoid (!) $\tilde{\mathcal{G}}$ is constructed. In [1, Section 3.2.1], for such a tight action, a locally compact étale groupoid $\mathcal{H}$ is constructed. By [1, Theorem 3.30], $\mathcal{H}$ is a (bicategorial) colimit (see [1, Section 2.6.1]) for the original proper action in Albandik's "$\mathfrak{Gr}$". With the direction of 1-arrows in the version $\mathfrak{Gr}$ of we use, this would be a limit. We do not consider bicategorial limits here. When we apply Albandik's construction described above to $D_{\mathfrak{Gr}}$ under the conditions given above, by [20, Remark 8.22][14], $\mathcal{H}$ is identical to our groupoid model $\mathcal{L}$.

Remark 5.46. Suppose that P fulfils the Ore conditions and that $\theta_p(G) \leq G$ is finite for all $p \in P$. Then Ω is compact by Lemma 5.10 and *a fortiori* locally compact[15]. Furthermore, by Proposition 7.2, $I(D_{\mathfrak{Gr}}) \ltimes \Omega$ is Hausdorff. Thus, by [8, Theorem 7.6], there is an isomorphism

$$C^*(I(D_{\mathfrak{Gr}}) \ltimes \Omega) \cong I(D_{\mathfrak{Gr}}) \ltimes_{\vartheta_*} C_0(\Omega), \tag{86}$$

where the right side is a crossed product as constructed by Sieben ([24]) for actions of unital inverse semigroups on C^*-algebras by partial automorphisms. It seems very plausible that, using Sieben's theory ([24, Sections 3 and 4]) and the definition of the semigroup crossed product ([16, Proposition 2.1]), one can show that there is an isomorphism

$$I(D_{\mathfrak{Gr}}) \ltimes_{\vartheta_*} C_0(\Omega) \cong C(\Omega) \rtimes_{\vartheta_*} \mathcal{S}^{\times}(D_{\mathfrak{Gr}}). \tag{87}$$

If this is true, and, as suggested in Remark 5.31, $C(\Omega) \rtimes_{\vartheta_*} \mathcal{S}^{\times}(D_{\mathfrak{Gr}})$ is isomorphic to $\mathcal{O}[G, P, \theta]$, then there is an isomorphism

$$C^*(I(D_{\mathfrak{Gr}}) \ltimes \Omega) \cong \mathcal{O}[G, P, \theta], \tag{88}$$

or, put differently, $I(D_{\mathfrak{Gr}}) \rtimes \Omega$ is a groupoid model for Stammeier's C^*-algebra $\mathcal{O}[G, P, \theta]$. In fact, this will be shown via product systems in Section 6, see Corollary 6.14.

[14] This refers to the version of the preprint following the version in the references.

[15] Local compactness also follows from Proposition 5.44.

6 Translation to C^*-correspondences

Essential product systems in the sense of Fowler ([12]) can be considered (strictly unital) monoid shaped diagrams in $\mathfrak{Corr}$, a bicategory of C^*-correspondences whose original version was introduced by Buss, Meyer and Zhu in [10].

The dynamical systems in $\mathfrak{Gr}$ (see Definition 4.32) induce certain essential product systems (see Remark 6.7), which can be considered *dynamical systems* in $\mathfrak{Corr}$. Namely, those are obtained by postcomposing to the dynamical systems in $\mathfrak{Gr}$ a homomorphism of bicategories from $\mathfrak{Gr}$ to $\mathfrak{Corr}$ which has been first described by Albandik ([1]), related to constructions in [13], [21] and [22].

Applying this homomorphism of bicategories to the groupoid models $\mathcal{L}$ (see Section 5.5) in the case that P fulfils the right Ore conditions (see Definition 5.15) yields the groupoid C^*-algebras $C^*(\mathcal{L})$. When P is Ore and $\theta_p(G) \leq G$ has finite index for all $p \in P$, then those groupoid C^*-algebras are the Cuntz-Pimsner algebras of the aforementioned product systems coming from $D_{\mathfrak{Gr}}$; this is a special case of a proposition by Albandik ([1]).

In Stammeier's case, the product system coincides with a product system described by Stammeier; Stammeier shows that in the finite-type case, the C^*-algebra $\mathcal{O}[G, P, \theta]$ associated to his irreversible algebraic dynamical system coincides with the Cuntz-Pimsner algebra of this product system. Hence, in Stammeier's finite-type case, $\mathcal{L}$ is a groupoid model for Stammeier's C^*-algebra.

6.1 The bicategory $\mathfrak{Corr}$ of C^*-correspondences

We recall the definition of a bicategory $\mathfrak{Corr}$ of C^*-correspondences. It was introduced by Buss, Meyer and Zhu in [10, Section 2.2]. This original version is also used by Albandik ([1]) but we need the slightly different version described in [5, end of Section 6]. In the latter version (the one we use), the 1-arrows go in the other direction and the 2-arrows are not required to be invertible, as in the original bicategory.

The objects of the bicategory $\mathfrak{Corr}$ are C^*-algebras, the arrows are C^*-correspondences:

Definition 6.1 (compare [10, Definition 2.6])**.** Let A and B be C^*-algebras. A C^*-correspondence $\mathcal{E}\colon A \leftarrow B$ is a Hilbert B-module $\mathcal{H}$ with a non-degenerate $*$-representation of A on $\mathcal{H}$ by adjointable operators. If A acts on $\mathcal{H}$ by compact operators, the C^*-correspondence $\mathcal{E}$ is called *proper*.

Let A, B be C^*-algebras. The 2-arrows in $\mathfrak{Corr}$ between two C^*-correspondences $\mathcal{E}, \mathcal{F}\colon A \leftarrow B$ are isometric A, B-bimodule maps $\alpha\colon \mathcal{E} \to \mathcal{F}$. Here, "isometric" means that $\langle \alpha(x) \,|\, \alpha(y)\rangle = \langle x \,|\, y\rangle$ for $x, y \in \mathcal{E}$. C^*-correspondences $B \leftarrow A$ and biequivariant isometries between them with the usual composition as maps form the Hom-category $\mathfrak{Corr}(A, B)$. See [5, end of Section 6].

Given two composable C^*-correspondences $\mathcal{E}\colon A \leftarrow B$, $\mathcal{F}: B \leftarrow C$, we define their composition $\mathcal{E} \otimes \mathcal{F}: A \leftarrow C$. The algebraic tensor product $\mathcal{E} \otimes_{\text{alg}} \mathcal{F}$ is the universal vector space with generating set $\mathcal{E} \times \mathcal{F}$ and relations

$$(e \cdot b, f) \sim (e, b \cdot f), \tag{89}$$

$$(ec + e', f) \sim (e, f) \cdot c + (e', f), \tag{90}$$

$$(e, fc + f') \sim (e, f) \cdot c + (e, f'), \tag{91}$$

for $e, e' \in \mathcal{E}$, $f, f' \in \mathcal{F}$, $b \in B$, $c \in \mathbb{C}$, equipped with left A- and right C-actions inherited from $\mathcal{E}$ and $\mathcal{F}$, respectively, in the obvious way. The congruence class of (e, f) is denoted by $e \otimes f$. Now $\mathcal{E} \otimes \mathcal{F}$ is the completion of $\mathcal{E} \otimes_{\mathrm{alg}} \mathcal{F}$ with respect to the norm induced by the C-valued inner product

$$\langle e \otimes f \,|\, e' \otimes f' \rangle := \left\langle f \middle| \, \langle e \,|\, e' \rangle_B \, f' \right\rangle_C, \tag{92}$$

for $e, e' \in \mathcal{E}$, $f, f' \in \mathcal{F}$. This completion together with the continuous extensions of the left A-and right C-actions and the C-valued inner product (92) is a C^*-correspondence $\mathcal{E} \otimes \mathcal{F} : A \leftarrow C$. The horizontal composition of 2-arrows (isometric bimodule maps) $\alpha \colon \mathcal{E} \to \mathcal{E}'$ and $\beta \colon \mathcal{F} \to \mathcal{F}'$, for C^*-correspondences $\mathcal{E}, \mathcal{E}' : A \leftarrow B$ and $\mathcal{F}, \mathcal{F}' : B \leftarrow C$, is given by

$$\alpha \otimes \beta \colon \mathcal{E} \otimes \mathcal{F} \to \mathcal{E}' \otimes \mathcal{F}' \tag{93}$$

$$e \otimes f \mapsto \alpha(e) \otimes \beta(f). \tag{94}$$

Compare [10, Section 2.2].

For C^*-algebras A, B, C, D and correspondences $\mathcal{E} \colon A \leftarrow B$, $\mathcal{F} : B \leftarrow C$, $\mathcal{K} \colon C \leftarrow D$,

$$(e \otimes f) \otimes k \mapsto e \otimes (f \otimes k) \tag{95}$$

induces a unitary bimodule map

$$\mathrm{assoc}_{\mathcal{E},\mathcal{F},\mathcal{K}} \colon (\mathcal{E} \otimes \mathcal{F}) \otimes \mathcal{K} \to \mathcal{E} \otimes (\mathcal{F} \otimes \mathcal{K}). \tag{96}$$

Such maps constitute the components of the associators in $\mathfrak{Corr}$. For a C^*-algebra B, the "unit" 1-arrow is given by B interpreted as a C^*-correspondence with the left and right regular actions of B on itself and

$$\langle b_1 \,|\, b_2 \rangle := b_1^* b_2 \tag{97}$$

as B-inner product. For C^*-algebras A, C, the components of the associated unitors at B are given by

$$\mathcal{E} \otimes B \to \mathcal{E} : \qquad e \otimes b \to e \cdot b \tag{98}$$

$$B \otimes \mathcal{F} \to \mathcal{F} : \qquad b \otimes f \to b \cdot f \tag{99}$$

for correspondences $\mathcal{E} \colon A \leftarrow B$ and $\mathcal{F} : B \leftarrow C$. See [5, end of Section 6].

6.2 The homomorphism from $\mathfrak{Gr}$ to $\mathfrak{Corr}$

We recall the definition of a homomorphism $\mathfrak{Gr} \to \mathfrak{Corr}$ described in [5, Section 7] going back to Albandik ([1]).

Let X be a locally compact, locally Hausdorff topological space. For an open, Hausdorff set $U \subseteq X$, denote by $C_c(U)$ the complex vector space of compactly supported continuous functions on U. For $f \in C_c(U)$, extend f to X by $f(x) = 0$ for $x \in X \setminus U$. Let $\mathfrak{S}(X)$ be the linear span of such functions (for variable U) in the space of functions on X. Such functions do not need to be continuous, if X is not Hausdorff. (See [5, paragraph before Proposition 7.1].)

First, we recall the definition of the C^*-algebra for a locally compact étale groupoid $\mathcal{G}$. For $\xi, \eta \in \mathfrak{S}(\mathcal{G}^1)$, let

$$\xi * \eta(g) = \sum_{h \in \mathcal{G}^{\mathbf{r}(g)}} \xi(h)\eta(h^{-1}g), \tag{100}$$

$$\xi^*(g) = \overline{\xi(g^{-1})}. \tag{101}$$

This turns $\mathfrak{S}(\mathcal{G}^1)$ into a *-algebra allowing a maximal C^*-seminorm, which is a C^*-norm. (See [5, between Definitions 7.2 and 7.3].)

Definition 6.2 ([5, Definition 7.3])**.** The groupoid C^*-algebra $C^*(\mathcal{G})$ of $\mathcal{G}$ is the completion of $\mathfrak{S}(\mathcal{G}^1)$ in the largest C^*-norm.

Now, let $\mathcal{G}$, $\mathcal{H}$ be étale groupoids. We recall the definition of the C^*-algebra $C^*(\mathcal{X})$ associated to a locally compact étale groupoid correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$. For $\xi, \eta \in \mathfrak{S}(\mathcal{X})$, $\gamma \in \mathfrak{S}(\mathcal{G}^1)$, $\zeta \in \mathfrak{S}(\mathcal{H})$ and $x \in \mathcal{X}$, $g \in \mathcal{G}^1$ define

$$\xi * \gamma(x) := \sum_{g \in \mathcal{G}^{\mathbf{s}(x)}} \xi(x \cdot g)\gamma(g^{-1}), \tag{102}$$

$$\langle \xi \,|\, \eta \rangle (g) := \sum_{\{x \in \mathcal{X} \,|\, \mathbf{s}(x) = \mathbf{r}(g)\}} \overline{\xi(x)}\eta(x \cdot g) \tag{103}$$

$$\zeta * \xi(x) := \sum_{h \in \mathcal{H}^{\mathbf{r}(x)}} \zeta(h)\xi(h^{-1}x), \tag{104}$$

see [5, (7.1),(7.2),(7.3) and Lemma 7.4]. This turns $\mathfrak{S}(\mathcal{X})$ into a $\mathfrak{S}(\mathcal{H}^1)$-$\mathfrak{S}(\mathcal{G}^1)$-bimodule with a $\mathfrak{S}(\mathcal{H}^1)$-valued inner product satisfying certain equations (see [5, Lemma 7.5]) such that

$$\|\xi\| := \|\langle \xi \,|\, \xi \rangle\|^{1/2}_{C^*(\mathcal{G})} \tag{105}$$

defines a norm on $\mathfrak{S}(\mathcal{X})$ such that the completion $C^*(\mathcal{X})$ of $\mathfrak{S}(\mathcal{X})$ with respect to this norm allows extensions of the left and the right action and the inner product making it a C^*-correspondence $C^*(\mathcal{X})\colon C^*(\mathcal{G}) \leftarrow C^*(\mathcal{H})$. See [5, Lemma 7.5 to Lemma 7.7 and discussion thereafter].

Let $\mathcal{X}, \mathcal{X}'\colon \mathcal{H} \leftarrow \mathcal{G}$ be locally compact étale groupoid correspondences. A 2-arrow $\alpha\colon \mathcal{X} \to \mathcal{X}'$ is an injective, $\mathcal{H}, \mathcal{G}$-equivariant, continuous map from $\mathcal{X}$ to $\mathcal{X}'$. By Lemma 4.16, α is a homeomorphism onto an open subset U of $\mathcal{X}'$. Extension by zero gives an injective map $\mathfrak{S}(\mathcal{X}) \to \mathfrak{S}(\mathcal{X}')$, which extends uniquely to an isometric map $C^*(\alpha)\colon C^*(\mathcal{X}) \to C^*(\mathcal{X}')$. This map $C^*(\alpha)$ is isometric and $C^*(\mathcal{H}), C^*(\mathcal{G})$-equivariant and thus a 2-arrow from $C^*(\mathcal{X})$ to $C^*(\mathcal{X}')$. (See [5, paragraph following Example 7.10].)

The following lemma provides the multiplication-related data for the homomorphism of bicategories $\mathfrak{Gr} \to \mathfrak{Corr}$.

Lemma 6.3 ([5, Definition 7.12])**.** *Let $\mathcal{X}\colon \mathcal{K} \leftarrow \mathcal{H}$ and $\mathcal{Y}\colon \mathcal{H} \leftarrow \mathcal{G}$ be composable groupoid correspondences. There is a well defined map*

$$\phi^0_{\mathcal{X},\mathcal{Y}}\colon \mathfrak{S}(\mathcal{X}) \otimes \mathfrak{S}(\mathcal{Y}) \to \mathfrak{S}(\mathcal{X} \circ_{\mathcal{H}} \mathcal{Y}),$$
$$\phi^0_{\mathcal{X},\mathcal{Y}}(f_1 \otimes f_2)([x,y]) = \sum_{\{h \in \mathcal{H}^0 \,|\, \mathbf{r}(h) = \mathbf{s}(x)\}} f_1(xh) \cdot f_2(h^{-1}y).$$

It extends to an isomorphism of $C^*(\mathcal{K})$-$C^*(\mathcal{G})$-correspondences

$$\phi_{\mathcal{X},\mathcal{Y}}\colon C^*(\mathcal{X}) \otimes_{C^*(\mathcal{H})} C^*(\mathcal{Y}) \to C^*(\mathcal{X} \circ_{\mathcal{H}} \mathcal{Y}).$$

This isomorphism is natural with respect to the 2-arrows in $\mathfrak{Gr}$.

Lemma 6.4 (see [5, Theorem 7.13])**.** *The data given above defines a strictly unital homomorphism of bicategories $C^*\colon \mathfrak{Gr} \to \mathfrak{Corr}$.*

Remark 6.5. In [1, Section 2.5], Albandik describes a homomorphism of bicategories (“functor”, see [1, Definition 2.2]) from a version of $\mathfrak{Gr}$ to the version of the bicategory $\mathfrak{Corr}$ of C^*-correspondences originally described in [10]. In both involved bicategories, the 1-arrows go in the other direction than in the respective versions that we consider. Except for this, the different versions of $\mathfrak{Corr}$ only differ in whether 2-arrows need to be invertible, which is irrelevant for homomorphisms from categories to “$\mathfrak{Corr}$”, compare Remark 4.33. The versions of $\mathfrak{Gr}$ also only differ in the direction of 1-arrows and in whether the 2-arrows are required to be invertible, see Remark 4.24. All this means that the strictly unital homomorphism $\mathfrak{Gr} \to \mathfrak{Corr}$ described above induces a strictly unital homomorphism between the bicategories “$\mathfrak{Gr}$” and “$\mathfrak{Corr}$” used by Albandik, which deviates essentially from ours only in that its action on 2-arrows is a restriction to invertibles. Indeed, this induced homomorphism is the one described by Albandik in [1, Section 2.5].

6.3 Dynamical systems as product systems

Fowler ([12]) defines product systems, which is a slightly more general notion than our monoid shaped diagrams in $\mathfrak{Corr}$. Stammeier ([26]) associates product systems to his irreversible algebraic dynamical systems. Suppose that our dynamical system $D_{\mathfrak{Gr}}$ comes from Stammeier's, that is, G, P, θ are as in Definition 3.4. Then indeed, composing the homomorphism of bicategories $\mathfrak{Gr} \to \mathfrak{Corr}$ described in Section 6.2 with $D_{\mathfrak{Gr}}$ gives the same product system as Stammeier's. We will use this in Section 6.5 to obtain Stammeier's C^*-algebra from our reinterpretation of his irreversible algebraic dynamical systems as $D_{\mathfrak{Gr}}$ and $D_{\mathfrak{Corr}}$ in the finite-type case.

Lemma 6.6 (compare [2, first paragraph of Section 3], [3, Proposition 6.2] and [1, Proposition 2.44])**.** *A monoid-shaped diagram in $\mathfrak{Corr}$ is of the form $D = (A, P, \mathcal{E}_p, \mu_{p,q})$ with the data*

- *a C^*-algebra A,*
- *a monoid P,*
- *a C^*-correspondence $\mathcal{E}_p \colon A \leftarrow A$ for each $p \in P$,*
- *a unitary A-bimodule map $\mu_{p,q} \colon \mathcal{E}_p \otimes \mathcal{E}_q \to \mathcal{E}_{p \cdot q}$ for each pair $p, q \in P$,*

fulfilling

$$\mathcal{E}_1 = A, \tag{106}$$

$$\mu_{pq,r} \circ (\mu_{p,q} * \mathrm{id}_{\mathcal{E}_r}) = \mu_{p,qr} \circ (\mathrm{id}_{\mathcal{E}_p} * \mu_{q,r}) \circ \mathrm{assoc}_{p,q,r}, \tag{107}$$

$$\mathfrak{l}_{\mathcal{E}_p} = \mu_{1,p} \text{ and} \tag{108}$$

$$\mathfrak{r}_{\mathcal{E}_p} = \mu_{p,1} \tag{109}$$

for all $p, q, r \in P$.

Proof. This directly follows from 2.9 and the definition of $\mathfrak{Corr}$ in Section 6.1. ∎

Remark 6.7 (see [3, discussion following Proposition 6.4], [2, first paragraph of Section 3] and [1, Section 3.2.3])**.** Monoid shaped diagrams in $\mathfrak{Corr}$ as in Lemma 6.6 are product systems of essential "Hilbert bimodules"[16], or essential product systems, in the sense of Fowler [12, Definition 2.1 and last paragraph of Section 1]. Here, *essential* means that the left actions on the "Hilbert bimodules" (roughly C^*-correspondences) are non-degenerate. Strictly speaking, Fowler requires countable monoids. There is however no reason not to consider Fowler's definition in the generality of arbitrary shape monoids; countability is neither required for the definition of a Cuntz-Pimsner algebra nor for the part of Albandik's use of Fowler's theory which we need, as experts confirm. Hence we can say that essential product systems over P are the same as P-shaped diagrams in $\mathfrak{Corr}$.

Lemma 6.8 (compare [1, first paragraph of Section 3.2.3] and [5, second paragraph after Theorem 7.13] and [5, Example 7.9])**.** *Applying the homomorphism of bicategories $\mathfrak{Gr} \to \mathfrak{Corr}$ from Lemma 6.4 to a dynamical system in $\mathfrak{Gr}$ (that is, a monoid-shaped diagram in $\mathfrak{Gr}$ coming from a monoid shaped diagram in $\mathfrak{GrpMn}$) $D_{\mathfrak{Gr}} = (G, P, \mathcal{X}_p, \mu_{p,q})$ as in Lemma 4.30 results in the monoid-shaped diagram $D_{\mathfrak{Corr}} = (C^*(G), P, C^*(\mathcal{X}_p), \mu_{p,q})$ in $\mathfrak{Corr}$ with the following data:*

- *$C^*(G)$ is the group C^*-algebra for G (see Definition 6.2).*
- *The correspondence $C^*(\mathcal{X}_p)\colon C^*(G) \leftarrow C^*(G)$, for $p \in P$, is defined as follows: Let $C_c(\mathcal{X}_p)$ be the complex vector space of finitely supported functions $\mathcal{X}_p \to \mathbb{C}$ and denote by δ_x, for $x \in \mathcal{X}_p$, the function*

$$\delta_x(y) := \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise}, \end{cases}$$

for $y \in \mathcal{X}_p$. Then equip $C_c(\mathcal{X}_p)$ with left and right actions by $C_c(G)$ given by

$$\delta_g.\delta_x := \delta_{g\cdot x} \text{ and} \tag{110}$$

$$\delta_x.\delta_g := \delta_{x\cdot\theta_p(g)}, \tag{111}$$

respectively, for $x \in \mathcal{X}_p = G$, $g \in G$, for each $p \in P$, and the $C^*(G)$-valued inner product given by

$$\langle \delta_x \mid \delta_y \rangle := \begin{Bmatrix} \delta_{\langle x \mid y \rangle_p}, & \text{if } [x]_p = [y]_p, \\ 0, & \text{otherwise}, \end{Bmatrix} = \mathbb{1}_{[x]_p = [y]_p} \delta_{\theta_p^{-1}(x^{-1}y)} \tag{112}$$

for $x, y \in \mathcal{X}_p$, $p \in P$, where in the rightmost term, x and y are used as elements of G and $\mathbb{1}$ is the indicator function. Then $\|f\| := \|\langle f \mid f \rangle\|_{C^*(G)}^{1/2}$ defines a norm on $C_c(G)$ with respect to which $C^*(\mathcal{X}_p)$ is the completion. The left and right $C_c(G)$-actions on $C_c(\mathcal{X}_p)$ given by (110) and (111), respectively, extend to $C^*(G)$-actions on $C^*(\mathcal{X}_p)$, turning $C^*(\mathcal{X}_p)$ into a C^*-correspondence $C^*(G) \leftarrow C^*(G)$.

[16] The term *there* means something very close to C^*-correspondences, but is used differently elsewhere, see [9, Definition 4.1].

- *For each $p, q \in P$, $\mu_{p,q}$ is the $C^*(G)$-biequivariant unitary map*

$$C^*(\mathcal{X}_p) \otimes C^*(\mathcal{X}_q) \to C^*(\mathcal{X}_{pq}), \tag{113}$$
$$\delta_g \otimes \delta_h \mapsto \delta_{g\theta_p(h)}, \tag{114}$$

for each pair $p, q \in P$.

Definition 6.9 (compare [26, Definition 5.2])**.** Let A be a C^*-algebra and $\mathcal{E}$ a Hilbert A-module. Let $e_i \in \mathcal{E}$, for $i \in I$, for some index set I. Then $(e_i)_{i\in I}$ is called an *orthonormal basis* for $\mathcal{E}$, if

$$\langle e_i \,|\, e_j\rangle = \begin{cases} 1_A & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for all } i, j \in I, \tag{115}$$
$$e = \sum_{i\in I} e_i \langle e_i \,|\, e\rangle \qquad \text{for all } e \in \mathcal{E}. \tag{116}$$

Lemma 6.10 (compare [26, proof of Proposition 5.6])**.** *For each $p \in P$, an orthonormal basis for $C^*(\mathcal{X}_p)$ is given by $(\delta_{g_i})_{i\in I}$, where $(g_i)_{i\in I}$ is a family of representants of the cosets of $\theta_p(G)$ in G such that $([g_i])_{i\in I}$ partitions G.*

Proof. Fix $p \in P$. For $i, j \in I$, $\left\langle \delta_{g_i} \,\middle|\, \delta_{g_j}\right\rangle$ is $\delta_{\langle g_i \,|\, g_j\rangle_p}$ if $[g_i]_p = [g_j]_p$ and 0 otherwise. The former is the case if and only if $i = j$ by the choice of the family $(g_i)_{i\in I}$. If $i = j$, $\langle g_i \,|\, g_j\rangle_p = 1_G$. Hence, (115) is fulfilled for $(\delta_{g_i})_{i\in I}$.

Suppose that $f \in C_c(\mathcal{X}_p)$. Then there exist $n \in \mathbb{N}$ and $h_1, \dots, h_n$ such that $f = \sum_{k=1}^n \delta_{h_k} f_k$. Then

$$\sum_{i\in I} \delta_{g_i} \langle \delta_{g_i} \,|\, f\rangle = \sum_{i\in I;k=1,\dots,n} \delta_{g_i} \langle \delta_{g_i} \,|\, \delta_{h_k}\rangle f_k \tag{117}$$
$$= \sum_{i\in I;k=1,\dots,n;[g_i]_p=[h_k]_p} \delta_{g_i}\delta_{\langle g_i \,|\, h_k\rangle_p} f_k = \sum_{k=1}^n \delta_{h_k} f_k = f. \tag{118}$$

In the second-to-last sum, for each $k = 1, \dots, n$, at most one $i \in I$ can occur together with k in a summand due to the condition $[g_i]_p = [h_k]_p$ and the fact that $[g_i]_p \neq [g_j]_p$ for distinct $i, j \in I$; and for each $k = 1, \dots, n$, at least one $i \in I$ occurs together with k in a summand, since $([g_i])_{i\in I}$ covers G. Furthermore, for $i \in I$, $k \in \{1, \dots, n\}$ with $[g_i]_p = [h_k]_p$, $g_i \langle g_i \,|\, h_k\rangle_p = h_k$. This explains the simplification resulting in the next sum. Thus (116) holds for $(\delta_{g_i})_{i\in I}$. This concludes the proof. ∎

Stammeier associates a product system to his irreversible algebraic dynamical system in [26, Section 5]. We show that it is a special case of $D_{\mathfrak{Corr}}$.

Proposition 6.11. *Assume Stammeier's finite-type case. Then the dynamical system $D_{\mathfrak{Corr}}$ described in Lemma 6.8 coincides with the product system described in* [26, Proposition 5.6].

Proof. The assertion can be verified by comparing [26, Proposition 5.6] and Lemma 6.8. Deviating from Lemma 6.8, Stammeier uses the group C^*-algebra $C^*(G)$ as underlying Banach space for the C^*-correspondence associated to $p \in P$. Consider the bijective linear map

$$C_c(G) \to C_c(\mathcal{X}_p)\colon \delta_g \to \delta_g \tag{119}$$

induced by the identity of sets $G = \mathcal{X}_p$. We show that this map and its inverse are each restrictions of bounded linear maps between the respective norm closures $C^*(G)$ and $C^*(\mathcal{X}_p)$. This then implies that the map in (119) extends to a linear homeomorphism between $C^*(G)$ and $C^*(\mathcal{X}_p)$. Then comparing [26, Lemma 6.8] and Lemma 6.8 reveals that those linear homeomorphisms, for all $p \in P$, preserve the C^*-correspondence-structure, and hence identify the product systems, as required for the proposition. Note that the $C^*(G)$-valued inner products Stammeier defines on $C^*(G)$, for $p \in P$, do not necessarily induce the canonical C^*-norm on $C^*(G)$.

We first prove that the map in (119) is a bounded linear map. The left $C^*(G)$-action on $C^*(\mathcal{X}_p)$ consists of a bounded linear map

$$C^*(G) \to \mathbb{B}(C^*(\mathcal{X}_p)).$$

Evaluating a bounded $C^*(G)$-linear operator on $C^*(\mathcal{X}_p)$ at $\delta_{1_G} \in C_c(G) = C_c(\mathcal{X}_p)$ is a bounded map

$$\mathbb{B}(C^*(\mathcal{X}_p)) \to C^*(\mathcal{X}_p).$$

The composition of the last two maps, a bounded linear map, does evidently restrict to the map in (119). Now we show that the inverse of (119) is the restriction of a bounded linear map $C^*(\mathcal{X}_p) \to C^*(G)$. Lemma 6.10 provides us with an orthonormal basis for $C^*(\mathcal{X}_p)$. Taking into account the requirement that $\theta_p(G) \leq G$ is finite, this orthonormal basis is of the form $\delta_{g_1}, \dots, \delta_{g_n}$ for some $n \in \mathbb{N}$ with $g_1, \dots, g_n \in G$. It induces an isomorphism of Hilbert $C^*(G)$-modules

$$w\colon C^*(\mathcal{X}_p) \to C^*(G)^n \colon f \to \left(\langle \delta_{g_i} \mid f \rangle\right)_{i=1,\dots,n} \tag{120}$$

which is clearly bounded. Here, $C^*(G)^n$ is the direct sum of Hilbert $C^*(G)$-modules (see [17]). For $i \in \{1, \dots, n\}$, define a map

$$v_i\colon C^*(G) \to C^*(G) \colon a \mapsto \delta_{g_i} a.$$

For $i = 1, \dots, n$, those maps are bounded, and so is the induced map

$$v\colon C^*(G)^n \to C^*(G) \colon (a_i)_{i=1,\dots,n} \mapsto \sum_{i=1}^n v_i(a_i) = \sum_{i=1}^n \delta_{g_i} a_i.$$

It can be easily seen that $v \circ w$, a bounded map, restricts to the inverse of (119) on $C_c(X_p)$. This concludes the proof. □

6.4 Stammeier's C^*-algebras as Cuntz-Pimsner algebras

To a product system (see Lemma 6.6 and Remark 6.7 and [12, Definition 2.1]) Fowler associates a Cuntz-Pimsner algebra (see [12, Proposition 2.9]).

Stammeier uses this to give an alternative description of his C^*-algebra in the case of irreversible algebraic dynamical systems of finite type. Hence, the next proposition links Stammeier's C^*-algebra for the finite-type case to our reinterpretation of his irreversible algebraic dynamical systems as dynamical systems in $\mathfrak{Gr}$ and $\mathfrak{Corr}$.

Proposition 6.12 (compare [26, Theorem 5.9])**.** *Assume Stammeier's finite-type case. Then Stammeier's C^*-algebra $\mathcal{O}[G, P, \theta_p]$ is the Cuntz-Pimsner algebra of the dynamical system $D_{\mathfrak{Corr}}$ described in Lemma 6.8 interpreted as a product system.*

Proof. Assume Stammeier's finite-type case. Then, by [26, Theorem 5.9] Stammeier's C^*-algebra $\mathcal{O}[G, P, \theta_p]$ is the Cuntz-Pimsner algebra associated to the product system in [26, Proposition 5.6]. By Proposition 6.11, this product system coincides with the dynamical system $D_{\mathfrak{Corr}}$ described in Lemma 6.8. $\square$

Of course, even in the general case, we can associate to a dynamical system $D_{\mathfrak{GrpMn}}$ as in Definition 3.12 the Cuntz-Pimsner algebra of $D_{\mathfrak{Corr}}$. It coincides with Stammeier's C^*-algebra in Stammeier's finite-type case. We denote it by $\mathcal{O}^{\mathfrak{Corr}}[G, P, \theta]$, in order to avoid the impression that it is the generalisation of the C^*-algebra from Definition 3.8 to the case when Stammeier's conditions – and the finite-type condition – are relaxed.

In Section 6.5, we will see that if P fulfils the right Ore conditions and $\theta_p(G) \leq G$ has finite index for all $p \in P$, then $\mathcal{O}^{\mathfrak{Corr}}[G, P, \theta]$ is isomorphic to $C^*(\mathcal{L})$ for the groupoid model $\mathcal{L}$ from Section 5.

6.5 Stammeier's C^*-algebras as groupoid C^*-algebras

In Section 6.3, we showed that the dynamical system $D_{\mathfrak{Corr}}$ in $\mathfrak{Corr}$ is a product system which – in the case that it comes from Stammeier's irreversible algebraic dynamical systems – coincides with the product system constructed by Stammeier. In the finite-type case, Stammeier's C^*-algebra is the Cuntz-Pimsner algebra of this product system.

In this section, applying a theorem by Albandik, we show – provided that $\theta_p(G) \leq G$ has finite index for $p \in P$ and that P fulfils the right Ore conditions – that the Cuntz-Pimsner algebra of the dynamical system $D_{\mathfrak{Corr}}$ (as a product system) is the groupoid C^*-algebra of the groupoid model $\mathcal{L}$ constructed in Section 5. In the case that our dynamical system comes from Stammeier's, this implies that $\mathcal{L}$ is a groupoid model for Stammeier's C^*-algebra.

Proposition 6.13. *Suppose that the requirements in Remark 5.45 are fulfilled, that is, P fulfils the right Ore conditions and $\theta_p(G) \leq G$ has finite index for all $p \in P$. Then $\mathcal{L}$ is indeed a groupoid model for $\mathcal{O}^{\mathfrak{Corr}}[G, P, \theta]$, the Cuntz-Pimsner algebra of $D_{\mathfrak{Corr}}$, that is, the latter is the groupoid C^*-algebra of $\mathcal{L}$.*

Proof. Suppose the given conditions are fulfilled. Then $D_{\mathfrak{Gr}}$ is an action of P in "$\mathfrak{Gr}$" in Albandik's sense. Furthermore, the groupoid model constructed in Section 5 is identical to the groupoid "$\mathcal{H}$" constructed in [1, Section 3.2.1]. See Remark 5.45. Then, getting from $D_{\mathfrak{Gr}}$ to $D_{\mathfrak{Corr}}$ corresponds to postcomposing the "functor $\mathfrak{Gr} \to \mathfrak{Corr}$" in Albandik's terms to $D_{\mathfrak{Gr}}$ reinterpreted in Albandik's terms. Then [1, Theorem 3.36] implies that $\mathcal{O}^{\mathfrak{Corr}}[G, P, \theta]$, the Cuntz-Pimsner algebra of $D_{\mathfrak{Corr}}$, is the groupoid C^*-algebra (see Section 6.2) of $\mathcal{L}$. $\square$

Corollary 6.14. *Assume Stammeier's finite-type case. Then Stammeier's C^*-algebra $\mathcal{O}[G, P, \theta]$ is the groupoid C^*-algebra of $\mathcal{L}$.*

Proof. Suppose that the given conditions are fulfilled. Then $\theta_p(G) \leq G$ has finite index for all $p \in P$ and the Ore conditions are fulfilled (see Remark 5.16). Thus the corollary follows from Proposition 6.13 and Proposition 6.12. $\square$

7 Properties of the groupoid model

In this section we give sufficient criteria for the groupoid model to be Hausdorff, effective, minimal and locally contracting, respectively.[17] In Proposition 5.44, we gave a sufficient criterion for the groupoid model to be locally compact. Using those criteria and applying a theorem by Brown, Clark, Farthing and Sims [7, Theorem 5.1], we prove the following simplicity criterion for $C^*(\mathcal{L})$ (Corollary 7.20): If P fulfils the right Ore conditions and is countable, G is countable, $\theta_p(G) \leq G$ has finite index for all $p \in P$, $C^*(\mathcal{L}) = C^*_{\mathrm{red}}(\mathcal{L})$, $\theta\colon P \to \mathrm{Mono}(G)$ is injective and $\bigcap_{p\in P} \theta_p(G) = \{1_G\}$, then $C^*(\mathcal{L})$ is simple. We study which of the optional properties which Stammeier defines for his irreversible algebraic dynamical systems allow the application of each of our criteria.

Before we go into the specific properties of the groupoid model, we prove an auxiliary lemma.

Lemma 7.1. *If P fulfils the right Ore conditions, then Ω's topology has a basis consisting only of the closed (and open) sets*

$$Z^{\Omega}_{(x,q)} = \left\{([g_p])_{p\in P} \in \Omega \,\middle|\, [g_q]_q = [x]_q\right\} \tag{121}$$

for $x \in \mathcal{X}_q$ and $q \in P$.

Proof. Suppose that P fulfils the right Ore conditions. Let $n > 2$ and let $q_i \in P$, $h_i \in \mathcal{X}_{q_i}$, $i = 1, \dots, n$. Then

$$Z^{\Omega}_{(h_1,q_1),\dots,(h_n,q_n)} = Z^{\Omega}_{(h_1,q_1)} \cap \dots \cap Z^{\Omega}_{(h_n,q_n)}. \tag{122}$$

Suppose that there exists $([g_p]_p)_p \in Z^{\Omega}_{(h_1,q_1),\dots,(h_n,q_n)}$. Then for all $i = 1, \dots, n$,

$$[g_{q_i}]_{q_i} = [h_i]_{q_i}. \tag{123}$$

By the first Ore condition, there are $s_1, \dots, s_n \in P$ such that $q := q_i s_i$ is constant for varying $i = 1, \dots, n$. By the coherence condition in the definition of Ω, (123) implies that for all $i = 1, \dots, n$,

$$[g_q]_{q_i} = [h_i]_{q_i}. \tag{124}$$

Now, $Z^{\Omega}_{(g_q,q)} \subseteq Z^{\Omega}_{(h_1,q_1)}$ for all $i \in \{1, \dots, n\}$: Suppose that $([\tilde{g}_p]_p)_p \in Z^{\Omega}_{(g_q,q)}$. This implies $[\tilde{g}_q]_q = [g_q]_q$. Applying the coherence conditions in the definition of Ω to the left side and relaxing both sides to cosets with respect to a larger subgroup of G, we obtain $[\tilde{g}_{q_i}]_{q_i} = [g_q]_{q_i}$ for all $i = 1, \dots, n$. This, together with (124), yields $[\tilde{g}_{q_i}]_{q_i} = [h_i]_{q_i}$ for all $i = 1, \dots, n$. We have thus shown that if the set in (122) is non-empty, then it contains a set of the desired form (121). With Lemma 5.8, sets as in (121) are closed and open and by what we have shown, form a basis of Ω. □

7.1 When is the groupoid model Hausdorff?

Proposition 7.2. *Suppose that P fulfils the right Ore conditions and that $\theta_p(G) \leq G$ has finite index for all $p \in P$. Then the groupoid model $\mathcal{L}$ is Hausdorff.*

In particular, in Stammeier's finite-type case, $\mathcal{L}$ is Hausdorff.

[17] In fact, minimality is automatic if P fulfils the right Ore conditions, which we need anyway for the construction of the groupoid model which we use.

Proof. Suppose that the given conditions hold. Then, as explained in Remark 5.45, the groupoid model $\mathcal{L}$ is identical to a limit[18] construction $\mathcal{H}$ by Albandik for a diagram of tight locally compact étale groupoid correspondences $\tilde{\mathcal{X}}_p$ over a groupoid $\tilde{\mathcal{G}}$ associated to the diagram $(\mathcal{X}_p)$ corresponding to $D_{\mathfrak{Gr}}$.[19] For each $p \in P$, the left G-action on $\mathcal{X}_p$ is free. This is because, as a set, $\mathcal{X}_p = G$ and the left action on $\mathcal{X}_p$ corresponds along this identification to the regular action of the group on itself. Then, by [1, Lemma 3.28], in the tightened diagram $(\tilde{\mathcal{X}}_p)$, the left actions of the groupoid on the correspondences are also free. Furthermore, Ω is Hausdorff by Lemma 5.44. Then Albandik's $\mathcal{H}$, and thus the groupoid model $\mathcal{L}$, is Hausdorff by [1, Proposition 3.20].

Forget all assumptions in this proof. In Stammeier's finite-type case, $\theta_p(G) \leq G$ has finite index for all $p \in P$ and the Ore conditions are fulfilled (see Remark 5.16). Hence, in that case, the first part of the proposition is applicable and yields that $\mathcal{L}$ is Hausdorff. $\square$

7.2 When is the groupoid model effective?

The isotropy subgroupoid $\mathrm{Iso}(\mathcal{G})$ of a groupoid $\mathcal{G}$ is defined as

$$\mathrm{Iso}(\mathcal{G}) = \{g \in \mathcal{G} \mid \mathbf{s}(g) = \mathbf{r}(g)\},$$

see for example [25, Section 2.2].

Definition 7.3 ([25, Definition 4.2.1])**.** An étale topological groupoid $\mathcal{G}$ is called *effective*, if the interior of $\mathrm{Iso}(\mathcal{G})$ consists only of units of $\mathcal{G}$.

Lemma 7.4 (compare [11])**.** *Let S be an inverse semigroup with an action $\vartheta\colon S \to \mathcal{I}(Y)$ by partial homeomorphisms. Then units of $S \ltimes_\vartheta Y$ are of the form $[e, y]$ where $e \in E(S)$ and $y \in D_e$.*

Proof. Any element of $S \ltimes Y$ can be written as $[s, y]$ with $s \in S$ and $y \in D_{s^*s}$. For any object $y \in Y$, there is $s \in S$ such that $y \in \mathrm{dom}(\vartheta_s)$ (see Definition 5.40).

We show that $[s^*s, y] \in S \ltimes Y$ is the unit for y. Let $y, z \in Y$ and $s, r, t \in S$ such that $y \in D_{t^*t} \cap D_{s^*s}$ and $z \in D_{r^*r}$ with $y = \vartheta_r(z)$. Then $[t, y][s^*s, y] = [ts^*s, y] = [t, y]$. By (3), $s^*sr = rr^*s^*sr$. Since r^*s^*sr is idempotent, $[s^*s, y][r, z] = [s^*sr, z] = [r, z]$. We have shown that $[s^*s, y]$ is the unit for the object y in $S \ltimes Y$. Since s^*s is idempotent, this proves the assertion. $\square$

Lemma 7.5. *An element $[s, \omega] \in S \ltimes_\vartheta Y$ is in the interior of the isotropy subgroupoid if and only if there exists $U \subseteq D_{s^*s}$ open in Y with $\omega \in U$ such that $\vartheta_s|_U = \mathrm{id}_U$.*

Proof. The assertion follows easily from Lemma 5.42 and the definition of the isotropy subgroupoid. $\square$

Before we give sufficient criteria for the groupoid model's being effective, we provide a technical statement about a sufficient set of criteria for transformation groupoids of inverse semigroup actions to be effective.

[18] colimit in Albandik's version of $\mathfrak{Gr}$

[19] More data than $\mathcal{X}_p$, $p \in P$, is needed for the diagram $(\mathcal{X}_p)$, but since it is not of interest in the proof, we use such shorthands in the proof to avoid the necessity to handle different notations coming from different texts.

Lemma 7.6. *Let S be an inverse semigroup and let $A \subseteq S$ be a generating (in the sense of inverse semigroups) subset which is closed under multiplication. Let $\vartheta\colon S \to \mathcal{I}(Y)$ be an action of S by partial homeomorphisms on Y such that $\vartheta|_A\colon A \to \mathcal{I}(Y)$ is injective. Suppose that for every $s \in S$ and $y \in D_{s^*s}$, there are $e \in E(S)$ with $y \in D_e$ and $a, b \in A$ such that $(ab^*)^*ab^*e = e$ and $ab^*e = se$. Suppose that $\{D_{cc^*} | c \in A\}$ is a basis for Y. Then $S \ltimes_\vartheta Y$ is effective.*

Proof. In this proof, we will, for idempotents $e \in E(S)$, ask the reader to "make e smaller" in order for it to fulfil certain conditions, while retaining others. This is to be understood as follows: e is an idempotent with certain properties (those to be retained), and it is possible to find another idempotent e' with certain properties (those to be gained), such that ee' has all the desired properties. Then, after the "shrinking", e is used to denote what would otherwise be denoted by ee'.

Suppose the given prerequisites are fulfilled. Suppose $s \in S$ and $y \in D_{s^*s}$ such that $[s, y]$ is in the interior of the isotropy subgroupoid of $S \ltimes_\vartheta Y$. Then, by Lemma 7.5 and the fact that D_e, $e \in E(S)$, form a basis of Y (as follows from a stronger requirement in the lemma), there is $e \in E(S)$ with $y \in D_e$ such that $\vartheta_{se} = \vartheta_e$. Since $y \in D_{s^*s}$, by a requirement in the lemma, we can pick $a, b \in A$ and make e smaller, while retaining $y \in D_e$, in such a manner that $(ab^*)^*ab^*e = e$ (this implies $bb^*e = bb^*ba^*ab^*e = ba^*ab^*e = e$) and $ab^*e = se$. Then

$$\vartheta_{ab^*e} = \vartheta_{se} = \vartheta_e. \tag{125}$$

Let $\tilde{e} := b^*eb$. Then, due to (125),

$$\vartheta_{a\tilde{e}b^*} = \vartheta_{ab^*ebb^*} = \vartheta_{ebb^*} = \vartheta_{bb^*e}.$$

Multiplying both sides of the equation with ϑ_b from the right yields

$$\vartheta_{a\tilde{e}} = \vartheta_{bb^*eb} = \vartheta_{b\tilde{e}} \tag{126}$$

By Lemma 5.37, $\tilde{e}$ is idempotent and $\vartheta_{b^*}(D_e) = \vartheta_{b^*}(D_e \cap D_{bb^*}) = D_{b^*eb} = D_{\tilde{e}}$. Then, since $\vartheta_{b^*}(y) \in \vartheta_{b^*}(D_e) = D_{\tilde{e}}$, by the requirements in the lemma, there is $c \in A$ such that $\vartheta_{b^*}(y) \in D_{cc^*}$ and $D_{cc^*} \subseteq D_{\tilde{e}}$ and hence $\vartheta_{\tilde{e}cc^*} = \vartheta_{cc^*}$. Then multiplying (126) from the right with ϑ_{cc^*c} yields $\vartheta_{ac} = \vartheta_{bc}$. Since $ac, bc \in A$ (since A is closed under products) and $\vartheta|_A\colon A \to \mathcal{I}(Y)$ is injective, this implies $ac = bc$. Then

$$a(b^*b)(cc^*)b^* = a(cc^*)(b^*b)b^* = bcc^*b^*. \tag{127}$$

From $\vartheta_{b^*}(y) \in D_{cc^*}$, it follows that $y \in \vartheta_b(D_{cc^*})$, and, since $\vartheta_{b^*bcc^*} = \vartheta_{b^*b\tilde{e}cc^*} = \vartheta_{\tilde{e}cc^*} = \vartheta_{cc^*}$, that $\vartheta_{b^*}(y) \in D_{b^*b}$, hence $y \in D_{bb^*}$. By Lemma 5.37 and since ϑ_t, $t \in S$, are homeomorphisms,

$$\vartheta_b(D_{cc^*}) \cap D_{bb^*} = \vartheta_b(D_{cc^*}) \cap \vartheta_b(D_{b^*b}) = \vartheta_b(D_{cc^*} \cap D_{b^*b}) = D_{bcc^*b^*}.$$

By what we have shown above, this implies $y \in D_{bcc^*b^*}$. Then (127) implies that $[ab^*, y]$ is a unit, and since $ab^*e = se$, $[ab^*, y] = [s, y]$. We thus showed that an element in the interior of the isotropy groupoid of $S \ltimes Y$ is a unit. Hence $S \ltimes Y$ is effective.

□

Now we give lemmata providing the prerequisites for an application of Lemma 7.6 to the groupoid model.

Lemma 7.7. *Suppose that P fulfils the right Ore conditions and that $\bigcap_{p\in P}\theta_p(G) = \{1_G\}$. Let $g, h \in G$ and $r \in P$. If $[g]_{rs} = [h]_{rs}$ for all $s \in P$, then $g = h$.*

Proof. Suppose that the requirements in the lemma hold. Let $g, h \in G$. Suppose that $g \neq h$. Then, since $\bigcap_{p\in P}\theta_p(G) = \{1_G\}$, there is $p \in P$ such that $g^{-1}h \notin \theta_p(G)$. By the Ore conditions there are $s, q \in P$ such that $rs = pq$. Then $\theta_{rs}(G) = \theta_{pq}(G) \subseteq \theta_p(G)$. Hence $g^{-1}h \notin \theta_{rs}(G)$. Then $[g]_{rs} \neq [h]_{rs}$. This concludes the proof. □

The next lemma helps telling elements of $I(D_{\mathfrak{Gr}})$ apart.

Lemma 7.8. *Suppose that P fulfils the Ore conditions, that the monoid homomorphism $\theta\colon P \to \mathrm{Mono}(G)$ is injective and that $\bigcap_{p\in P}\theta_p(G) = \{1_G\}$. If $\vartheta_{(x,p)} = \vartheta_{(x',p')}$, then $x = x'$ and $p' = p$.*

Proof. Suppose that the requirements in the lemma hold. Let $p_i \in P$ and $x_i \in \mathcal{X}_{p_i}$, $i = 1, 2$. Suppose $\vartheta_{(x_1,p_1)} = \vartheta_{(x_2,p_2)}$. Let $g \in G$ be arbitrary and define $g_r := g$ for all $r \in P$. Then $([g_r]_r)_r \in \Omega$. Since P fulfils the right Ore conditions, there are $q_1, q_2 \in P$ such that $p_1q_1 = p_2q_2$. Then

$$[x_1\theta_{p_1}(g)]_{p_1q_1s} = (\vartheta_{(x_1,p_1)}([g]_r))_{p_1q_1s} = (\vartheta_{(x_2,p_2)}([g]_r))_{p_2q_2s} = [x_2\theta_{p_2}(g)]_{p_2q_2s} \tag{128}$$

for all $s \in P$. By Lemma 7.7, this implies that $x_1\theta_{p_1}(g) = x_2\theta_{p_2}(g)$. This holds for all $g \in G$, since $g \in G$ was assumed arbitrary. Plugging in $g := 1$ yields $x_1 = x_2$. Then, letting $g \in G$ vary again, $\theta_{p_1}(g) = \theta_{p_2}(g)$, for $g \in G$, follows. Hence $\theta_{p_1} = \theta_{p_2}$. By injectivity of θ, $p_1 = p_2$. We have thus shown that $p_1 = p_2$ and $x_1 = x_2$, which concludes the proof. □

Lemma 7.9 (compare [20, Lemma 8.11])**.** *Suppose that P fulfils the right Ore conditions. Let $p, q \in P$, $x \in \mathcal{X}_p$, $y \in \mathcal{X}_q$ and $\omega \in \mathrm{dom}(\vartheta^*_{(x,p)}\vartheta_{y,q})$. Then there are $k, l \in P$ and $w \in \mathcal{X}_k$, $z \in \mathcal{X}_l$, such that $[(x,p)^* \cdot (y,q), \omega] = [(w,k)\cdot(z,l)^*, \omega]$ in $I(D_{\mathfrak{Gr}}) \ltimes \Omega$.*

Proof. This proof is analogous to the proof of [20, Lemma 8.11]. By the first Ore condition, there are $k, l \in P$ with $pk = ql$. The codomains of $\vartheta_{(z,l)}$, namely $Z^{\Omega}_{\vartheta_{(z,l)}}$, for $z \in \mathcal{X}_l$, cover Ω, as is easy to see (see Lemma 5.8). Pick $z \in \mathcal{X}_l$ such that $\omega \in Z^{\Omega}_{(z,l)}$. Since $\mathcal{X}_p \circ \mathcal{X}_k \cong \mathcal{X}_q \circ \mathcal{X}_l$, there are $w_1 \in \mathcal{X}_p$, $w_2 \in \mathcal{X}_k$ such that $\mu_{p,k}([w_1, w_2]) = \mu_{q,l}([y, z])$ and thus (see Lemma 5.23), $(w_1, p)(w_2, k) = (y, q)(z, l)$. Then

$$(x,p)^*(y,q)(z,l) = (x,p)^*(w_1,p)(w_2,k)$$

Since $(x,p)^*(w_1,p)(w_2,k) = (\langle x \,|\, w_1\rangle \cdot w_2, k) := (w,k)$, we have

$$(x,p)^*(y,q)(z,l)(z,l)^* = (w,k)(z,l)^*$$

where $w := \langle x \,|\, w_1\rangle \cdot w_2$. The idempotent $(z,1)(z,1)^*$, enjoying the property $\omega \in \mathrm{codom}(\vartheta_{(z,l)}) = \mathrm{dom}(\vartheta_{(z,l)(z,l)^*})$, witnesses that $[(x,p)^*(y,q), \omega] = [(w,k)(z,l)^*, \omega]$ in $I(D_{\mathfrak{Gr}})$. □

Lemma 7.10. *Suppose that P fulfils the right Ore conditions. Then $I(D_{\mathfrak{Gr}}) \ltimes \Omega$ is covered by subsets of the form*

$$\Theta((x_2,p_2)(x_1,p_1)^*, Z^{\Omega}_{(x_1,p_1)}) = \left\{[(x_2,p_2)(x_1,p_1)^*, \omega] \,\middle|\, \omega \in Z^{\Omega}_{(x_1,p_1)}\right\},$$

for $p_1, p_2 \in P, x_1 \in \mathcal{X}_{p_1}, x_2 \in \mathcal{X}_{p_2}$.

Proof. For any element $[s,\omega] \in I(D_{\mathfrak{Gr}}) \ltimes \Omega$, s can be written as a word of elements of $I(D_{\mathfrak{Gr}})$ each having the form (x,p) or $(x,p)^*$ (this being a choice for each single element). As long as this word is of the form

$$a(x,p)^*(y,q)b \tag{129}$$

for subwords a, b of s and $p, q \in P$, $x \in \mathcal{X}_p$, $y \in \mathcal{X}_q$, it can be replaced by $s' := a(w,k)(z,l)^*b$ for certain $k, l \in P$, $w \in \mathcal{X}_k$, $z \in \mathcal{X}_l$, such that $[s,\omega] = [s',\omega]$ in $I(D_{\mathfrak{Gr}})$: Indeed, by Lemma 7.9, there are such k, l, w, z, such that $[(x,p)^* \cdot (y,q), \omega] = [(w,k)\cdot(z,l)^*, \omega]$. Hence

$$\begin{aligned}
[s,\omega] &= [a(x,p)^*(y,q)b,\omega] \\
&= [a,\omega''] \cdot [(x,p)^*(y,q),\omega'] \cdot [b,\omega] \\
&= [a,\omega''] \cdot [(w,k)(z,l)^*,\omega'] \cdot [b,\omega] \\
&= [a(w,k)(z,l)^*b,\omega] \\
&= [s',\omega],
\end{aligned}$$

with $\omega' := \vartheta_b(\omega)$ and $\omega'' := \vartheta_{(x,p)^*(y,q)b}(\omega)$.

After finitely many steps, the resulting word will not be of the form (129). Since then there is no "starred pair" left of a non-starred one in it, and the set of pairs $(\tilde{x},\tilde{p})$, $\tilde{p} \in P$, $\tilde{x} \in \mathcal{X}_{\tilde{p}}$, is closed under taking products, it can be contracted to the desired form $(x_2,p_2)(x_1,p_1)^*$ for $p_i \in P$, $x_i \in \mathcal{X}_{p_i}$. Hence $[s,\omega] = [(x_2,p_2)(x_1,p_1)^*,\omega]$. Furthermore, $Z^{\Omega}_{(p_1,x_1)}$ is the domain of the homeomorphism $\vartheta_{(x_2,p_2)(x_1,p_1)^*}$. In particular, $\omega \in Z^{\Omega}_{(p_1,x_1)}$. Thus, the assertion of the lemma follows.

□

Lemma 7.11. *Suppose that P fulfils the Ore conditions, $\theta\colon P \to \mathrm{Mono}(G)$ is injective and that $\bigcap_{p\in P}\theta_p(G) = \{1_G\}$. Then the canonical homomorphism of semigroups $\mathcal{S}^{\times}(D_{\mathfrak{Gr}}) \to I(D_{\mathfrak{Gr}})$ is injective, that is, $\mathcal{S}^{\times}(D_{\mathfrak{Gr}})$ can be seen as a subsemigroup of $I(D_{\mathfrak{Gr}})$.*

Proof. I expect that the result holds in the general case, since the relations (84) should not identify any of the generating pairs (x,p), $p \in P$, $x \in \mathcal{X}_p$. Since we did not prove nor need this in the general case, we show it indirectly in the case of the assumptions in the lemma. Those assumptions imply, by Lemma 7.8, that for $p, p' \in P$, $x \in \mathcal{X}_p$, $x' \in \mathcal{X}_{p'}$, if $\vartheta_{(x,p)} = \vartheta_{(x',p')}$, then $x = x'$ and $p' = p$. So $\vartheta\colon I(D_{\mathfrak{Gr}}) \to \mathcal{I}(\Omega)$ witnesses that no pairs (x,p),(x',p'), which are distinct in $\mathcal{S}^{\times}(D_{\mathfrak{Gr}})$, are identified in $I(D_{\mathfrak{Gr}})$. This implies the statement of the lemma. □

Proposition 7.12. *Suppose that P fulfils the right Ore conditions, that $\theta\colon P \to \mathrm{Mono}(G)$ is injective and that $\bigcap_{p\in P}\theta_p(G) = \{1_G\}$. Then the groupoid model $\mathcal{L} = I(D_{\mathfrak{Gr}}) \ltimes \Omega$ is effective.*

In particular, in Stammeier's "minimal", finite-type case, $\mathcal{L}$ is effective.

Proof. Suppose that the requirements in the proposition hold. We prove the proposition by plugging in $I(D_{\mathfrak{Gr}})$ for S, Ω for Y, $\vartheta\colon I(D_{\mathfrak{Gr}}) \to \mathcal{I}(\Omega)$ for $\vartheta\colon S \to \mathcal{I}(Y)$ and $\mathcal{S}(D_{\mathfrak{Gr}})$ for A in Lemma 7.6 and proving that the requirements of the lemma are fulfilled. The semigroup $\mathcal{I}(Y)$ is a generating subsemigroup of

$I(D_{\mathfrak{Gr}})$ (see Lemma 5.23, Definition 5.38 and Lemma 7.11) and hence closed under multiplication. Furthermore, $\vartheta\colon I(D_{\mathfrak{Gr}}) \to \mathcal{I}(\Omega)$ is an action of $I(D_{\mathfrak{Gr}})$ by partial homeomorphisms on Ω, and by Lemma 7.8, its restriction to $\mathcal{S}(D_{\mathfrak{Gr}})$ is injective.

We further need that for every $s \in I(D_{\mathfrak{Gr}})$ and $\omega \in D_{s^*s}$, there are $e \in E(S)$ with $\omega \in D_e$ and there are $p_i \in P$, $x_i \in \mathcal{X}_{p_i}$, $i = 1, 2$, such that

$$((x_2, p_2)(x_1, p_1)^*)^*(x_2, p_2)(x_1, p_1)^* e = e \text{ and} \tag{130}$$

$$(x_2, p_2)(x_1, p_1)^* e = se. \tag{131}$$

For arbitrary $s \in I(D_{\mathfrak{Gr}})$ and $\omega \in D_{s^*s}$, Lemma 7.10 provides $e \in E$ with $\omega \in D_e$ and $p_i \in P$, $x_i \in \mathcal{X}_{p_i}$, $i = 1, 2$, which fulfil (131) and furthermore $\omega \in D_{((x_2,p_2)(x_1,p_1)^*)^*(x_2,p_2)(x_1,p_1)^*}$. This last property allows us to shrink the idempotent e so that (130) holds.

Lastly, the sets $D_{(x,p)(x,p)^*} = Z^{\Omega}_{(x,p)}$, for $p \in P$, $x \in \mathcal{X}_p$, form a basis of Ω by Lemma 7.1. We have thus shown that the data we plugged in fulfil all conditions in Lemma 7.6; thus, $I(D_{\mathfrak{Gr}}) \ltimes_{\vartheta} \Omega$ is effective.

Now forget all assumptions in this proof and instead suppose that Stammeier's conditions hold and suppose that Stammeier's "minimality" condition holds. The latter means that $\bigcap_{p\in P} \theta_p(G) = \{1_G\}$. Furthermore, the Ore conditions are fulfilled (see Remark 5.16) and θ is injective (see Remark 3.14). Hence the first part of this proposition is applicable and $\mathcal{L}$ is effective. □

7.3 When is the groupoid model minimal?

A subset $U \subseteq \mathcal{G}^0$ of the object space of a groupoid $\mathcal{G}$ is *invariant*, if for all arrows $g \in \mathcal{G}^1$, $\mathbf{s}(g) \in U$ implies $\mathbf{r}(g) \in U$.

Definition 7.13 ([2, Definition 6.4])**.** A topological groupoid $\mathcal{G}$ is *minimal* if its object space $\mathcal{G}^0$ has no open, invariant subsets besides $\emptyset$ and $\mathcal{G}^0$.

Proposition 7.14. *Suppose that P fulfils the right Ore conditions. Then the groupoid model $\mathcal{L} = I(D_{\mathfrak{Gr}}) \ltimes \Omega$ is minimal.*

In particular, in Stammeier's finite-type case, $\mathcal{L}$ is minimal.

Proof. Suppose that $U \subseteq \Omega$ is a non-empty invariant open subset. Then by Lemma 7.1, there are $q \in P$ and $x \in \mathcal{X}_q$ such that $Z^{\Omega}_{(x,q)} \subseteq U$. By its definition in Lemma 5.27, $\vartheta_{(x,q)}$ is a homeomorphism with domain Ω and codomain $Z^{\Omega}_{(x,q)} \subseteq U$. The inverse of this homeomorphism induces a slice $\mathcal{U} := [(x,q)^*, Z^{\Omega}_{(x,q)}]$ in $I(D_{\mathfrak{Gr}}) \ltimes \Omega$ with $\mathbf{s}(\mathcal{U}) = Z^{\Omega}_{(x,q)} \subseteq U$ and $\mathbf{r}(\mathcal{U}) = \Omega$, see Lemma 5.42. Since U is invariant, this implies $\Omega \subseteq U$, that is, $\Omega = U$. This concludes the proof that there are no invariant subsets of Ω except for $\emptyset$ and Ω.

Now forget all assumptions in the proof and instead assume Stammeier's conditions hold. Then P fulfils the right Ore conditions by Remark 5.16. Thus the first part of the proposition is applicable and $\mathcal{L}$ is minimal. □

7.4 When is the groupoid model locally contracting?

Another question which can be interesting when studying groupoid C^*-algebras is whether the groupoids are locally contracting, see for example [4, Proposition 2.4]. We give a sufficient criterion for the groupoid model $\mathcal{L}$ to be locally contracting.

Definition 7.15 (compare [4, Definition 2.1])**.** A topological groupoid $\mathcal{G}$ is *locally contracting* if for every non-empty open subset O of $\mathcal{G}^0$, there exist an open subset V in O and a slice $\mathcal{U}$ of $\mathcal{G}$ with $\overline{V} \subseteq \mathbf{s}(\mathcal{U})$ and $\mathcal{U}_*(\overline{V}) \subsetneq V$. Here, $\mathcal{U}_*$ is the homeomorphism $\mathbf{s}(\mathcal{U}) \to \mathbf{r}(\mathcal{U})$ induced by the slice.

Lemma 7.16. *Suppose that P fulfils the right Ore conditions. Then the groupoid model $\mathcal{L} = I(D_{\mathfrak{Gr}}) \ltimes Y$ is locally contracting if and only if the set of the sets $Z^{\Omega}_{(x,p)}$, $p \in P$, $x \in \mathcal{X}_p$, ordered by inclusion, has no minimal elements.*

Proof. We first make some observations. For each $p \in P$ and $x \in \mathcal{X}_p$, $(\mathcal{U}_{(x,p)})_* = \vartheta_{(x,p)}$ is the homeomorphism with domain Ω and codomain $Z^{\Omega}_{(x,p)}$ coming from the slice $\mathcal{U}_{(x,p)} := \Theta((x,p), Z^{\Omega}_{(x,p)})$. For any $p_i \in P$, $x_i \in P$, $i = 1, 2$, the slice $\mathcal{U}_{(x_2,p_2)(x_1,p_1)^*}$ comes with a homeomorphism $(\mathcal{U}_{(x_2,p_2)(x_1,p_1)^*})_* \colon Z^{\Omega}_{(x_1,p_1)} \to Z^{\Omega}_{(x_2,p_2)}$. Sets of the form $Z^{\Omega}_{(x,p)}$, $p \in P$ and $x \in \mathcal{X}_p$, are closed and open and form a basis of Ω by Lemma 7.1.

Now, in order to prove one direction of the equivalence in the lemma, suppose that $\left(\left\{Z^{\Omega}_{(x,p)} \,\middle|\, p \in P, x \in \mathcal{X}_p\right\}, \subseteq\right)$ has no minimal elements. Then, given any open set $\mathcal{O} \in \Omega$, there are $p_i \in P$, $x_i \in \mathcal{X}_{p_i}$, $i = 1, 2$, such that $Z^{\Omega}_{(x_2,p_2)} \subsetneq Z^{\Omega}_{(x_1,p_1)} \subseteq \mathcal{O}$. Since $Z^{\Omega}_{(x_1,p_1)}$ is already closed,

$$(\mathcal{U}_{(x_2,p_2)(x_1,p_1)^*})_*(\overline{Z^{\Omega}_{(x_1,p_1)}}) = Z^{\Omega}_{(x_2,p_2)} \subsetneq Z^{\Omega}_{(x_1,p_1)}$$

Since $\mathcal{O}$ was chosen as an arbitrary open subset of Ω, this proves that $I(D_{\mathfrak{Gr}}) \ltimes \Omega$ is locally contracting.

Conversely, suppose that $I(D_{\mathfrak{Gr}}) \ltimes \Omega$ is locally contracting. Assume that there exist $p \in P$ and $x \in \mathcal{X}_p$ such that $Z^{\Omega}_{(x,p)}$ is minimal with respect to the subset relation. Since $Z^{\Omega}_{(x,p)}$ is open, there must be an open subset $V \subseteq Z^{\Omega}_{(x,p)}$ and a slice $\mathcal{U}$ such that $\mathcal{U}_*(\overline{V}) \subsetneq V$. *A fortiori*, $\mathcal{U}_*(V) \subsetneq V$. Since $\mathcal{U}_*(V)$ is open, there must be a basic set $Z^{\Omega}_{(y,q)}$ with $q \in P$ and $y \in \mathcal{X}_q$ such that

$$Z^{\Omega}_{(y,q)} \subseteq \mathcal{U}_*(V) \subsetneq V \subseteq Z^{\Omega}_{(x,p)}.$$

This contradicts the minimality of $Z^{\Omega}_{(x,p)}$. Hence, the assumption that there exists a minimal set of the form $Z^{\Omega}_{(x,p)}$, $p \in P$, $x \in \mathcal{X}_p$, is wrong. This concludes the proof. □

Proposition 7.17. *Suppose that P fulfils the right Ore conditions, that $\bigcap_{p \in P} \theta_p(G) = \{1_G\}$ and that $\theta_p(G) \neq \{1_G\}$ for all $p \in P$. Then the groupoid model $\mathcal{L}$ is locally contracting.*

In particular, in Stammeier's "minimal", finite-type case, $\mathcal{L}$ is locally contracting.

Proof. For $p \in P$ and $([g_q]_q)_q \in \Omega$, $\omega := ([g_q]_q)_q$,

$$\omega \in Z^{\Omega}_{(1_G,p)} \text{ if and only if } g_p \in \theta_p(G). \tag{132}$$

Suppose that the requirements in the proposition hold. Assume that there exists $p \in P$ and $x \in \mathcal{X}_p$ such that $Z^{\Omega}_{(x,p)}$ is minimal in the sense that it does not properly contain any other set of this form. Then $Z^{\Omega}_{(1_G,p)}$ is also minimal, since any set of the form $Z^{\Omega}_{(y,q)}$, $q \in P$, $y \in \mathcal{X}_q$, with $Z^{\Omega}_{(y,q)} \subsetneq Z^{\Omega}_{(1_G,p)}$ could be shifted by multiplying by x from the left to result in a set contradicting the minimality of $Z^{\Omega}_{(x,p)}$: Indeed, for $\tilde{x} \in G$, $\tilde{p} \in P$, $\tilde{y} \in \mathcal{X}_{\tilde{p}}$, $\omega := ([g_q]_q)_q \in \Omega$ and $\tilde{x}\omega := ([\tilde{x}g_q]_q)_q$, $\omega \in Z^{\Omega}_{(\tilde{y},\tilde{p})}$, meaning that $[g_{\tilde{p}}]_{\tilde{p}} = [\tilde{y}]_{\tilde{p}}$, holds if and only if $\tilde{x}\omega \in Z^{\Omega}_{(\tilde{x}\tilde{y},\tilde{p})}$, meaning that $[\tilde{x}g_{\tilde{p}}]_{\tilde{p}} = [\tilde{x}\tilde{y}]_{\tilde{p}}$, holds; hence $Z^{\Omega}_{(y,q)} \subsetneq Z^{\Omega}_{(1_G,p)}$ implies $Z^{\Omega}_{(xy,q)} \subsetneq Z^{\Omega}_{(x,p)}$.

One of the requirements in the proposition implies that there exists $1_G \neq g \in \theta_p(G)$. Let $\omega := ([g]_{\tilde{q}})_{\tilde{q}}$. Then, by (132),

$$\omega \in Z^{\Omega}_{(1_G,p)}. \tag{133}$$

By the requirement that $\bigcap_{r\in P} \theta_r(G) = \{1_G\}$, there must be $r \in P$ such that $g \notin \theta_r(G)$. By the right Ore conditions, there are $q, s \in P$ such that $pq = rs$. Then, since $\theta_{rs}(G) \subseteq \theta_r(G)$, $g \notin \theta_{pq}(G)$. Then, by (132), $\omega \notin Z^{\Omega}_{(1_G,pq)}$. Thus, by (133), $Z^{\Omega}_{(1_G,pq)} \subsetneq Z^{\Omega}_{(1_G,p)}$, a contradiction to the minimality of $Z^{\Omega}_{(1_G,p)}$. Thus, the assumption that there exists a minimal basic set of the form $Z^{\Omega}_{(x,p)}$, $p \in P$, $x \in \mathcal{X}_p$, is wrong. Hence, by Lemma 7.16, $\mathcal{L}$ is locally contracting.

Now forget all assumptions in the proof and instead suppose that Stammeier's conditions hold and suppose that Stammeier's "minimality" and finite-type conditions are fulfilled. The "minimality" condition means that $\bigcap_{p\in P} \theta_p(G) = \{1_G\}$. Furthermore, P fulfils the right Ore conditions by Remark 5.16. Furthermore, Stammeier's conditions imply that G is infinite, and the finite-type condition means that $\theta_p(G) \leq G$ has finite index for each $p \in P$; this implies that for each $p \in P$, $\theta_p(G)$ is infinite and hence not the trivial group. Hence the first part of the proposition is applicable and $\mathcal{L}$ is locally contracting. □

7.5 A criterion for simplicity of the C^*-algebra

We provide some statements about criteria for simplicity of $C^*(\mathcal{L})$. For a criterion in Stammeier's work, see [26, Theorem 3.26]. Beware that we only remodeled his C^*-algebra $\mathcal{O}[G, P, \theta]$ by $C^*(\mathcal{L})$ in Stammeier's finite-type case, see Corollary 6.14.

Lemma 7.18. *Suppose that P fulfils the right Ore conditions and is countable. Further suppose that G is countable. Then $\mathcal{L}$ is second countable.*

Proof. Suppose the requirements in the lemma hold. Then by Lemma 7.1, a basis for Ω is given by the sets $Z^{\Omega}_{(x,p)}$, $p \in P$, $x \in \mathcal{X}_p = G$. Since P and G are countable, there are only countably many such sets. This concludes the proof. □

Theorem 7.19 (compare [7, Theorem 5.1])**.** *Suppose that P fulfils the right Ore conditions and is countable, that G is countable and that $\theta_p(G) \leq G$ has finite index for all $p \in P$. Then $C^*(\mathcal{L})$ is simple if and only if*

1. *$C^*(\mathcal{L}) = C^*_{\mathrm{red}}(\mathcal{L})$,*
2. *$\mathcal{L}$ is effective.*

Proof. Suppose that the required conditions hold. Then, by Lemma 7.18, $\mathcal{L}$ is second countable. Furthermore, by Lemma 5.44, $\mathcal{L}$ is locally compact, and, by

Proposition 7.2, $\mathcal{L}$ is Hausdorff. In summary, $\mathcal{L}$ is a second countable, locally compact, Hausdorff, étale groupoid. Thus by [7, Lemma 3.3], $\mathcal{L}$ is "topologically principal", if and only if the interior of the isotropy subgroupoid of $\mathcal{L}$ is its space of units, that is, if and only if it is effective in the sense we use. Thus, by [7, Theorem 5.1], $C^*(\mathcal{L})$ is simple if and only if

1. $C^*(\mathcal{L}) = C^*_{\text{red}}(\mathcal{L})$,
2. $\mathcal{L}$ is effective and
3. $\mathcal{L}$ is minimal.

By Proposition, 7.14 is minimal. This concludes the proof. □

Corollary 7.20. *Suppose that P fulfils the right Ore conditions and is countable, that G is countable, that $\theta_p(G) \leq G$ has finite index for all $p \in P$, that $C^*(\mathcal{L}) = C^*_{\text{red}}(\mathcal{L})$, that $\theta \colon P \to \text{Mono}(G)$ is injective and that $\bigcap_{p\in P} \theta_p(G) = \{1_G\}$. Then $C^*(\mathcal{L})$ is simple.*

In particular, in Stammeier's "minimal", finite-type case, $C^*(\mathcal{L}) = C^*_{\text{red}}(\mathcal{L})$ implies that $C^*(\mathcal{L})$ is simple.

Proof. The first part follows from Theorem 7.19 and Proposition 7.12.

Now suppose that Stammeier's conditions hold. This implies that P fulfils the right Ore conditions and is countable, that G is countable and, see Remark 3.14, that θ is injective. Further suppose that Stammeier's "minimality" and finite-type conditions hold. This means, respectively, that $\bigcap_{p\in P} \theta_p(G) = \{1_G\}$ and that $\theta_p(G) \leq G$ has finite index for all $p \in P$. Now all prerequisites of the first part of the corollary except for $C^*(\mathcal{L}) = C^*_{\text{red}}(\mathcal{L})$ are fulfilled. Hence $C^*(\mathcal{L})$ is simple, if $C^*(\mathcal{L}) = C^*_{\text{red}}(\mathcal{L})$. □

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