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Bicategorical perspective on Steinberg algebras

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BICATEGORICAL PERSPECTIVE ON STEINBERG ALGEBRAS

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Abstract. We define a bicategory with ample groupoids as objects, groupoid correspondences (namely spaces with commuting left and right actions) as 1-arrows, and continuous equivariant maps as 2-arrows. We extend the construction of the Steinberg algebra for ample groupoids to a homomorphism from this bicategory to the bicategory of rings with local units, smooth bimodules, and bimodule homomorphisms. Then, we find an explicit construction of a covariance ring for a finitely generated and projective diagram over an Ore monoid in the subbicategory of unital rings. We recall the construction of a groupoid model for a tight diagram over an Ore monoid. Finally, we prove that the covariance ring of a diagram of bimodules, obtained from a tight diagram of correspondences over an Ore monoid, is given by the Steinberg algebra of the groupoid model.

Contents

Key words and phrases. ample groupoid; groupoid correspondence; bicategory; strictly unital homomorphism of bicategories; Steinberg algebra; filtered colimit; covariance ring; Ore monoid; groupoid model.

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1. Introduction

The principle of algebraization suggests to transform abstract, intangible concepts into algebraic expressions because this allows the researcher to mathematically model and solve complex problems using algebraic methods and tools. By this method, existing structures and relations can be analyzed and subsequently abstracted to their essential properties, so that a profound comprehension of the actual mathematical objects in study can be gained. According to [Ara+18] the trend of algebraization of concepts from operator theory into algebra started with von Neumann and Kaplansky (in 1986) and their students Berberian and Rickart, who strived to explore what properties in operator algebra theory arise naturally from discrete underlying structures. Quite recently, Steinberg [Ste09] introduced his Steinberg algebras as an algebraization of the groupoid C^*-algebras first studied by Renault [Ren80]. Currently, Meyer's research is focused on investigating these groupoid C*-algebras from a bicategorical standpoint. The objective of this thesis is to identify algebraic parallels to some of his most recent findings.

In [AKM22; AM15; Alb15] groupoid C^*-algebras and related constructions of C^*-algebras are put into a bicategorical perspective and studied from this point of view. In [AKM22] the bicategory $\mathsf{Gr}_{\mathsf{inj}}$ of (locally compact, étale) groupoids, (locally compact, étale) correspondences, and injective continuous equivariant maps is introduced to extend the definition of a groupoid C^*-algebra to a strictly unital homomorphism $\mathsf{Gr}_{\mathsf{inj}} \to \mathsf{Corr}$ to the bicategory Corr of C^*-correspondences. This enables studying diagrams in Corr (that is, product systems) that were obtained from diagrams in $\mathsf{Gr}_{\mathsf{inj}}$ by composing with this homomorphism. In particular, one can study certain bicategorical limits (that is, absolute Cuntz-Pimsner algebras) of such diagrams, which capture many constructions of C^*-algebras coming from combinatorial or dynamical data. In [AM15] the absolute Cuntz-Pimsner algebra of a proper, non-degenerate product system over an Ore monoid P is constructed explicitly through filtered colimits and a related diagram over a group (that is, a Fell bundle over a group). Finally, in [Alb15; AM15], it is shown how to realize this bicategorical limit of a diagram in Corr (that is, the absolute Cuntz-Pimsner algebra of a product system) obtained from a tight Ore diagram in $\mathsf{Gr}_{\mathsf{inj}}$, as a groupoid C^*-algebra.

This thesis identifies an algebraic counterpart to this theory, with a focus on viewing Steinberg algebras of ample groupoids and associated constructions from a bicategorical perspective. Since we are interested in the Steinberg algebra of a groupoid, we restrict ourselves to *ample* groupoids, which have a sufficiently rich base for their topology (that we call an *ample base*), so that the Steinberg algebra is manageable and interesting to consider. We first define the bicategory Gr_{a} of ample groupoids, ample groupoid correspondences, and continuous equivariant maps. Now, the Steinberg algebra is a ring with local units, that is, an object in the bicategory Rings of rings with local units, smooth bimodules, and bimodule homomorphisms, which serves as the algebraic analogue to the bicategory Corr of C^*-correspondences. So we extend the definition of the Steinberg algebra to a strictly unital homomorphism $A\colon \mathsf{Gr}_{\mathsf{a}} \to \mathsf{Rings}$. Note that by composing with this homomorphism, every diagram in Gr_{a} induces a diagram in Rings. Next, we explicitly construct a bicategorical limit, namely the covariance ring, of a finitely generated and projective (*fgp*) Ore diagram in Rings through filtered colimits and a related lax diagram over a group. Finally, we prove that for a diagram in Rings that is induced by a tight Ore diagram in Gr_{a}, the covariance ring can be realized as the Steinberg algebra of a groupoid.

The main goal of this thesis is to formulate these algebraic versions of the results and to work out the proofs properly. It remains for future studies to consider concrete examples such as group actions on spaces, (higher-rank) graphs, and self-similar groups and to apply the obtained results to them.

This thesis is structured as follows. In Section 2, we introduce ample groupoids and discuss that the compact slices form an ample base for their topology. In Section 3 we introduce groupoid actions on spaces and define ample groupoid correspondences as spaces with commuting left and right groupoid actions, where the right action is particularly well-behaved. We show that the compact slices form an ample base for their topology. Finally, we take a closer look at proper and tight correspondences. In Section 4, we introduce the remaining necessary data to define the bicategory Gr_a of ample correspondences and also introduce the subbicategories of *cocompact* groupoids and *proper/tight* ample correspondences. These Sections 2–4 were structured analogously to [AKM22, Section 2-6], where the bicategory Gr_inj is introduced. In Section 5, we recall the bicategory Rings of smooth bimodules and the subbicategory $\mathsf{Rings}_\mathsf{u}$ of unital rings, as well as the subbicategory $\mathsf{Rings}_\mathsf{fgp}$ of unital rings and fgp bimodules that were all introduced in [Mey22a] in detail. We also prove some standard results on fgp modules. In Section 6, we define the Steinberg module of a topological space and prove that the Steinberg module of a topological space with an ample base is particularly manageable. We then define multiplicative structures (given by convolution) to turn the Steinberg module of an ample groupoid into an algebra and of an ample correspondence into a bimodule. After that, we prove that the Steinberg bimodule of a proper correspondence over a cocompact groupoid is fgp. In Section 7, we extend the construction of the Steinberg algebra into a strictly unital homomorphism $A\colon \mathsf{Gr}_\mathsf{a} \to \mathsf{Rings}$ of bicategories, analogously to the construction of the homomorphism $\mathsf{Gr}_\mathsf{inj} \to \mathsf{Corr}$ in [AKM22]. In Section 8, we define filtered colimits and explicitly construct them in the categories $\mathtt{Top}$, R-$\mathtt{Mod}$, $\mathtt{Ring}$, $\mathtt{AbGroup}$, and $\mathtt{Set}$. We do this as a preparation since we need these constructions in the following sections. In Section 9, we recall the definitions of (lax) diagrams, (lax) covariant representations, and (lax) covariance rings in $\mathsf{Rings}_\mathsf{u}$ from [Mey22a]. We explicitly construct a covariance ring for an fgp Ore diagram $\mathcal{F}$ in $\mathsf{Rings}_\mathsf{u}$ by building a lax diagram $\mathcal{O}$ out of filtered colimits and proving that the covariance ring of $\mathcal{F}$ is given by the lax covariance ring of $\mathcal{O}$, analogously to the construction of an absolute Cuntz-Pimsner algebra in [AM15, Theorem 3.16]. In Section 10, we give a brief review of diagrams in Gr_a and groupoid models studied in [Mey22b]. We recall the explicit construction of a groupoid model for a tight Ore diagram in Gr_a, which is done for Gr_inj in [Alb15]. In Section 11, we prove that the Steinberg algebra of the groupoid model of a tight Ore diagram is the covariance ring of the induced fgp Ore diagram in $\mathsf{Rings}_\mathsf{u}$, analogously to the proof in [Alb15].

Note that throughout this thesis a ring is not necessarily commutative nor unital unless explicitly stated, except for the ring R, introduced at the beginning of Section 6, which we always assume to be commutative and unital.

2. Ample groupoids

We start by introducing ample groupoids $\mathcal{G}$, which are topological groupoids with a sufficiently well-behaved topological structure, so that the Steinberg module (introduced in Section 6) is an interesting object. Namely, we want to find an ample base for the topology on $\mathcal{G}$. We introduce *slices*, which are open subsets respecting the structure, and we show that the compact slices form an ample base for the topology on $\mathcal{G}$. The ample groupoids are the objects in the bicategory $\mathsf{Gr_a}$.

We start by defining a topological groupoid.

Definition 2.1. A *groupoid* $\mathcal{G}$ is a small category where every morphism is invertible. A *topological groupoid* is a groupoid $\mathcal{G}$ with a topology on the object set $\mathcal{G}^0$ and on the morphism set $\mathcal{G}$ such that the maps

- $r, s \colon \mathcal{G} \to \mathcal{G}^0$ the range and source maps;
- $\mathcal{G} \times_{s,\mathcal{G}^0,r} \mathcal{G} \to \mathcal{G}$, $(g,h) \mapsto g \cdot h$, the composition map;
- $\mathcal{G} \to \mathcal{G}$, $g \mapsto g^{-1}$, the inverse map; and
- $\mathcal{G}^0 \to \mathcal{G}$, $x \mapsto 1_x$, the unit map

are continuous.

One can think of a (topological) groupoid as a generalization of a (topological) group. In a group, we can multiply any two elements. In a groupoid, this does not always work. We may only multiply two elements that have a fitting range and source. So we can think of a groupoid as a group where multiplication is only partially defined.

Example 2.2. A (topological) groupoid $\mathcal{G}$ where the category has only one object is the same as a (topological) group.

We denote the preimages of the range and source maps of $x \in \mathcal{G}^0$, respectively, as

$$\mathcal{G}^x := r^{-1}(x) \text{ and } \mathcal{G}_x := s^{-1}(x).$$

From now on, when we say "groupoid", we always mean a topological groupoid.

Definition 2.3. A topological space X is called *totally disconnected*, if the only connected subsets are singletons.

Definition 2.4. A groupoid $\mathcal{G}$ is called

- *cocompact*, if $\mathcal{G}^0$ is compact;
- *étale*, if r and s are local homeomorphisms;
- *locally compact*, if $\mathcal{G}^0$ is Hausdorff and locally compact and $\mathcal{G}$ is locally compact;
- *totally disconnected*, if $\mathcal{G}^0$ is totally disconnected.

We call a groupoid *ample*, if it is étale, locally compact, and totally disconnected.

First, we collect some basic properties of the relevant topological terms.

Lemma 2.5. *For topological spaces X, Y and a local homeomorphism $f \colon X \to Y$, we have*

- *if X is Hausdor , then any subset $A \subset X$ is Hausdor ;*
- *if X is totally disconnected, then any subset $A \subset X$ is totally disconnected;*
- *if Y is totally disconnected, then X is totally disconnected; and*
- *if Y is locally compact and Hausdor , then X is locally compact and locally Hausdor .*

Proof. The first and second statements are immediate from the definition.

For the third statement, we take a connected subset $A \subset X$. Since f is a local homeomorphism and hence continuous also $f(A) \subset Y$ is connected. Now, Y is totally disconnected and thus $f(A)$ is a singleton set, that is, $f(A) = \{y\}$ for some $y \in Y$. Thus, $A \subset f^{-1}(f(A)) = f^{-1}(y)$ is a connected subset of the fiber of $y \in Y$. Now, fibers of local homeomorphisms are discrete, and hence A is also a singleton set.

For the fourth statement, take $x \in X$ and an open neighborhood $U \subset X$ of x such that $f|_U : U \to f(U)$ is a homeomorphism. Since f is a local homeomorphism, it is an open map. Now, since Y is locally compact and Hausdorff, the open subset $f(U)$ is locally compact as well. Furthermore, as a subset of a Hausdorff space, it is Hausdorff. Thus, U is locally compact and Hausdorff. Hence, X is locally compact and locally Hausdorff.

Remark 2.6. Note that for an ample groupoid $\mathcal{G}$, the object space $\mathcal{G}^0$ is locally compact, Hausdorff and totally disconnected. Since we have local homeomorphisms $r, s: \mathcal{G} \to \mathcal{G}^0$, Lemma 2.5 shows that the groupoid $\mathcal{G}$ itself is locally compact, locally Hausdorff and totally disconnected. But $\mathcal{G}$ is not necessarily Hausdorff.

Definition 2.7. A *slice*[1] of a groupoid $\mathcal{G}$ is an open subset $U \subset \mathcal{G}$ such that $r|_U$ and $s|_U$ are homeomorphisms onto an open subset of $\mathcal{G}^0$. We denote the set of all slices as $\mathcal{G}^{\text{op}}$ and the subset of all compact slices as $\mathcal{G}^{\text{a}}$.

We get the following alternative classification of étale groupoids.

Lemma 2.8. *A groupoid $\mathcal{G}$ is étale if and only if the topology on $\mathcal{G}$ has a base of slices.*

Proof. Given an étale groupoid $\mathcal{G}$ and a point $x \in \mathcal{G}$, we find open neighborhoods $x \in U, V$ of x such that $r|_U$ and $s|_V$ are homeomorphisms onto their images, which are open, as local homeomorphisms are open maps. Hence, $W := U \cap V$ is an open neighborhood of x and a slice. Note that open subsets of slices are still slices. Hence, the slices form a base for the topology. The other direction is immediate.

There is always the trivial *unit slice*.

Lemma 2.9. *If $\mathcal{G}$ is a locally compact, étale groupoid, then $\mathcal{G}^0 \subset \mathcal{G}$ is an open subset and thus a slice, which is called the* unit slice.

Proof. See [Exe08, Proposition 3.2].

Furthermore, we get a result about ample groupoids, providing an equivalent definition.

Proposition 2.10. *A locally compact, étale groupoid is ample if and only if the compact slices $\mathcal{G}^{\text{a}}$ form a base for the topology on $\mathcal{G}$.*

Proposition 2.10 is immediate from the following two lemmas.

Lemma 2.11. *Take a locally compact, Hausdorff space X. Then the following are equivalent:*

- *X is totally disconnected;*
- *there is a clopen*[2] *base for the topology on X;*
- *there is a compact open base $\mathcal{B}$ for the topology on X.*

Proof. For the forward direction in the first equivalence, we refer to [AT08, Propositon 3.1.7, p.136]. For the other direction, take a subset $A \subset X$ with $x, y \in A$ and $x \neq y$. Now, as X is Hausdorff, $\{y\}^c$ is open, and thus we find a clopen set $B \in \mathcal{B}$ such that $x \in B \subset \{y\}^c$. Thus, $A = (A \cap B) \cup (A \cap B^c)$ with $x \in A \cap B$ and $y \in A \cap B^c$ are both non-empty open subsets of A. Hence, A is not connected.

[1] In some literature, a slice is also called a *bisection*.

[2] Short for closed and open.

For the forward direction in the second equivalence, we show that the set of all compact open subsets is a base. For $x \in U \subset X$ open, because of local compactness, we find some compact K and open V such that $c \in V \subset K \subset U$. Using the assumption, we find a clopen subset $W \in \mathcal{B}$ with $x \in W \subset V \subset K$. As a closed subset of the compact set K, the space W is compact as well. The other direction is immediate since every compact subset in a Hausdorff space is closed.

Lemma 2.12. *For a locally compact, étale groupoid $\mathcal{G}$, there exists a base of compact open sets for the topology on $\mathcal{G}^0$ if and only if the compact slices $\mathcal{G}^{\mathrm{a}}$ form a base for the topology on $\mathcal{G}$.*

Proof. See [Ste09, Proposition 3.6, p.696].

We now take a closer look at the sets $\mathcal{G}^{\mathrm{op}}$ of slices and $\mathcal{G}^{\mathrm{a}} \subset \mathcal{G}^{\mathrm{op}}$ of compact slices for a locally compact, étale groupoid $\mathcal{G}$. The following Proposition 2.13 shows that they form inverse semigroups (note that associativity of the multiplication of the slices is inherited from $\mathcal{G}$). Furthermore, $\mathcal{G}^{\mathrm{op}}$ is even an inverse monoid where the identity element is given by the unit slice $\mathcal{G}^0 \in \mathcal{G}^{\mathrm{op}}$.

Proposition 2.13. *Given a locally compact, étale groupoid $\mathcal{G}$ and two slices $U, V \subset \mathcal{G}$, then*

$$UV := \{uv \in \mathcal{G} \mid u \in U,\ v \in V \text{ with } s(u) = r(v)\}$$
$$U^{-1} := \{u^{-1} \in \mathcal{G} \mid u \in U\}$$

are again slices. Furthermore, $U^{-1} \in \mathcal{G}^{\mathrm{op}}$ is the unique element, so that $U = UU^{-1}U$ and $U^{-1} = U^{-1}UU^{-1}$.

If $\mathcal{G}$ is ample and $U, V \subset \mathcal{G}$ are compact slices, then UV and U^{-1} are again compact slices.

Proof. For the first part, see [Exe08, Proposition 3.8]. For the second part, note that as $u = uu^{-1}u$, we immediately get $U \subset UU^{-1}U$. For the other direction, take $uv^{-1}w \in UU^{-1}U$. Now, as U is a slice, the composability of u, v^{-1}, w already implies $u = v = w$ and hence $uv^{-1}w = u \in U$. The proof of $U^{-1} = U^{-1}UU^{-1}$ is analogous.

Finally, for the third part, we take two compact slices $U, V \subset \mathcal{G}$. Since $\mathcal{G}^0$ is Hausdorff, any pullback over $\mathcal{G}^0$ is a closed subset of the product. Hence, $\mathcal{G} \times_{s,\mathrm{G}^0,r} \mathcal{G} \subset \mathcal{G} \times \mathcal{G}$ is closed. Now, as $U, V \subset \mathcal{G}$ are compact, so is the space $U \times V \subset \mathcal{G} \times \mathcal{G}$, and as $\mathcal{G} \times_{s,\mathrm{G}^0,r} \mathcal{G}$ is closed, the intersection $U \times_{s,\mathrm{G}^0,r} V$ is compact as well. Thus, the set UV, which is the image of $U \times_{s,\mathrm{G}^0,r} V$ under the continuous composition map, is compact. Similarly, the set U^{-1}, which is the image of U under the continuous inversion map, is compact.

Furthermore, in Section 6, we are interested in the Steinberg module $A_R(\mathcal{G})$ of an ample groupoid $\mathcal{G}$. To get an explicit presentation of the module, we need to find a sufficiently well-behaved base for the topology (to be able to apply Propositions 6.3 and 6.4). We call such a base an *ample base*.

Definition 2.14. For a topological space X, we call a base $\mathcal{B}$ *ample* if it is stable under taking compact open subsets (that is, if $B \in \mathcal{B}$ and $A \subset B$ is a compact open subset, then $A \in \mathcal{B}$) and if its sets $U \in \mathcal{B}$ are compact and Hausdorff.

Note that the base of compact slices $\mathcal{G}^a$ of an ample groupoid $\mathcal{G}$ is ample, as an open subset of a slice is again a slice, since r and s stay injective on a subset. Furthermore, any slice is homeomorphic to a subset of the Hausdorff space $\mathcal{G}^0$ and is hence Hausdorff.

Example 2.15. Consider the groupoid $\mathcal{G}$ given by a topological group. Now, the object space of $\mathcal{G}$ is given by the singleton set, and hence a slice of $\mathcal{G}$ is given by a point in $\mathcal{G}$. Thus, $\mathcal{G}$ is étale if and only if $\mathcal{G}$ is discrete (using Lemma 2.8). Since a discrete group $\mathcal{G}$ is locally compact and the singleton set is locally compact, Hausdorff, and totally disconnected, a topological group $\mathcal{G}$ is an ample groupoid if and only if it is discrete. The compact slices of a discrete group are given by $\{g\}$ for all $g \in \mathcal{G}$.

3. Groupoid actions and ample groupoid correspondences

In this section, we introduce groupoid actions on topological spaces. One can think of a groupoid action on a topological space as a generalization of a group action on a topological space, where the multiplication is only partially defined. Since we are only interested in locally compact, étale groupoids, we usually drop these adjectives. So from now on, when we say "groupoid", we always mean an étale, locally compact groupoid.

Afterward, we introduce groupoid correspondences, which are topological spaces with commuting left and right groupoid actions, so that the right action is particularly well-behaved. We follow closely the definitions and results from [AKM22], where groupoids and groupoid correspondences are introduced. Since we are mainly interested in ample groupoids, we investigate correspondences between ample groupoids and see what properties they inherit from their ample groupoids. We define slices on correspondences $\mathcal{X}$ and as it turns out the compact slices $\mathcal{X}^a$ form an ample base for the topology on $\mathcal{X}$ and hence the Steinberg module of $\mathcal{X}$ is interesting to study. The ample correspondences are the 1-arrows in the bicategory Gr_a.

We start with all the relevant definitions.

Definition 3.1 ([AKM22, Definition 2.3]). Let $\mathcal{G}$ be a groupoid. A *right $\mathcal{G}$-space* is a topological space $\mathcal{X}$ with a continuous map $s\colon \mathcal{X} \to \mathcal{G}^0$, called the *anchor map*, and a continuous map

$$\text{mult}\colon \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G} \to \mathcal{X}, \qquad \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G} := \{(x,g) \in \mathcal{X} \times \mathcal{G} \mid s(x) = r(g)\},$$

which we denote multiplicatively as $\cdot$, such that

(1) $s(x \cdot g) = s(g)$ for $x \in \mathcal{X}$, $g \in \mathcal{G}$ with $s(x) = r(g)$;
(2) $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 \cdot g_2)$ for $x \in \mathcal{X}$, $g_1, g_2 \in \mathcal{G}$ with $s(x) = r(g_1)$, $s(g_1) = r(g_2)$;
(3) $x \cdot s(x) = x$ for all $x \in \mathcal{X}$.

Similarly, one can define a *left $\mathcal{G}$-space* with $r\colon \mathcal{X} \to \mathcal{G}^0$ as the anchor map.

Sometimes, we just write "xg" and mean "$x \cdot g$". We sometimes also write s_X, r_X and s_G, r_G to distinguish between the respective range and source maps, if there is a chance of confusing them.

Definition 3.2 ([AKM22, Definition 2.4]). The *orbit space* $\mathcal{X}/\mathcal{G}$ is the quotient $\mathcal{X}/{\sim_\mathsf{G}}$ with the quotient topology, where $x \sim_\mathsf{G} y$ if there is an element $g \in \mathcal{G}$ with $s(x) = r(g)$ and $x \cdot g = y$. We always write $p\colon \mathcal{X} \to \mathcal{X}/\mathcal{G}$ for the orbit space projection.

Definition 3.3. A *right $\mathcal{G}$-space* $\mathcal{X}$ is called *cocompact*, if the orbit space $\mathcal{X}/\mathcal{G}$ is compact.

Definition 3.4 ([AKM22, Definition 2.5]). Let $\mathcal{X}$ and $\mathcal{Y}$ be right $\mathcal{G}$-spaces. A continuous map $f\colon \mathcal{X} \to \mathcal{Y}$ is *$\mathcal{G}$-equivariant* if $s\big(f(x)\big) = s(x)$ for all $x \in \mathcal{X}$ and $f(x \cdot g) = f(x) \cdot g$ for all $x \in \mathcal{X}$, $g \in \mathcal{G}$ with $s(x) = r(g)$.

Definition 3.5 ([AKM22, Definition 2.6]). Let $\mathcal{X}$ be a right $\mathcal{G}$-space and $\mathcal{Z}$ a space. A continuous map $f\colon \mathcal{X} \to \mathcal{Z}$ is *$\mathcal{G}$-invariant* if $f(x \cdot g) = f(x)$ for all $x \in \mathcal{X}$, $g \in \mathcal{G}$ with $s(x) = r(g)$.

Definition 3.6. Let $\mathcal{G}$ be a groupoid and $\mathcal{X}$ a right $\mathcal{G}$-space. An open subset $U \subset \mathcal{X}$ such that the projection map $p|_U\colon U \to \mathcal{X}/\mathcal{G}$ is a homeomorphism is called a *fundamental domain* of $\mathcal{X}$.

Next, we establish a technical result.

Lemma 3.7. *For a groupoid $\mathcal{G}$, a right $\mathcal{G}$-space $\mathcal{X}$ and a subset $V \subset \mathcal{X}$ such that $s|_V\colon V \to s(V)$ is a homeomorphism, the projection map*

$$\pi_\mathsf{G}\colon V \times_{s,\mathcal{G}^0,r} \mathcal{G} \to r_\mathsf{G}^{-1}\big(s_\mathsf{X}(V)\big) \subset \mathcal{G}, \qquad (x,g) \mapsto g,$$

is a homeomorphism.

Proof. The map is continuous by definition. Furthermore, the canonical embedding $\iota\colon r_G^{-1}(s_X(V)) \to \mathcal{G}$ and the continuous map

$$r_G^{-1}(s_X(V)) \xrightarrow{r} s(V) \xrightarrow{s|_V^{-1}} V$$

make the relevant fiber product square commute, so we get a unique continuous map

$$r_G^{-1}(s_X(V)) \to V \times_{s,G^0,r} \mathcal{G}, \qquad g \mapsto \left(s|_V^{-1}(r(g)), g\right).$$

Now, this map defines a continuous inverse to π_G and hence π_G is a homeomorphism.

3.1. **Basic groupoid actions.** We are mainly interested in spaces $\mathcal{X}$ with a well-behaved right $\mathcal{G}$-action, which in our case means a *basic* right $\mathcal{G}$-action with a Hausdorff orbit space $\mathcal{X}/\mathcal{G}$ and a locally homeomorphic anchor map $s\colon \mathcal{X} \to \mathcal{G}^0$. We start by exploring *basic* right actions.

Definition 3.8 ([AKM22, Definition 2.7])**.** A right $\mathcal{G}$-space $\mathcal{X}$ is *basic* if the following map is a homeomorphism onto its image with the subspace topology from $\mathcal{X} \times \mathcal{X}$:

$$f\colon \mathcal{X} \times_{s,G^0,r} \mathcal{G} \to \mathcal{X} \times \mathcal{X}, \qquad (x, g) \mapsto (x \cdot g, x). \tag{3.9}$$

Definition 3.10 ([AKM22, Definition 2.13])**.** A right $\mathcal{G}$-space is *free* if the map in (3.9) is injective; equivalently, $x \cdot g = x$ for $x \in \mathcal{X}$, $g \in \mathcal{G}$ with $s(x) = r(g)$ implies $g = s(x)$.

So, in particular, every basic action is free, that is, for $x, y \in \mathcal{X}$ an element $g \in \mathcal{G}$ with $xg = y$ is unique. One can think of a basic action as a free action, where this unique $g \in \mathcal{G}$ is chosen in a continuous way, as the following Lemma 3.13 shows.

Definition 3.11 (compare [AKM22, Definition and Lemma 3.4])**.** Let $\mathcal{X}$ be a space with a basic right $\mathcal{G}$-action. Let $p\colon \mathcal{X} \to \mathcal{X}/\mathcal{G}$ be the orbit space projection. The image of the map (3.9) is the subset $\mathcal{X} \times_{X/G} \mathcal{X} = \mathcal{X} \times_{p,X/G,p} \mathcal{X}$ of all $(x_1, x_2) \in \mathcal{X} \times \mathcal{X}$ with $p(x_1) = p(x_2)$. The inverse to the map in (3.9) induces a continuous map

$$\mathcal{X} \times_{X/G} \mathcal{X} \xrightarrow{\cong} \mathcal{X} \times_{s,G^0,r} \mathcal{G} \xrightarrow{\mathrm{pr}_G} \mathcal{G}, \qquad (x_1, x_2) \mapsto \langle x_2 \,|\, x_1 \rangle. \tag{3.12}$$

That is, $\langle x_1 \,|\, x_2 \rangle$ is defined for $x_1, x_2 \in \mathcal{X}$ with $p(x_1) = p(x_2)$ in $\mathcal{X}/\mathcal{G}$, and it is the unique $g \in \mathcal{G}$ with $s(x_1) = r(g)$ and $x_2 = x_1 g$.

Now, this map gives rise to an equivalent characterization of a right $\mathcal{G}$-action being basic:

Lemma 3.13 (compare [AKM22, Definition and Lemma 3.4.])**.** *A right $\mathcal{G}$-action on a topological space $\mathcal{X}$ is basic if and only if the $g \in \mathcal{G}$ with $x_2 = x_1 g$ for $x_1, x_2 \in \mathcal{X}$ with $p(x_1) = p(x_2)$ is unique and the resulting map $\mathcal{X} \times_{X/G} \mathcal{X} \to \mathcal{G}$, $(x_1, x_2) \mapsto g$, is continuous.*

Proof. See [AKM22, Definition and Lemma 3.4].

Furthermore, for basic actions, the orbit space projection map is well-behaved:

Lemma 3.14 (compare [AKM22, Lemma 2.12])**.** *Let $\mathcal{G}$ be a groupoid. The orbit space projection $p\colon \mathcal{X} \to \mathcal{X}/\mathcal{G}$ for a basic $\mathcal{G}$-action is a surjective local homeomorphism.*

Proof. See [AKM22, Lemma 2.12].

A fundamental domain of a basic right $\mathcal{G}$-space $\mathcal{X}$ already fully describes it:

Lemma 3.15. *Let $\mathcal{X}$ be a basic right $\mathcal{G}$-space and $U \subset \mathcal{X}$ a fundamental domain. Then the multiplication map*

$$U \times_{s,\mathcal{G}^0,r} \mathcal{G} \to \mathcal{X}, \qquad (u,g) \mapsto ug,$$

is a homeomorphism.

Proof. The above-defined map is continuous by definition, and we can define a continuous inverse by

$$\mathcal{X} \to U \times_{s,\mathcal{G}^0,r} \mathcal{G}, \qquad x \mapsto \big(u_x, \langle u_x \,|\, x\rangle\big),$$

where $u_x := p|_U^{-1}\big(p(x)\big) \in U$ is the unique element in U with $p(u_x) = p(x)$.

Note that if we define a right $\mathcal{G}$-space structure on $U \times_{s,\mathcal{G}^0,r} \mathcal{G}$ by multiplying in the second component, this homeomorphism is actually a homeomorphism of $\mathcal{G}$-spaces, that is, a $\mathcal{G}$-equivariant homeomorphism.

3.2. **Ample groupoid correspondences.** Now, we are ready to introduce ample groupoid correspondences. These are the 1-arrows in the bicategory of ample groupoids Gr_a. In [AKM22] groupoid correspondences are introduced. We recall the important definitions and results and investigate ample correspondences, that is, correspondences between ample groupoids. We define *slices* as open subsets of $\mathcal{X}$ that are well-behaved with the right $\mathcal{G}$-structure. As it turns out, an ample correspondence $\mathcal{X}$ inherits enough of the structure on the ample groupoid $\mathcal{G}$, implying that the compact slices form an ample base for the topology on $\mathcal{X}$. Hence, we get an explicit presentation of the Steinberg module $A_R(\mathcal{X})$ in Section 6. We start with the definition of a groupoid correspondence.

Definition 3.16 (compare [AKM22, Definition 3.1])**.** Let $\mathcal{H}$ and $\mathcal{G}$ be groupoids. A *groupoid correspondence* $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ from $\mathcal{G}$ to $\mathcal{H}$ is a space $\mathcal{X}$ with commuting actions of $\mathcal{H}$ on the left and $\mathcal{G}$ on the right such that

- the right anchor map $s\colon \mathcal{X} \to \mathcal{G}^0$ is a local homeomorphism;
- the right $\mathcal{G}$-action is basic; and
- the right orbit space $\mathcal{X}/\mathcal{G}$ is Hausdorff.

That the actions of $\mathcal{H}$ and $\mathcal{G}$ commute means that $s(h\cdot x) = s(x)$, $r(x\cdot g) = r(x)$, and $(h\cdot x)\cdot g = h\cdot(x\cdot g)$ for all $g \in \mathcal{G}$, $x \in \mathcal{X}$, $h \in \mathcal{H}$ with $s(h) = r(x)$ and $s(x) = r(g)$, where $s\colon \mathcal{X} \to \mathcal{G}^0$ and $r\colon \mathcal{X} \to \mathcal{H}^0$ are the anchor maps.

If both $\mathcal{H}$ and $\mathcal{G}$ are ample, we call $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ an *ample groupoid correspondence*.

We sometimes just write "correspondence", and mean "groupoid correspondence".

Definition 3.17 (compare [AKM22, Definition 7.2])**.** Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. A *slice* of a groupoid correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ is an open subset $V \subseteq \mathcal{X}$ such that the right anchor map $s|_V$ and the orbit space projection $p|_V$ are homeomorphisms onto open subsets of $\mathcal{G}^0$ and $\mathcal{X}/\mathcal{G}$, respectively. Denote the set of all slices as $\mathcal{X}^{\mathrm{op}}$ and the subset of all compact slices as $\mathcal{X}^{\mathrm{a}}$.

Example 3.18. For a groupoid $\mathcal{G}$, we can canonically define a groupoid correspondence $\mathcal{G}\colon \mathcal{G} \leftarrow \mathcal{G}$ with the obvious left and right actions of $\mathcal{G}$ by multiplication and the range and source maps $r, s\colon \mathcal{G} \to \mathcal{G}^0$ as left and right anchor maps. Thus, the right anchor map s is a local homeomorphism, as $\mathcal{G}$ is étale. Furthermore, for $p\colon \mathcal{G} \to \mathcal{G}/\mathcal{G}$ it is easy to check that $p(x) = p(y) \Leftrightarrow r(x) = r(y)$, hence r induces an isomorphism $\mathcal{G}/\mathcal{G} \cong \mathcal{G}^0$. Thus, $\mathcal{G}/\mathcal{G}$ is Hausdorff. Additionally, the right action is basic, as the map

$$f\colon \mathcal{G} \times_{s,\mathcal{G}^0,r} \mathcal{G} \to \mathcal{G} \times \mathcal{G}, \qquad (x,g) \mapsto (x\cdot g, x),$$

has image $\mathcal{G} \times_{r,\mathcal{G}^0,r} \mathcal{G}$ and we can define a continuous inverse map

$$\mathcal{G} \times_{r,\mathcal{G}^0,r} \mathcal{G} \to \mathcal{G} \times_{s,\mathcal{G}^0,r} \mathcal{G}, \qquad (x,y) \mapsto (y, y^{-1}x).$$

Hence, $\mathcal{G}$ is indeed a groupoid correspondence, and the definition of a slice of $\mathcal{G}$ as a groupoid and of $\mathcal{G}$ as a groupoid correspondence coincide.

Remark 3.19. For a groupoid correspondence $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$, the right anchor map $s: \mathcal{X} \to \mathcal{G}^0$ is a local homeomorphism by definition and the orbit space projection $p: \mathcal{X} \to \mathcal{X}/\mathcal{G}$ is a surjective local homeomorphism by Lemma 3.14. This implies that every point in $\mathcal{X}$ has a slice as an open neighborhood. Since open subsets of slices are again slices (similar to Lemma 2.8), we get that the topology on $\mathcal{X}$ has a base of slices $\mathcal{X}^{\mathrm{op}}$.

By Proposition 2.10, a groupoid is ample if and only if the compact slices form a base for the topology on $\mathcal{G}$. We get a similar result for groupoid correspondences $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ if $\mathcal{G}$ is an ample groupoid. Thus, in particular, this holds for ample correspondences.

Proposition 3.20. *Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids and $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ a groupoid correspondence. If $\mathcal{G}$ is ample, then the set of compact slices $\mathcal{X}^{\mathrm{a}}$ is an ample base for the topology on $\mathcal{X}$.*

Proof. The proof is inspired by and thus analogous to the proof of Lemma 2.12.

In Remark 3.19 we establish that the slices form a base for the topology on $\mathcal{X}$, hence we only need to prove that every slice is a union of compact slices. A slice $U \subset \mathcal{X}$ is homeomorphic via $s|_U$ to the open set $s|_U(U) \subset \mathcal{G}^0$, which is equal to a union of compact open subsets $A_i \subset \mathcal{G}^0$ for some $i \in I$ (by Lemma 2.11, using that $\mathcal{G}$ is ample). Thus, $U = (s|_U)^{-1}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} (s|_U)^{-1}(A_i)$ is a union of compact open subsets of the slice U, and hence they are compact slices.

Finally, this base is ample, as any $U \in \mathcal{X}^a$ is by definition compact and homeomorphic to a subset of the Hausdorff space $\mathcal{G}^0$ and hence Hausdorff. Additionally, any compact open subset of a slice is again a compact slice, since p and s stay injective on a subset of a slice.

Furthermore, we get a result similar to Proposition 2.13.

Lemma 3.21 (compare [AKM22, Lemma 7.7]). *Let $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ be a groupoid correspondence. Consider slices $V_1, V_2 \subseteq \mathcal{X}$, $W \subseteq \mathcal{G}$ and $Z \subseteq \mathcal{H}$. Then the following subsets are also slices:*

$$V_1 W := \{xg \mid x \in V_1,\ g \in W,\ s(x) = r(g)\} \subseteq \mathcal{X},$$
$$\langle V_1 \mid V_2 \rangle := \{\langle x_1 \mid x_2 \rangle \mid x_1 \in V_1,\ x_2 \in V_2,\ p(x_1) = p(x_2)\} \subseteq \mathcal{G},$$
$$Z V_1 := \{hx \mid h \in Z,\ x \in V_1,\ s(h) = r(x)\} \subseteq \mathcal{X}.$$

If $\mathcal{X}$ is an ample correspondence and $V_1, V_2 \subseteq \mathcal{X}$, $W \subseteq \mathcal{G}$ and $Z \subseteq \mathcal{H}$ are all compact slices, then $V_1 W$, $\langle V_1 \mid V_2 \rangle$ and $Z V_1$ are compact slices again.

Proof. For the first part, see [AKM22, Lemma 7.7]. For the second part, note that since $\mathcal{G}^0, \mathcal{H}^0, \mathcal{X}/\mathcal{G}$ are Hausdorff, any pullback over them is a closed subset of the product, for example, $\mathcal{H} \times_{s,\mathrm{H}^0,r} \mathcal{X} \subset \mathcal{H} \times \mathcal{X}$ is closed. Thus, the closed subsets $V_1 \times_{s,\mathrm{H}^0,r} W$, $V_1 \times_{p,\mathrm{X}/\mathrm{G},p} V_2$, $Z \times_{s,\mathrm{G}^0,r} V_2$ of the compact sets $V_1 \times W$, $V_1 \times V_2$, $Z \times V_2$ are again compact. Now, the defined sets are just images of these compact sets under the canonical continuous maps

$$\mathcal{H} \times_{s,\mathrm{H}^0,r} \mathcal{X} \to \mathcal{X}; \quad \mathcal{X} \times_{p,\mathrm{X}/\mathrm{G},p} \mathcal{X} \to \mathcal{G}; \quad \mathcal{X} \times_{s,\mathrm{G}^0,r} \mathcal{G} \mapsto \mathcal{X},$$

and hence are compact as well.

We get a similar result as in Remark 2.6 for ample correspondences.

Remark 3.22. Note that for an ample groupoid correspondence $\mathcal{X}$, the object space $\mathcal{G}^0$ is a locally compact, totally disconnected, Hausdorff space. Since we have a local homeomorphism $s: \mathcal{X} \to \mathcal{G}^0$, Lemma 2.5 implies that the correspondence $\mathcal{X}$ itself is locally compact, locally Hausdorff and totally disconnected. But $\mathcal{X}$ is not necessarily Hausdorff.

3.3. **On proper and tight correspondences.** We now consider particularly well-behaved ample correspondences, namely *proper* and *tight* ample correspondences. We are interested in proper ample correspondences $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ over cocompact groupoids, since then $\mathcal{G}$ is ample and the right $\mathcal{G}$-action is cocompact and given these properties, we can prove our main result of this subsection, Theorem 3.29. The theorem states that correspondences of this form are given by a disjoint union of open $\mathcal{G}$-subsets of $\mathcal{G}$, as a right $\mathcal{G}$-space. In simple terms, as a right $\mathcal{G}$-space $\mathcal{X}$ has a particularly manageable form and is rather easy to deal with (for example, in Subsection 6.4). So first, we define proper and tight correspondences.

Definition 3.23 (compare [AKM22, Definition 3.3])**.** A correspondence $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ is *proper* if the map $\tilde{r}: \mathcal{X}/\mathcal{G} \to \mathcal{H}^0$ induced by r is proper, that is, the preimage of every compact set is compact. It is *tight* if $\tilde{r}$ is a homeomorphism.

Note that any tight correspondence is proper, therefore the results of this section apply to them as well. For a groupoid correspondence $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ where $\mathcal{H}$ is cocompact, there is an equivalent definition of $\mathcal{X}$ being proper.

Lemma 3.24. *Let $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ be a groupoid correspondence where $\mathcal{H}$ is cocompact. Then $\mathcal{X}$ is proper if and only if the right $\mathcal{G}$-action on $\mathcal{X}$ is cocompact.*

Proof. For $\mathcal{X}$ proper, the map $\tilde{r}: \mathcal{X}/\mathcal{G} \to \mathcal{H}^0$ is proper. Since $\mathcal{H}^0$ is compact, the preimage of $\mathcal{H}^0$ under $\tilde{r}$ given by $\mathcal{X}/\mathcal{G}$ is compact, as well. For the other direction, let $K \subset \mathcal{H}^0$ be a compact subset. Since $\mathcal{H}^0$ is Hausdorff, this implies that K is closed. Now, as $\tilde{r}$ is continuous, $r^{-1}(K) \subset \mathcal{X}/\mathcal{G}$ is closed in the compact space $\mathcal{X}/\mathcal{G}$ and hence compact.

Now, we want to investigate correspondences $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ where $\mathcal{G}$ is ample and the right $\mathcal{G}$-action is cocompact, since these are the key properties we use in the following results. Note that any proper correspondence $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ where $\mathcal{H}$ is cocompact and $\mathcal{G}$ is ample, is of this form (by Lemma 3.24). For a correspondence $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ where $\mathcal{G}$ is ample and the right $\mathcal{G}$-action is cocompact, we know by definition that the orbit space $\mathcal{X}/\mathcal{G}$ is compact and Hausdorff. Furthermore, since $\mathcal{G}$ is ample, applying the following Lemma 3.25 to the orbit space projection $p: \mathcal{X} \to \mathcal{X}/\mathcal{G}$ shows that $\mathcal{X}/\mathcal{G}$ is totally disconnected. We may apply this lemma, since compactness implies local compactness, by Lemma 3.14 the projection is a surjective local homeomorphism, and by Proposition 3.20 the compact slices form a compact open base for the topology on $\mathcal{X}$.

Lemma 3.25. *Consider two topological spaces X, Y with Y locally compact and Hausdorff and a surjective local homeomorphism $f: X \to Y$. If there is a base of compact open subsets for the topology on X, then Y is totally disconnected.*

Proof. We want to prove that there is a base of compact open subsets for the topology on Y, since by Lemma 2.11, this implies that Y is totally disconnected. Consider an open subset $U \subset Y$ and a point $y \in U$. Since f is surjective, we find $x \in X$ with $f(x) = y$. Now, as f is a local homeomorphism and thus continuous, $f^{-1}(U) \subset X$ is an open neighborhood of x. Hence, we find a compact open subset $K \subset X$ with $x \in K \subset f^{-1}(U)$. Now, as f is a local homeomorphism, it is open and continuous, and thus $f(K) \subset Y$ is open and compact. Furthermore, $y \in f(K) \subset f\big(f^{-1}(U)\big) = U$. Hence, the image of the compact open base on X is a compact open base on Y.

So, by the above discussion for a correspondence $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ where $\mathcal{G}$ is ample and the right $\mathcal{G}$-action is cocompact, we know that $\mathcal{X}/\mathcal{G}$ is a compact, Hausdorff and totally disconnected space, that is, a so-called *Stone space*.

Lemma 3.26. *A compact, Hausdorff and totally disconnected topological space X is finitely ultraparacompact, that is, every open cover has a finite disjoint clopen refinement.*

Proof. Consider an open cover $X = \bigcup_{i \in I} U_i$ of X. By Lemma 2.11, there is a base $\mathcal{B}$ of clopen sets for the topology on X. Hence, we can write every U_i as a union of elements of the base and get $X = \bigcup_{j \in J} B_j$, where for all j, there is an i such that $B_j \subset U_i$ and $B_j \in \mathcal{B}$ clopen in X. Hence, this is a clopen refinement. Now, as X is compact we get $X = \bigcup_{j=1}^{n} B_j$. We define $V_j := B_j \setminus (\bigsqcup_{i=1}^{j-1} B_j)$, which is a finite intersection of clopen sets and hence clopen in X. Furthermore, it is a finite refinement of the open cover $\{U_i\}_{i \in I}$, and the V_j are disjoint by definition.

Now, we can use these results to construct a fundamental domain of a correspondence $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$, if $\mathcal{G}$ is ample and the right $\mathcal{G}$-action is cocompact.

Proposition 3.27. *Consider a correspondence $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ where $\mathcal{G}$ is ample and the right $\mathcal{G}$-action is cocompact. Then there are disjoint compact slices $U_1, \dots, U_n$ of $\mathcal{X}$ such that $\mathcal{X}/\mathcal{G} = \bigsqcup_{i=1}^{n} p(U_i)$ and $U := \bigsqcup_{i=1}^{n} U_i$ is a fundamental domain of $\mathcal{X}$, that is, the projection $p|_U: U \to \mathcal{X}/\mathcal{G}$ is a homeomorphism.*

Proof. There is an open cover of compact slices $\mathcal{X} = \bigcup_{i \in I} W_i$ of $\mathcal{X}$ by Proposition 3.20. As $p: \mathcal{X} \to \mathcal{X}/\mathcal{G}$ is a surjective local homeomorphism (and thus an open map), we get an open cover $\mathcal{X}/\mathcal{G} = \bigcup_{i \in I} p(W_i)$ of the compact, Hausdorff and totally disconnected space $\mathcal{X}/\mathcal{G}$. Then, by Lemma 3.26, we find a finite disjoint clopen refinement $\mathcal{X}/\mathcal{G} = \bigsqcup_{i=1}^{n} V_i$. As $V_i \subset \mathcal{X}/\mathcal{G}$ is a closed subset of a compact space, it is compact itself. Furthermore, since it is a refinement for each V_i, there is a $j \in I$ such that $V_i \subset p(W_j)$. Now, as W_j is a slice, $p|_{W_j}$ is a homeomorphism and hence $U_i := p|_{W_j}^{-1}(V_i) \cong V_i$ is a compact open subset of W_j, and hence a compact slice in $\mathcal{X}$ with $p(U_i) = V_i$. Furthermore, for $i \neq j$ we have

$$U_i \cap U_j \subset p^{-1}\big(p(U_i \cap U_j)\big) \subset p^{-1}\big(p(U_i) \cap p(U_j)\big) \subset p^{-1}(\varnothing) = \varnothing$$

and hence the U_i are disjoint compact slices of $\mathcal{X}$, so that for $U := \bigsqcup_{i=1}^{n} U_i$ the projection $p|_U: U \to \mathcal{X}/\mathcal{G}$ is a homeomorphism.

Remark 3.28. Note that the fundamental domain $U := \bigsqcup_{i=1}^{n} U_i \subset \mathcal{X}$ is a compact open subset such that $p|_U: U \to \mathcal{X}/\mathcal{G}$ is a homeomorphism, but it is not necessarily a slice of $\mathcal{X}$, since $s|_U$ might not be injective.

Finally, we get our main result. By Lemma 3.24, the theorem applies to proper correspondences $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ where $\mathcal{H}$ is cocompact and $\mathcal{G}$ is ample. Thus, it also applies to proper ample correspondences $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ where $\mathcal{H}$ and $\mathcal{G}$ are cocompact (as we use it in Subsection 6.4) and to tight correspondences $\mathcal{X}: \mathcal{G} \leftarrow \mathcal{G}$ where $\mathcal{G}$ is cocompact (as we use it in Section 11).

Theorem 3.29. *For a correspondence $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ where $\mathcal{G}$ is ample and the right $\mathcal{G}$-action is cocompact, there are compact open subsets $K_1, \dots, K_n \subset \mathcal{G}^0$ such that the correspondence $\mathcal{X}$ is given by*

$$\mathcal{X} \cong \bigsqcup_{i=1}^{n} r_{\mathrm{G}}^{-1}(K_i) \subset \bigsqcup_{i=1}^{n} \mathcal{G}$$

as a right $\mathcal{G}$-space.

Proof. By Proposition 3.27 we find disjoint compact slices $U_1, \dots, U_n$ of $\mathcal{X}$ such that $\mathcal{X}/\mathcal{G} = \bigsqcup_{i=1}^n p(U_i)$ and $U := \bigsqcup_{i_1}^n U_i$ is a fundamental domain of $\mathcal{X}$. Now, the correspondence $\mathcal{X}$ is given by

$$\mathcal{X} = U \times_{s,\mathrm{G}^0,r} \mathcal{G} = \left(\bigsqcup_{i=1}^n U_i\right) \times_{s,\mathrm{G}^0,r} \mathcal{G} = \bigsqcup_{i=1}^n (U_i \times_{s,\mathrm{G}^0,r} \mathcal{G})$$

as a right $\mathcal{G}$-space using Lemma 3.15 and the fact that disjoint union behaves well with fiber products. Furthermore, by Lemma 3.7 we have $U_i \times_{s,\mathrm{G}^0,r} \mathcal{G} \cong r_{\mathrm{G}}^{-1}\big(s_{\mathrm{X}}(U_i)\big)$ as right $\mathcal{G}$-spaces. Now, $K_i := s_{\mathrm{X}}(U_i) \subset \mathcal{G}^0$ is compact and open, since s is a local homeomorphism and U_i is compact and open, and we get

$$\mathcal{X} \cong \bigsqcup_{i=1}^n r_{\mathrm{G}}^{-1}(K_i)$$

as right $\mathcal{G}$-spaces.

4. The bicategory of ample groupoid correspondences

We now want to introduce the necessary information to make ample groupoids and ample groupoid correspondences into a bicategory $\mathsf{Gr_a}$. In [Mey22b] the bicategory GR of (étale) groupoids, (étale) groupoid correspondences, and continuous equivariant maps is introduced in detail. In [AKM22] the bicategory $\mathsf{Gr_{inj}} \subset \mathsf{GR}$ of (locally compact, étale) groupoids, (locally compact, étale) correspondences, and *injective* continuous equivariant maps is introduced. Furthermore, it is hinted that one could also define the bicategory Gr of (locally compact, étale) groupoids, (locally compact, étale) correspondences, and continuous equivariant maps (see [AKM22, Remark 6.2]). Thus, we have a chain of bicategories

$$\mathsf{Gr_{inj}} \subset \mathsf{Gr} \subset \mathsf{GR}.$$

Now, we are interested in the subbicategory

$$\mathsf{Gr_a} \subset \mathsf{Gr} \subset \mathsf{GR}$$

of ample groupoids, ample correspondences, and continuous equivariant maps. Note that compared to the bicategory Gr, we only restrict ourselves on the object level by only considering ample groupoids. As 1-arrows and 2-arrows, we just take all the 1-arrows and 2-arrows in Gr. In this section, we give a short recap on all the important data involved in the bicategory $\mathsf{Gr_a}$ of ample groupoids, ample groupoid correspondences, and equivariant continuous maps. For explicit proofs that this data indeed defines a bicategory, we refer to [AKM22], where it is proven in detail that $\mathsf{Gr_{inj}}$ is a bicategory, and it is hinted that Gr is a bicategory as well. Now, as $\mathsf{Gr_a} \subset \mathsf{Gr}$ is just restricting the objects, it is a bicategory as well. Finally, we also mention the important subbicategory of cocompact groupoids and proper/tight correspondences

$$\mathsf{Gr_{co,tight}} \subset \mathsf{Gr_{co,proper}} \subset \mathsf{Gr_a}.$$

We start by recalling all the relevant data for the bicategory $\mathsf{Gr_a}$. The *objects* are given by ample groupoids $\mathcal{G}$. The 1-*arrows* are ample groupoid correspondences $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ and we take continuous $\mathcal{H}$,$\mathcal{G}$-equivariant maps $\mathcal{X} \to \mathcal{Y}$ as 2-*arrows* $\mathcal{X} \Rightarrow \mathcal{Y}$. Note that 2-arrows are always local homeomorphisms.

Lemma 4.1 (compare [AKM22, Lemma 6.1])**.** *Let $\mathcal{X}, \mathcal{Y}\colon \mathcal{H} \leftleftarrows \mathcal{G}$ be groupoid correspondences. Any continuous $\mathcal{H},\mathcal{G}$-equivariant map $f\colon \mathcal{X} \to \mathcal{Y}$ is a local homeomorphism and injective on $U \in \mathcal{X}^a$.*

Proof. For the first part, see [AKM22, Lemma 6.1]. Furthermore, for $x, y \in U \in \mathcal{X}^a$ with $f(x) = f(y)$, we get $s(x) = s\big(f(x)\big) = s\big(f(y)\big) = s(y)$ and hence $x = y$. So f is injective on $U \in \mathcal{X}^a$.

Now, for two ample groupoids $\mathcal{G}, \mathcal{H}$, the ample correspondences $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ and continuous $\mathcal{H}$,$\mathcal{G}$-equivariant maps $\mathcal{X} \to \mathcal{Y}$ indeed form a category $\mathsf{Gr_a}(\mathcal{G}, \mathcal{H})$ where unit 2-arrows 1_{X} are given by the identity maps and the (vertical) product of 2-arrows is given by usual composition of maps, which is associative.

Next, we want to construct a functor $\circ_{\mathsf{G}}\colon \mathsf{Gr_a}(\mathcal{G}, \mathcal{H}) \times \mathsf{Gr_a}(\mathcal{K}, \mathcal{G}) \to \mathsf{Gr_a}(\mathcal{K}, \mathcal{H})$, defining a product on 1-arrows and a (horizontal) product on 2-arrows. First, we define a product on 1-arrows. For ample groupoids $\mathcal{H}$ and $\mathcal{G}$ and ample groupoid correspondences $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ and $\mathcal{Y}\colon \mathcal{G} \leftarrow \mathcal{K}$, the composition groupoid correspondence $\mathcal{X} \circ_{\mathsf{G}} \mathcal{Y}$ is defined by the following construction (compare [AKM22, Section 5]). Let

$$\mathcal{X} \times_{\mathsf{G}^0} \mathcal{Y} := \mathcal{X} \times_{s,\mathsf{G}^0,r} \mathcal{Y} := \big\{(x,y) \in \mathcal{X} \times \mathcal{Y} \mid s(x) = r(y)\big\}.$$

Let $\mathcal{G}$ act on $\mathcal{X} \times_{\mathsf{G}^0} \mathcal{Y}$ by the *diagonal action*

$$g \cdot (x,y) := (x \cdot g^{-1}, g \cdot y)$$

for $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $g \in \mathcal{G}$ with $s(g) = r(y) = s(x)$. Let $\mathcal{X} \circ_G \mathcal{Y}$ be the orbit space of this action, with $[x, y] \in \mathcal{X} \circ_G \mathcal{Y}$ denoting the orbit of $(x, y) \in \mathcal{X} \times_{G^0} \mathcal{Y}$. Note that since the right $\mathcal{G}$-action on $\mathcal{X}$ is basic, the diagonal action is basic as well. Hence, by Lemma 3.14, the orbit space projection $\mathcal{X} \times_{s,G^0,r} \mathcal{Y} \to \mathcal{X} \circ_G \mathcal{Y}$ is a surjective local homeomorphism.

The maps $r(x, y) := r(x)$ and $s(x, y) := s(y)$ on $\mathcal{X} \times_{G^0} \mathcal{Y}$ are invariant for this action, and thus induce maps $r: \mathcal{X} \circ_G \mathcal{Y} \to \mathcal{H}^0$ and $s: \mathcal{X} \circ_G \mathcal{Y} \to \mathcal{K}^0$. These are the anchor maps for the commuting actions of $\mathcal{H}$ on the left and $\mathcal{K}$ on the right, which we define by

$$h \cdot [x, y] := [h \cdot x, y], \qquad [x, y] \cdot k := [x, y \cdot k]$$

for all $h \in \mathcal{H}$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $k \in \mathcal{K}$ with $s(h) = r(x)$, $s(x) = r(y)$, and $s(y) = r(k)$. This is well-defined because $[h \cdot x \cdot g^{-1}, g \cdot y] = [h \cdot x, y]$ and $[x \cdot g^{-1}, g \cdot y \cdot k] = [x, y \cdot k]$ for $g \in \mathcal{G}$ with $s(g) = s(x) = r(y)$.

This construction indeed gives us an ample correspondence.

Proposition 4.2 (compare [AKM22, Proposition 5.7]). *The actions of $\mathcal{H}$ and $\mathcal{K}$ on $\mathcal{X} \circ_G \mathcal{Y}$ are well-defined and turn this into an ample groupoid correspondence $\mathcal{H} \leftarrow \mathcal{K}$. If both correspondences $\mathcal{X}$ and $\mathcal{Y}$ are proper or tight, then so is $\mathcal{X} \circ_G \mathcal{Y}$.*

Proof. See [AKM22, Proposition 5.7].

Furthermore, we want to find an ample base for the topology on these spaces. We start with a technical Lemma 4.3. It immediately implies Proposition 4.4, which gives us an ample base. Note that we formulate this technical Lemma 4.3 in a more general form than we need here so that we can use it in a slightly different situation in Section 10.

Lemma 4.3. *Consider an ample correspondence $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$, a left $\mathcal{G}$-space $\mathcal{Y}$ with an ample base $\mathcal{B}$ for its topology and the orbit space projection $\pi: \mathcal{X} \times_{s,G^0,r} \mathcal{Y} \to \mathcal{X} \circ_G \mathcal{Y}$ of the diagonal action as defined above. For $U \in \mathcal{X}^a$ and $V \in \mathcal{B}$ such that $s(U) \supset r(V)$, we denote $UV := \pi(U \times_{s,G^0,r} V)$ and let $\mathcal{B}_{X \circ_G Y}$ denote the set of all these UV. Then $\mathcal{B}_{X \circ_G Y}$ is an ample base for the topology on $\mathcal{X} \circ_G \mathcal{Y}$.*

Proof. First, note that since the right $\mathcal{G}$-action on $\mathcal{X}$ is basic, the diagonal action is basic as well, and hence by Lemma 3.14, the orbit space projection is a surjective local homeomorphism. Now, $U \times_{s,G^0,r} V$ is open as U and V are open, and hence UV is open as well (since local homeomorphisms are open maps). Furthermore, since $\mathcal{G}^0$ is Hausdorff, any pullback over $\mathcal{G}^0$ is a closed subset of the product. Hence, $\mathcal{X} \times_{s,G^0,r} \mathcal{Y} \subset \mathcal{X} \times \mathcal{Y}$ is closed. Now, as $U \subset \mathcal{X}, V \subset \mathcal{Y}$ are compact, also $U \times V \subset \mathcal{X} \times \mathcal{Y}$ is compact, and as $\mathcal{X} \times_{s,G^0,r} \mathcal{Y}$ is closed the intersection $U \times_{s,G^0,r} V$ is compact as well. Thus, the image of $U \times_{s,G^0,r} V$ under the continuous quotient map given by UV is compact.

Next, we prove that $\pi_{U,V} := \pi|_{U \times_{s,G^0,r} V}: U \times_{s,G^0,r} V \to UV$ is a homeomorphism. Using that $p|_U$ is injective it is easy to check that $\pi_{U,V}$ is injective as well, and by definition $\pi_{U,V}$ is surjective. Hence, it is a bijective local homeomorphism, that is, a homeomorphism. Thus, $UV \cong U \times_{s,G^0,r} V$ is Hausdorff, as $U \times_{s,G^0,r} V \subset U \times V$ is a subset of the product of the Hausdorff sets U, V. Thus, the UV are indeed compact Hausdorff open subsets of $\mathcal{X} \circ_G \mathcal{Y}$.

To see that they form a base, we start with an open subset $W \subset \mathcal{X} \circ_G \mathcal{Y}$ and a point $[x, y] \in W$. Now, since the preimage under the quotient map is open in $\mathcal{X} \times_{s,G^0,r} \mathcal{Y}$, we find an open subset $\tilde{W} \subset \mathcal{X} \times \mathcal{Y}$ such that the preimage is given by $\tilde{W} \cap \mathcal{X} \times_{s,G^0,r} \mathcal{Y}$. Hence, $(x, y) \in \tilde{W}$ and thus we find a compact slice $U \in \mathcal{X}^a$ and a $V \in \mathcal{B}$ with $(x, y) \in U \times V \subset \tilde{W}$. Finally, we get $[x, y] \in UV \subset W$. Now, if we replace V with some $V \in \mathcal{B}$ that is a subset of $r_Y^{-1}(s(U)) \cap V$ containing y, we additionally get $s(U) \supset r(V)$.

Finally, we want to show that this base $\mathcal{B}_{X\circ_G Y}$ is stable under taking compact open subsets. For $UV \in \mathcal{B}_{X\circ_G Y}$ we have seen that $UV \cong U \times_{s,G^0,r} V$ via $\pi_{U,V}$. Now, the maps

$$U \times_{s,G^0,r} V \to V, \quad (u,v) \mapsto v, \quad \left(s|_U^{-1}(r(v)), v\right) \leftarrow v,$$

define a homeomorphism, hence $UV \cong V$ and thus any compact open subset of UV is given by UW for a compact open subset $W \subset V$. As $\mathcal{B}$ is stable under taking compact open subsets, we get $W \in \mathcal{B}$ and hence $UW \in \mathcal{B}_{X\circ_G Y}$.

Proposition 4.4. *Consider two ample groupoid correspondences $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ and $\mathcal{Y}: \mathcal{G} \leftarrow \mathcal{K}$. Then for $U \in \mathcal{X}^a, V \in \mathcal{Y}^a$ with $s(U) \supset r(V)$ the set of all*

$$UV := \left\{[x,y] \mid (x,y) \in U \times_{s,G^0,r} V\right\} \subset \mathcal{X} \circ_G \mathcal{Y}$$

is an ample base for the topology of the composition groupoid correspondence $\mathcal{X} \circ_G \mathcal{Y}$. Additionally, for $(x,y) \in X \times_{s,G^0,r} Y$ we have $[x,y] \in UV$ if and only if there is a unique $g \in \mathcal{G}$ with $s(g) = r(y) = s(x)$ such that $xg^{-1} \in U$ and $gy \in V$.

Proof. The first part is a corollary of Lemma 4.3, as the compact slices of $\mathcal{Y}$ form an ample base for the topology on $\mathcal{Y}$. The second statement is proven in [AKM22, Lemma 7.14].

We can now define the (horizontal) product of 2-arrows. For ample groupoid correspondences $\mathcal{X}_1, \mathcal{X}_2: \mathcal{H} \leftleftarrows \mathcal{G}$ and $\mathcal{Y}_1, \mathcal{Y}_2: \mathcal{G} \leftleftarrows \mathcal{K}$ and 2-arrows $f_1: \mathcal{X}_1 \Rightarrow \mathcal{X}_2$ and $f_2: \mathcal{Y}_1 \Rightarrow \mathcal{Y}_2$, we define their horizontal product as

$$f_1 \circ_G f_2: \mathcal{X}_1 \circ_G \mathcal{Y}_1 \Rightarrow \mathcal{X}_2 \circ_G \mathcal{Y}_2, \qquad [x,y] \mapsto \left[f_1(x), f_2(y)\right],$$

which is again a 2-arrow. Furthermore, it is easy to check that $1_X \circ_G 1_Y = 1_{X\circ_G Y}$, and that the horizontal product commutes with the vertical product. Hence, $\circ_G$ is indeed a functor.

For each ample groupoid $\mathcal{G}$, we define the *unit 1-arrow* $\mathcal{G}: \mathcal{G} \leftarrow \mathcal{G}$ as the trivial correspondence, as discussed in Example 3.18.

The following Lemma 4.5 describes the *uniters* and *associators*, which are invertible natural 2-arrows.

Lemma 4.5 (compare [AKM22, Lemma 6.3, 6.4])**.** *Let $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ and $\mathcal{X}_i: \mathcal{G}_i \leftarrow \mathcal{G}_{i+1}$ for $1 \le i \le 3$ be ample groupoid correspondences. The* uniters *are given by the maps*

$$\mathrm{l}_X: \mathcal{H} \circ_H \mathcal{X} \to \mathcal{X}, \qquad [h,x] \mapsto h \cdot x,$$
$$\mathrm{r}_X: \mathcal{X} \circ_G \mathcal{G} \to \mathcal{X}, \qquad [x,g] \mapsto x \cdot g,$$

which are continuous $\mathcal{H}, \mathcal{G}$-equivariant homeomorphisms, which are natural with respect to continuous $\mathcal{H}, \mathcal{G}$-equivariant maps $\mathcal{X} \to \mathcal{X}'$.

The associator *is given by the map*

$$\mathrm{assoc}: \mathcal{X}_1 \circ_{G_2} (\mathcal{X}_2 \circ_{G_3} \mathcal{X}_3) \to (\mathcal{X}_1 \circ_{G_2} \mathcal{X}_2) \circ_{G_3} \mathcal{X}_3, \qquad \left[x_1, [x_2, x_3]\right] \mapsto \left[[x_1, x_2], x_3\right],$$

which is a continuous $\mathcal{G}_1, \mathcal{G}_4$-equivariant homeomorphism, which is natural with respect to continuous $\mathcal{G}_i, \mathcal{G}_{i+1}$-equivariant maps $\alpha_i: \mathcal{X}_i \to \mathcal{X}_i'$ for $1 \le i \le 3$.

Proof. See [AKM22, Lemma 6.3, 6.4].

Finally, we have completed describing all the data involved in defining the bicategory Gr_a. In [AKM22, Proposition 6.5], it is discussed that Gr_inj is indeed a bicategory, that is, that the triangle and pentagon diagrams commute. Thus, also Gr_a is indeed a bicategory as the triangle and pentagon diagrams are the same.

Additionally, the ample groupoids and tight/proper ample correspondences form subbicategories $\mathsf{Gr}_\mathsf{tight} \subset \mathsf{Gr}_\mathsf{proper} \subset \mathsf{Gr}_\mathsf{a}$, respectively (using Proposition 4.2). If we restrict ourselves even further to cocompact ample groupoids and tight/proper ample correspondences, we get the subbicategories

$$\mathsf{Gr}_{\mathsf{co,tight}} \subset \mathsf{Gr}_{\mathsf{co,proper}} \subset \mathsf{Gr}_{\mathsf{a}},$$

which we use in Section 10 and Section 11.

5. The bicategory of smooth bimodules

Before we define the Steinberg algebra of an ample groupoid, let us introduce the bicategory in which it lives. This section briefly introduces the bicategory Rings of rings with local units, smooth bimodules, and bimodule homomorphisms. Note that a ring is not necessarily unital nor commutative. We want to consider the full subbicategory $\mathsf{Rings_u} \subset \mathsf{Rings}$ of unital rings and bimodules and the subbicategory $\mathsf{Rings_{fgp}} \subset \mathsf{Rings_u}$ of unital rings and finitely generated and projective bimodules. Hence, we have three relevant bicategories

$$\mathsf{Rings_{fgp}} \subset \mathsf{Rings_u} \subset \mathsf{Rings}$$

that we are interested in. For a more detailed introduction, we refer to [Mey22a], where the bicategory $\mathsf{Rings_u}$ of unital rings and bimodules is introduced and discussed in detail. All the relevant bicategories are actually subbicategories of the bicategory RNG of self-induced rings, smooth bimodules, and bimodule homomorphisms briefly introduced in [Mey22a, Exercise 4.6.24].

Since the Steinberg algebra of an ample groupoid is not necessarily a unital ring (see Proposition 6.12), we have to allow non-unital rings as well. Now, non-unital rings and arbitrary bimodules can get quite difficult to deal with and might not even form a bicategory. So, we want to restrict ourselves to a more manageable class of rings, as well as bimodules.

Definition 5.1. A ring S is called *self-induced*, if the canonical multiplication map $S \otimes_S S \to S, a \otimes b \mapsto ab$, is an isomorphism.

Definition 5.2. Let S be a ring and M a left S-module. We call M *smooth* if the canonical multiplication map $S \otimes_S M \to M$ is an isomorphism.

Similarly, we define *smooth* right and bimodules.

According to [Mey22a, Exercise 4.6.24] the self-induced rings, smooth bimodules, and bimodule homomorphisms form a bicategory RNG. Now, our Steinberg algebras are not just self-induced, but have an even stronger property, namely they have *local units*.

Definition 5.3. A ring S has *local units* $E \subset S$, if every $e \in E$ is an idempotent[3] and for any finite set $\{s_1, \dots, s_n\} \subset S$, we can find $e \in E$ such that $s_i e = s_i = e s_i$ for all $i = 1, \dots, n$.

The rings with local units form a subclass of the self-induced rings.

Proposition 5.4. *If a ring S has local units E, then it is self-induced.*

Proof. Given $s \in S$ we take a local unit $e \in E$ of s and then $e \otimes s \mapsto es = s$. Hence, the multiplication map is surjective. For the injectivity start with $\sum_{i=1}^n a_i b_i = \sum_{j=1}^m x_j y_j$ and take a local unit $e \in E$ of $a_1, \dots, a_n, x_1 \dots, x_m$. Then

$$\begin{aligned}\sum_{i=1}^n a_i \otimes b_i &= \sum_{i=1}^n e a_i \otimes b_i = \sum_{i=1}^n e \otimes a_i b_i = e \otimes \sum_{i=1}^n a_i b_i \\ &= e \otimes \sum_{j=1}^m x_j y_j = \sum_{j=1}^m e \otimes x_j y_j = \sum_{j=1}^m e x_j \otimes y_j = \sum_{j=1}^m x_j \otimes y_j\end{aligned}$$

and hence the map is indeed an isomorphism.

Furthermore, smooth bimodules over rings with local units are well-behaved. Note that one can formulate and prove an analogous statement to the following Proposition 5.5 for right S-modules.

[3] That is, $e \in S$ such that $e^2 = e$.

Proposition 5.5. *Consider a ring S with local units E and a left S-module M. Then, M is a smooth S-module if and only if for all $m \in M$ there is an $s \in S$ with $sm = m$.*

Proof. If M is smooth the map $\text{mult}\colon S \otimes_S M \to M$ is surjective and hence for $m \in M$ we find $s_i \in S$, $m_i \in M$ for $i = 1, \dots, n$ such that $m = \text{mult}(\sum_{i=1}^n s_i \otimes m_i) = \sum_{i=1}^n s_i m_i$. Now, if we take a local unit s for $s_1, \dots, s_n$, we have

$$\begin{aligned} sm &= \text{mult}(s \otimes m) = \text{mult}\left(s \otimes \sum_{i=1}^n s_i m_i\right) \\ &= \text{mult}\left(\sum_{i=1}^n s s_i \otimes m_i\right) = \text{mult}\left(\sum_{i=1}^n s_i \otimes m_i\right) = \sum_{i=1}^n s_i m_i = m. \end{aligned}$$

In the other direction, the surjectivity of $S \otimes_S M \to M$ is immediate as for $m \in M$ we take as a preimage $s \otimes m$. For the injectivity start with $a_i, x_j \in S$, $b_i, y_j \in M$ with $\sum_{i=1}^n a_i b_i = \sum_{j=1}^m x_j y_j$ and take a local unit $e \in E$ of $a_1, \dots, a_n, x_1 \dots, x_m$. Then by the same argument as in the proof of Proposition 5.4 we get $\sum_{i=1}^n a_i \otimes b_i = \sum_{j=1}^m x_j \otimes y_j$, and hence the map is indeed an isomorphism.

Finally, we can define the bicategory Rings of rings with local units, smooth bimodules, and bimodule homomorphisms as the full subbicategory $\mathsf{Rings} \subset \mathsf{RNG}$, where we restrict only the objects to rings with local units and take all the 1-arrows and 2-arrows. We quickly recall the relevant data from [Mey22a].

Theorem 5.6. *The following data defines the bicategory* Rings *of smooth bimodules:*

- *rings with local units S as objects;*
- *smooth S,T-bimodules $M\colon S \leftarrow T$ as 1-arrows, with the tensor product as the product and the canonical S,S-bimodule S with multiplication as the* unit arrow *for S;*
- *S,T-bimodule homomorphisms $f\colon M \to N$ as 2-arrow, with composition as (vertical) product and tensor product as (horizontal) product;*
- *the* associators *are given by the bimodule isomorphisms*
$$\begin{gathered}(M_1 \otimes_S M_2) \otimes_T M_3 \to M_1 \otimes_S (M_2 \otimes_T M_3), \\ (m_1 \otimes m_2) \otimes m_3 \mapsto m_1 \otimes (m_2 \otimes m_3); \text{ and}\end{gathered}$$
- *the* uniters *are given by the canonical multiplication maps*
$$\begin{aligned} S \otimes_S M &\to M, & s \otimes m &\mapsto sm, \\ M \otimes_T T &\to M, & m \otimes t &\mapsto mt, \end{aligned}$$
which are isomorphisms, since M is a smooth bimodule.

Proof. In [Mey22a, Exercise 4.6.24] the bicategory RNG of self-induced rings and smooth bimodules is introduced. Now, by Lemma 5.4 every ring with local units is self-induced and hence our bicategory Rings is just the full subbicategory of RNG where we restrict ourselves to rings with local units.

Now, in [Mey22a] the full subbicategory $\mathsf{Rings}_\mathsf{u} \subset \mathsf{Rings}$ of unital rings, bimodules, and bimodule homomorphism is introduced in detail. Furthermore, the concept of bicategorical limits is introduced and examined, which we use in Section 9 and Section 11. For this, we need to restrict ourselves even further to particularly well-behaved bimodules over unital rings, namely *finitely generated and projective* bimodules.

Definition 5.7. Let S be a unital ring and M a right S-module. Then we say

- M is *finitely generated*, if there are $k \in \mathbb{N}$ and a surjective right S-module homomorphism $S^k \to M$; and

- P is *projective*, if for every surjective right S-module homomorphism $f\colon N \to M$ and every right S-module homomorphism $g\colon P \to M$, there exists a right S-module homomorphism $h\colon P \to N$ such that $f \circ h = g$.

For unital rings S, T, we call an S,T-bimodule M *finitely generated and projective*, or in short *fgp*, if the right T-module M is finitely generated and projective.

Example 5.8. For a unital ring S, the right S-module S is fgp. It is finitely generated, as the identity map is surjective. To see that it is projective, we take $f\colon N \to M$ surjective and $g\colon S^k \to M$ two right S-module homomorphisms. Now, there exists $n \in N$ such that $f(n) = g(1)$ and hence we can define the right S-module homomorphism $h\colon S \to N; s \mapsto ns$. Finally, we have

$$(f \circ h)(s) = f\big(h(1)\big)s = f(n)s = g(1)s = g(s)$$

for all $s \in S$ and hence $f \circ h = g$.

Remark 5.9. We want to consider the bicategory $\mathsf{Rings_{fgp}}$ of unital rings, fgp bimodules, and bimodule homomorphisms. For $\mathsf{Rings_{fgp}}$ to actually be a well-defined subbicategory $\mathsf{Rings_{fgp}} \subset \mathsf{Rings_u}$, we need that the tensor product of fgp bimodules is again an fgp bimodule. This can be seen using the characterization from [Mey22a, Theorem 4.1.13] that states that for unital rings S, T, an S,T-bimodule M is fgp if and only if the functor $M \otimes_T -$ preserves limits.

Next, we want to establish some properties and technical results on fgp modules. We start with the most important one, which is the main reason why we want to consider fgp modules.

Theorem 5.10. *For any unital ring S and right S-modules M, N the dual module $M^{\vee} := \mathrm{Hom}_{-,S}(M, S)$ is a left S-module via $(s \cdot f)(x) := s \cdot f(x)$ and there is a natural abelian group homomorphism*

$$N \otimes_S M^{\vee} \to \mathrm{Hom}_{-,S}(M, N), \qquad n \otimes f \mapsto \big[m \mapsto n \cdot f(m)\big].$$

If M is fgp, this natural homomorphism is an isomorphism of abelian groups.

Proof. See [Mac63, Proposition 4.2 on p.147].

Note that this basically tells us that for an fgp module M, the abelian group $\mathrm{Hom}_{-,S}(M, N)$ is not that big and stays manageable. In Section 9 we work with these abelian groups and make use of this statement. Finally, we assemble all the technical results on this matter that we need.

Lemma 5.11. *For a unital ring S and fgp right S-modules $M_1, \ldots, M_n$, their direct sum $\bigoplus_{i=1}^n M_i$ is an fgp right S-module.*

Proof. Fix the canonical homomorphisms $\iota_j\colon M_j \to \bigoplus_{i=1}^n M_i$. Consider $f\colon N \to M$ surjective and $g\colon \bigoplus_{i=1}^n M_i \to M$ two right S-module homomorphisms. Then for each $i = 1, \ldots, n$ we find $h_i\colon M_i \to N$ such that $f \circ h_i = g \circ \iota_i$. Now, with the universal property of the direct sum, we find $h\colon \bigoplus_{i=1}^n M_i \to N$ such that $h \circ \iota_i = h_i$ for all $i = 1, \ldots, n$. Thus, we get

$$(f \circ h) \circ \iota_i = f \circ h_i = g \circ \iota_i$$

and hence by the uniqueness part of the universal property of the direct sum, it follows that $f \circ h = g$.

Lemma 5.12. *Consider two right S-modules P and Q such that Q is fgp and a right S-module homomorphism $r\colon Q \to P$ that admits a splitting map, that is, a map $s\colon P \to Q$ such that $r \circ s = \mathrm{id}_P$. Then P is fgp.*

Proof. Since $r \circ s = \mathrm{id}_P$ it follows that $r: Q \to P$ is surjective. Now, as Q is finitely generated there is a surjection $S^n \to Q$ and hence we get a surjection $S^n \to Q \to P$, and P is finitely generated as well.

Consider $f: N \to M$ surjective and $g: P \to M$ two right S-module homomorphisms. Then, as Q is projective, there is a map $h': Q \to N$ such that $f \circ h\prime = g \circ r$. Hence, for $h := h' \circ s: P \to Q \to N$ we have

$$f \circ h = f \circ h' \circ s = g \circ r \circ s = g.$$

Thus, P is projective as well, and hence P is fgp.

Lemma 5.13. *Let $e \in S$ be an idempotent. Then the right S-submodule $eS \subset S$ is fgp.*

Proof. By Example 5.8 the right S-module S is fgp. Now, we have a surjective right S-module homomorphism

$$m: S \to eS, \qquad r \mapsto er$$

and the canonical inclusion $\iota: eS \to S$ is a splitting for m, since $(m \circ \iota)(er) = e^2 r = er$ for all $r \in S$. Hence, by Lemma 5.12 the right S-module eS is fgp.

Lemma 5.14. *Let $e \in S$ be an idempotent and M a right S-module. Then the multiplication map*

$$M \otimes_S Se \to Me, \qquad m \otimes re \mapsto mre,$$

is an isomorphism of abelian groups.

Proof. The map defined above is induced by the S-balanced map

$$M \times Se \to Me, \qquad (m, re) \mapsto mre,$$

which is well-defined as for $r, r' \in S$ with $re = r'e$ we get $mre = mr'e$. Hence, it induces a well-defined map on the tensor product. Now, an inverse map is given by

$$Me \to M \otimes_S Se, \qquad me \mapsto m \otimes e.$$

This map is well-defined as $e = ee \in Se$ and for $me = m'e$ we get

$$m \otimes e = m \otimes ee = me \otimes e = m'e \otimes e = m' \otimes ee = m' \otimes e.$$

The defined maps are inverse to one another, since $mr \otimes e = m \otimes re$.

Lemma 5.15. *Let $e \in S$ be an idempotent. Then the map*

$$\mathrm{Hom}_{-,S}(eS, S) \to Se, \qquad f \mapsto f(e)$$

is an isomorphism of left S-modules.

Proof. Note first that $e = ee \in eS$, hence $f(e) \in S$ is defined, and

$$f(e) = f(ee) = f(e)e \in Se.$$

The map is injective, since any right S-module homomorphism $f: eS \to S$ is uniquely defined by $f(e)$, as for any $r \in S$ we have $f(er) = f(e)r$. Furthermore, it is surjective as any $re \in Se$ defines a right S-module homomorphism $eS \to S, er' \mapsto rer'$. The map is obviously left S-linear, where the left S-module structure on $\mathrm{Hom}_{-,S}(eS, S)$ is given by $(r \cdot f)(x) := r \cdot f(x)$.

Lemma 5.16. *Let $M_1, \dots, M_n$ be left S-modules and $M := \bigoplus_{i=1}^n M_i$ the direct sum with the canonical embeddings $\iota_i: M_i \to M$ and N a right S-module. Then the map*

$$N \otimes_S M \to \bigoplus_{i=1}^n N \otimes_S M_i, \qquad y \otimes (x_i)_{i=1}^n \mapsto (y \otimes x_i)_{i=1}^n$$

is an abelian group isomorphism.

Proof. It is easy to check that the map $\big(y,(x_i)_{i=1}^n\big) \mapsto (y \otimes x_i)_{i=1}^n$ is S-balanced and hence induces a well-defined map as such. For the other direction, it is easy to check, that the maps $N \times M_i \to N \otimes M$, $(y, x_i) \mapsto y \otimes \iota_i(x_i)$, are S-balanced and thus define group homomorphisms $N \otimes_S M_i \to N \otimes_S M$ that assemble into a group homomorphism

$$\bigoplus_{i=1}^n N \otimes_S M_i \to N \otimes_S M, \qquad (y \otimes x_i)_{i=1}^n \mapsto y \otimes (x_i)_{i=1}^n$$

that defines an inverse to our map.

Lemma 5.17. *Let $M_1, \dots, M_n, N$ be right S-modules and $M := \bigoplus_{i=1}^n M_i$ the direct sum with the canonical embeddings $\iota_i \colon M_i \to M$, then the map*

$$\mathrm{Hom}_{-,S}(M,N) \to \bigoplus_{i=1}^n \mathrm{Hom}_{-,S}(M_i,N), \qquad f \mapsto (f \circ \iota_i)_{i=1}^n$$

is a left S-module isomorphism.

Proof. The map is by definition left S-linear and well-defined, and an inverse map is given by

$$\bigoplus_{i=1}^n \mathrm{Hom}_{-,S}(M_i,N) \to \mathrm{Hom}_{-,S}(M,N), \qquad (f_i)_{i=1}^n \mapsto \tilde{f}$$

with $\tilde{f}\big((m_i)_{i=1}^n\big) := \sum_{i=1}^n f_i(m_i)$ for $(m_i)_{i=1}^n \in M$.

Finally, we get an equivalent characterization of a right S-module being fgp:

Proposition 5.18. *Consider a unital ring S and a right S-module M. Then M is fgp if and only if there are $k \in \mathbb{N}$ and an idempotent matrix $e \in \mathrm{Mat}_{k\times k}(S)$ (that is, $e^2 = e$) such that $M \cong eS^k$, that is M is isomorphic as a right S-module to the image of e.*

Proof. Take an fgp S-module M. Since it is finitely generated, we find $k \in \mathbb{N}$ and a surjective right S-module homomorphism $f : S^k \to M$. As M is projective, there exists a right S-module homomorphism $h \colon M \to S^k$ such that $f \circ h = \mathrm{id}$. Now, the right S-module homomorphism $h \circ f : S^k \to S^k$ is given by a matrix $e \in \mathrm{Mat}_{k\times k}(S)$, which is an idempotent matrix, as

$$e^2 = h \circ f \circ h \circ f = h \circ \mathrm{id} \circ f = e.$$

Finally, since $f \circ h = \mathrm{id}$, h is injective and hence M is isomorphic to the image of h. Since f is surjective the image of $e = h \circ f$ is given by the image of h and hence $M \cong eS^k$.

For the other direction, we note that the canonical embedding $eS^k \to S^k$ is a splitting map for the right S-module homomorphism $e \colon S^k \to eS^k$, since e is an idempotent matrix. Now, by Example 5.8 and Lemma 5.11 S^k is fgp, and hence by Lemma 5.12 also $M \cong eS^k$ is fgp.

6. Steinberg algebras and bimodules

In this section, we define the *Steinberg algebra* $A_R(\mathcal{G})$ of an ample groupoid $\mathcal{G}$, as Steinberg did in [Ste09, Chapter 4]. In particular, $A_R(\mathcal{G})$ is a (not necessarily commutative nor unital) ring. But we prove that it has local units, so it is an object in the bicategory of smooth bimodules Rings. We generalize the construction of the Steinberg algebra of an ample groupoid to ample groupoid correspondences $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ and make $A_R(\mathcal{X})$ into a smooth $A_R(\mathcal{H})$,$A_R(\mathcal{G})$-bimodule. Hence, $A_R(\mathcal{G})$ is an object, and $A_R(\mathcal{X})$ is a 1-arrow in the bicategory of smooth bimodules Rings. In the following Section 7, we also construct an $A_R(\mathcal{H})$,$A_R(\mathcal{G})$-bimodule homomorphism $A(f): A_R(\mathcal{X}) \to A_R(\mathcal{Y})$ for every 2-arrow $f: \mathcal{X} \Rightarrow \mathcal{Y}$ in $\mathsf{Gr_a}$ and embed these constructions into a strictly unital homomorphism $A: \mathsf{Gr_a} \to \mathsf{Rings}$ of bicategories.

We start in the greatest generality by defining the *Steinberg module* $A_R(X)$ of an arbitrary topological space X. Then, for $X = \mathcal{G}$ an ample groupoid, we can define a multiplicative structure on $A_R(\mathcal{G})$ to define the *Steinberg algebra* of $\mathcal{G}$. After that, for $X = \mathcal{X}$ an ample correspondence, we can define a left and right action on $A_R(\mathcal{X})$ to define the *Steinberg bimodule* of $\mathcal{X}$.

From now on until the end of this thesis, we fix a commutative, unital ring R with the discrete topology.

6.1. Steinberg modules of topological spaces. Fix a topological space X. Let R be the fixed commutative, unital ring with the discrete topology. The set of all maps $R^X := \{\xi: X \to R\}$ is an R-module by pointwise addition and scalar multiplication. For a subset $F \subset R^X$, we write $\langle F \rangle_R \subset R^X$ for *the smallest R-submodule of R^X generated by F*. Furthermore, for a subset $A \subset X$, we define 1_A, the *characteristic map of A*, via

$$1_A: X \to R, \qquad x \mapsto \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

This map is continuous if and only if $A \subset X$ is clopen. It is easy to check that a map $\xi \in R^X$ is continuous if and only if it is locally constant. For a subset $U \subset X$ and a map $\xi: U \to R$, we can define its *extension by zero* map $\tilde{\xi} \in R^X$ by putting $\tilde{\xi}|_U := \xi$ and $\tilde{\xi}|_{X \setminus U} := 0$. Hence, for any map $\xi \in \mathrm{C_c}(U, R)$ (that is, $\xi: U \to R$ is continuous with compact support $\operatorname{supp}(\xi) := \xi^{-1}(R \setminus \{0\}) \subset X$) we get $\tilde{\xi} \in R^X$.

Proposition and Definition 6.1. *Consider a topological space X. The* Steinberg module of X *is the R-submodule $A_R(X)$ of R^X described in the following equivalent ways:*

(1) $\langle \tilde{\xi} \mid \xi \in \mathrm{C_c}(U, R)$ *for $U \subset X$ a Hausdorff open subset*$\rangle_R$;
(2) $\{\xi \mid \operatorname{supp}(\xi)$ *compact Hausdorff open and $\xi|_{\operatorname{supp}(\xi)}$ continuous*$\}$;
(3) $\langle 1_U \mid U \subset X$ *a compact Hausdorff open subset*$\rangle_R$.

Proof. We prove that (1) ⊂ (2) ⊂ (3) ⊂ (1).

For the first inclusion, take a Hausdorff open subset $U \subset X$ and a map $\xi \in \mathrm{C_c}(U, R)$. Since subsets of Hausdorff spaces are Hausdorff, $\operatorname{supp}(\tilde{\xi}) = \operatorname{supp}(\xi)$ is Hausdorff. Furthermore, ξ is continuous and R discrete, hence $\operatorname{supp}(\xi)$ is open in U and as U is open in X, it is also open in X. Thus, $\operatorname{supp}(\tilde{\xi})$ is a compact Hausdorff open subset. As $\tilde{\xi}$ is continuous on U, it is also continuous on the open subset $\operatorname{supp}(\tilde{\xi}) \subset U$.

For the second inclusion, we take a map $\xi: X \to R$ with $U := \operatorname{supp}(\xi) \subset X$ compact Hausdorff open, such that $\xi|_U$ is continuous. We get that $\xi(U) \subset R$ is compact and hence (since R is discrete) $\xi(U) = \{r_1, \dots, r_n\}$ is finite. Now, the subset $U_i := \xi^{-1}(r_i) \subset \operatorname{supp}(\xi)$ is closed in a compact space and thus is a compact space itself. Furthermore, U_i is open in X (since $\xi|_{\operatorname{supp}(\xi)}$ is continuous and $\operatorname{supp}(\xi)$

is open). Additionally, as a subset of the Hausdorff space $\operatorname{supp}(\xi)$, it is Hausdorff itself. Furthermore, we have $U = \xi^{-1}(0) \sqcup U_1 \sqcup \cdots \sqcup U_n$ and hence get $\xi = \sum_{i=1}^{n} r_i \mathbb{1}_{U_i}$.

The third inclusion is immediate.

Note that the elements of the Steinberg module are not necessarily continuous. For example, the characteristic map $\mathbb{1}_U$ of a compact Hausdorff open subset $U \subset X$ does not need to be continuous, as U is not necessarily closed. If we assume X to be a Hausdorff space, then U is indeed closed, and hence $\mathbb{1}_U$ is actually continuous. In this case, we get $A_R(X) = \mathrm{C_c}(X, R)$, that is, the Steinberg module is exactly the space of all continuous maps with compact support.

Now, we want to establish some general properties of the Steinberg module. First, taking the Steinberg module commutes with coproducts.

Lemma 6.2. *For a disjoint union $X = \bigsqcup_{i \in I} X_i$, the Steinberg module over R*

$$A_R\left(\bigsqcup_{i \in I} X_i\right) \cong \bigoplus_{i \in I} A_R(X_i)$$

is given by the direct sum of R-modules.

Proof. We use Definition (2) of the Steinberg module. We start by defining the extension by zero maps

$$\iota_i \colon A_R(X_i) \to A_R(X), \qquad \xi \mapsto \tilde{\xi},$$

which are R-module homomorphisms. They are well-defined since $X_i \subset X$ is open. Now, for any family of R-module homomorphisms $f_i \colon A_R(X_i) \to B$ and $\xi \in A_R(X)$, take $S := \operatorname{supp}(\xi) \subset X$ compact Hausdorff open and define $S_i := S \cap X_i$. Then, as $X_i \subset X$ clopen, we get $S_i \subset X_i$ compact Hausdorff open. Hence, $S = \bigsqcup_i S_i$ and as S is compact, we can write $S = \bigsqcup_{i=1}^{n} S_i$. Now, $\xi_i := \xi|_{X_i}$ is continuous on its support S_i and thus $\xi_i \in A_R(X_i)$. We define the map

$$f \colon A_R(X) \to B, \qquad \xi \mapsto \sum_{i=1}^{n} f_i(\xi_i).$$

It is an R-module homomorphism, since f_i are R-module homomorphisms. Furthermore, we have $(f \circ \iota_i)(\xi) = f(\tilde{\xi}) = f_i(\tilde{\xi}|_{X_i}) = f_i(\xi)$ for all i and $\xi \in A_R(X_i)$, that is, $f \circ \iota_i = f_i$ for all i. For another R-module homomorphism $g \colon A_R(X) \to B$ such that $g \circ \iota_i = f_i$ we get

$$g(\xi) = g\left(\sum_{i=1}^{n} \tilde{\xi}_i\right) = \sum_{i=1}^{n} g(\tilde{\xi}_i) = \sum_{i=1}^{n} (g \circ \iota_i)(\xi_i) = \sum_{i=1}^{n} f_i(\xi_i) = f(\xi)$$

and hence f is unique.

Secondly, for topological spaces with an ample base (as defined in Definition 2.14), the Steinberg module is generated by the characteristic functions of the elements of the base.

Proposition 6.3. *Consider a topological space X with an ample base $\mathcal{B}$ for its topology. Then the Steinberg module is given by*

$$A_R(X) = \langle \mathbb{1}_B \mid B \in \mathcal{B} \rangle_R.$$

Proof. We use Definition (3) of the Steinberg module. Then the set on the right is obviously a subset of the one on the left. For the other direction, take a compact Hausdorff open $U \subset X$. Since $\mathcal{B}$ is a base and U is compact, we find finitely many $U_i \in \mathcal{B}$ with $i = 1, \ldots, n$ such that $U = \bigcup_{i=1}^{n} U_i$. Now, as U is Hausdorff and U_i compact, $U_i \subset U$ is closed. Thus, $U_i \subset U$ is clopen. Now, for $i = 1, \ldots, n$ define $W_i := U_i \setminus (\bigcup_{j=1}^{i} U_j)$, which are disjoint clopen subsets of U and $U = \bigsqcup_{i=1}^{n} W_i$. As U is open in X and $U_i \subset U$ is open, the W_i are open in X. Furthermore, as $W_i \subset U$ is closed and U is compact, the W_i are compact as well. Thus, $W_i \subset U_i$ is an open and compact subset and hence $W_i \in \mathcal{B}$. Finally, we have $1_U = \sum_{i=1}^n 1_{W_i}$ with $W_i \in \mathcal{B}$.

Furthermore, the Steinberg module is almost given by the free module over the characteristic functions of the elements of an ample base $\mathcal{B}$, except that we of course have $1_{U \sqcup V} = 1_U + 1_V$ for all $U \sqcup V, U, V \in \mathcal{B}$. The following result is inspired by [Li22, Lemma 2.2].

Proposition 6.4. *Consider a topological space X with an ample base $\mathcal{B}$ for its topology. Then the kernel of the surjective R-module homomorphism*

$$\pi\colon \bigoplus_{B \in \mathcal{B}} R \cdot 1_B \to A_R(X)$$

is given by the R-submodule

$$S := \langle 1_{U \sqcup V} - 1_U - 1_V \mid U, V, U \sqcup V \in \mathcal{B} \rangle_R$$

and hence $A_R(X)$ is given by the quotient of $\bigoplus_{B \in \mathcal{B}} R \cdot 1_B$ by S.

Proof. First, we note that the defined R-submodule is obviously in the kernel. Thus, we get an induced well-defined R-module homomorphism

$$\tilde{\pi}\colon \bigoplus_{B \in \mathcal{B}} R \cdot 1_B \,/_S \to A_R(X), \qquad f \mod S \mapsto \pi(f).$$

For the other direction, we take $f := \sum_{i=1}^n r_i 1_{B_i} \in \bigoplus_{B \in \mathcal{B}} R \cdot 1_B$ such that $\pi(f) = 0$ and now want to prove that $f \in S$. Now, we find $U_1, \dots, U_m \in \mathcal{B}$ such that for all $i = 1, \dots, n$ there is a $j = 1, \dots, m$ such that $B_i \subset U_j$ (at first just take $m = n$ and $U_i = B_i$). We now prove by induction over m that $f \equiv 0 \mod S$ and hence $f \in S$.

For $m = 1$ we get that $B_i \subset U_1$ for all $i = 1, \dots, n$. Now, the B_i are compact open subsets of the Hausdorff space U_1 and hence are clopen in U_1. We can make them disjoint by defining $B_i' := B_i \setminus (\bigcup_{j=1}^i B_j)$ for $i = 1, \dots, n$. The B_i' are disjoint clopen subsets of U_1, and hence are compact open subsets. Thus, as $U_1 \in \mathcal{B}$ is an element of the ample base $\mathcal{B}$ we get $B_i' \in \mathcal{B}$. Furthermore, we have $B_i = B_1' \sqcup \dots \sqcup B_i'$ and thus

$$r_i 1_{B_i} \equiv r_i (1_{B_1'} + \dots + 1_{B_i'}) \mod S$$

for all $i = 1, \dots, n$ (by induction on i, using that S is additively closed). Hence, we get

$$f \equiv \bigoplus_{i=1}^n \tilde{r}_i 1_{B_i'} \mod S$$

for fitting $\tilde{r}_i \in R$. Now, as $\pi(f) = 0$ we get $\tilde{\pi}(\bigoplus_{i=1}^n \tilde{r}_i 1_{B_i'}) = 0$ and hence as the B_i' are disjoint, we get $\tilde{r}_i = 0$ for all i and thus $f \equiv 0 \mod S$.

Now, for the induction step, we start with $f := \sum_{i=1}^n r_i 1_{B_i}$ with $r_i \in R \setminus \{0\}$ and want to find $\tilde{f}$ such that $f \equiv \tilde{f} \mod S$ and such that $m-1$-many U_j are sufficient for $\tilde{f}$. We first take all $B_i \subset U_m$ and make them disjoint (as done for $m = 1$). Now, $f \mod S$ is equal to the disjointed version modulo S, so by renaming we can assume that all $B_i \subset U_m$ are disjoint. Next, we fix one of the i such that $B_i \subset U_m$. Let $J \subset \{1, \dots, n\}$ denote all the $j = 1, \dots, n$ such that $B_j \not\subset U_m$. Then $B_i \subset \bigcup_{j \in J} B_j$, since otherwise there is an $x \in B_i$ such that $x \notin B_j$ for all $j \in J$ and hence $x \notin B_j$ for all $j \in \{1, \dots n\} \setminus \{i\}$ (as the remaining B_j are the subsets of U_m that are disjoint to B_i) and then $r_i = \pi(f)(x) = 0$, which is a contradiction to our assumptions. Now, as $B_i \subset \bigcup_{j \in J} B_j$, for all $x \in B_i$, we find $j \in J$ such that $x \in B_j$ and hence x is an element of the open subset $B_i \cap B_j \subset X$. So there is some $B_x' \in \mathcal{B}$ such that $x \in B_x' \subset B_i \cap B_j$. Now, $B_i = \bigcup_{x \in B_i} B_x'$ and for each B_x' there is a $j \in J$ such that $B_x' \subset B_j$. As B_i is compact we find finitely many such B_x' covering it, that is, we get $B_i = \bigcup_{k=1}^l B_k'$ with $B_k' \in \mathcal{B}$ and for all k there is a $j \in J$ such that $B_k' \subset B_j$. Since all the B_k' are in B_i we can make them disjoint (similar to $m = 1$) and after renaming we get $B_i = \bigsqcup_{k=1}^{l} B'_k$ with $B'_k \in \mathcal{B}$ and for all k there is a $j \in J$ such that $B'_k \subset B_j$. Hence, we have

$$f = r_i 1_{B_i} + \sum_{j \neq i} r_j 1_{B_j} \equiv r_i \sum_{k=1}^{l} 1_{B'_k} + \sum_{j \neq i} r_j 1_{B_j} \mod S$$

and after repeating this process for the other $B_i \subset U_m$ we have found our f' that only needs $U_1, \dots, U_{m-1}$ and for which the induction hypothesis applies. Hence, we get $f \equiv f' \equiv 0 \mod S$.

Next, we want to define an induced map on the Steinberg modules for sufficiently well-behaved maps on sufficiently well-behaved topological spaces.

Definition 6.5. Consider two topological spaces X, Y with an ample base $\mathcal{B}$ for the topology on X and a local homeomorphism $f: X \to Y$ that is injective on $U \in \mathcal{B}$. Define the map

$$f_{*}: A_R(X) \to A_R(Y), \qquad \alpha \mapsto \left[y \mapsto \sum_{x \in f^{-1}(y)} \alpha(x) \right].$$

Proposition 6.6. *The above-defined map f_{*} is a well-defined R-module homomorphism that sends 1_U to $1_{f(U)}$ for all $U \in \mathcal{B}$. Furthermore, we have $(f \circ f')_{*} = f_{*} \circ f'_{*}$ for two composable such maps f, f'.*

Proof. We first want to check that the map is well-defined. It is easy to check that the map is R-linear. Since $A_R(X)$ is generated by 1_U for $U \in \mathcal{B}$ (by Proposition 6.3) it is sufficient to check that it is well-defined on these. Since f is open and continuous, $f(U) \subset \mathcal{Y}$ is again a compact open subset. Furthermore, as f is injective on U, we have $f(U) \cong U$, and hence U is Hausdorff. Now, for $y \in f(U)$ we find a unique $x \in U$ such that $f(x) = y$ (as f is injective on U) and hence

$$f_{*}(1_U)(y) = \sum_{x \in f^{-1}(U)} 1_U(x) = 1.$$

For $y \notin f(U)$ any $x \in f^{-1}(y)$ is not in U and hence $1_U(x) = 0$ and

$$f_{*}(1_U)(y) = \sum_{x \in f^{-1}(U)} 1_U(x) = 0.$$

Thus, we get that 1_U is mapped to $1_{f(U)}$ for all $U \in \mathcal{B}$ and $1_{f(U)} \in A_R(Y)$ (using Definition (3) of the Steinberg module) and hence f_{*} is well-defined.

It is easy to check that the second part follows from

$$(f \circ f')^{-1}(z) = \bigcup_{y \in f^{-1}(z)} (f')^{-1}(y)$$

for all $z \in Z$.

6.2. **Steinberg algebras of ample groupoids.** Now, we turn our attention to an ample groupoid $\mathcal{G}$. We want to consider the Steinberg module $A_R(\mathcal{G})$ of $\mathcal{G}$ and define a multiplicative structure on it, turning it into an R-algebra. We start by investigating the given R-module $A_R(\mathcal{G})$. By Proposition 3.20 (and using Example 3.18) the compact slices $\mathcal{G}^a$ form an ample base for the topology on $\mathcal{G}$. Thus, we can apply Proposition 6.3 to $\mathcal{G}$ and the ample base $\mathcal{G}^a$.

Corollary 6.7. *For an ample groupoid $\mathcal{G}$, we have that*

$$A_R(\mathcal{G}) = \langle 1_U \mid U \in \mathcal{G}^{\mathrm{a}} \rangle_R .$$

Proof. Is immediate from Proposition 6.3 and the discussion above.

Next, we want to define a multiplicative structure on the Steinberg module $A_R(\mathcal{G})$ of an ample groupoid $\mathcal{G}$ that turns it into an R-algebra. That is, we need to define a multiplicative structure on $\mathcal{G}$ that turns it into a (not necessarily commutative nor unital) ring and behaves well with the R-module structure.

Definition 6.8 (compare [Ste09, Definition 4.4])**.** For an ample groupoid $\mathcal{G}$, a multiplicative structure on $A_R(\mathcal{G})$ is given by the *convolution*

$$(\xi * \eta)(g) = \sum_{h \in \mathcal{G}^{r(g)}} \xi(h)\eta(h^{-1}g)$$

for $\xi, \eta \in A_R(\mathcal{G})$ and $g \in \mathcal{G}$.

It is not obvious that for $\xi, \eta \in A_R(\mathcal{G})$ the convolution $\xi * \eta$ defines a function $\mathcal{G} \to R$, as the (seemingly infinite) sum does not need to be defined in R. Neither is it obvious that this function is in $A_R(\mathcal{G})$. But it is well known that the convolution is R-bilinear and hence it is sufficient to look at the convolution of two characteristic maps of compact slices since they generate the Steinberg algebra (by Corollary 6.7).

Proposition 6.9. *For an ample groupoid $\mathcal{G}$ and $U, V \subset \mathcal{G}$ compact slices, the composition UV is again a compact slice and we get $1_U * 1_V = 1_{UV}$.*

Proof. That UV is a compact slice follows from Proposition 2.13. The proof of $1_U * 1_V = 1_{UV}$ is done in [Ste09, Proposition 4.5]. In Proposition 6.19 we prove a more general statement (see Remark 6.20).

It is now immediate that the convolution is well-defined, that is, that the sum is finite for all $g \in \mathcal{G}$ and that the defined function is again in $A_R(\mathcal{G})$. Furthermore, it is well known that $A_R(\mathcal{G})$ with convolution is indeed an R-module with a compatible ring structure, that is, an R-algebra. Note that the ring structure is not necessarily commutative.

Definition 6.10 (compare [Ste09, Definition 4.12])**.** A map $\xi \in A_R(\mathcal{G})$ such that, firstly, for $g \in \mathcal{G}$ with $r(g) \neq s(g)$ we have $\xi(g) = 0$, and secondly, for $g, h \in \mathcal{G}$ with $s(g) = r(g) = s(h)$ we have $f(hgh^{-1}) = f(g)$, is called a *class function*.

Proposition 6.11 (compare [Ste09, Proposition 4.13])**.** *For an ample groupoid $\mathcal{G}$, the center of the Steinberg algebra $A_R(\mathcal{G})$ is given by the set of class functions.*

Proof. See [Ste09, Proposition 4.13].

Furthermore, the ring structure is not necessarily unital:

Proposition 6.12 (compare [Ste09, Proposition 4.11])**.** *For an ample groupoid $\mathcal{G}$ the Steinberg algebra $A_R(\mathcal{G})$ is unital if and only if the unit slice $\mathcal{G}^0$ is compact, that is, $\mathcal{G}$ is cocompact.*

Proof. The proof boils down to the fact that $1_{\mathcal{G}^0}$ is the unique map behaving like a unit with respect to convolution, and it is an element of $A_R(\mathcal{G})$ if and only if the unit slice $\mathcal{G}^0$ is compact (for details see [Ste09, Proposition 4.11]).

So our ring $A_R(\mathcal{G})$ is not necessarily unital, but at least we always find local units:

Proposition 6.13. *For an ample groupoid $\mathcal{G}$, the set*

$$E := \{1_U \mid U \subset \mathcal{G}^0 \text{ compact open}\}$$

is a set of local units of $A_R(\mathcal{G})$.

Proof. The set E is a subset of $A_R(\mathcal{G})$, since any compact open subset of $\mathcal{G}^0$ is a compact slice. Furthermore, the elements of E are idempotents, as

$$1_U * 1_U = 1_{UU} = 1_U.$$

Now, given finitely many $\xi_1, \ldots, \xi_n \in A_R(\mathcal{G})$, each given by $\xi_i = \sum_{V \in \Xi_i} r_V 1_V$ for finite subsets $\Xi_i \subset \mathcal{G}^{\mathrm{a}}$ and $r_V \in R$. We define

$$U := \bigcup_{i=1}^{n} \bigcup_{V \in \Xi_i} r(V) \cup s(V) \subset \mathcal{G}^0.$$

Since r, s are open continuous maps and the finite union of compact sets is compact, we get $1_U \in E$. Now, for all $i = 1, \ldots, n$ and $V \in \Xi_i$ we have $VU = V = UV$ and thus

$$\xi_i * 1_U = \sum_{V \in \Xi_i} r_V 1_{VU} = \xi_i = \sum_{V \in \Xi_i} r_V 1_{UV} = 1_U * \xi_i.$$

Hence, E is indeed a set of local units.

Thus, the Steinberg algebra $A_R(\mathcal{G})$ of an ample groupoid is indeed an object in the bicategory Rings of rings with local units and smooth bimodules.

Example 6.14. For an ample groupoid $\mathcal{G}$ given by a discrete group the compact slices are given by $\{g\}$ for all $g \in \mathcal{G}$ (see Example 2.15). Now, by Proposition 6.4 the Steinberg algebra of $\mathcal{G}$ is given by the free R-module generated by $\delta_g := 1_{\{g\}}$ with multiplication given by

$$\delta_g \cdot \delta_h := 1_{\{g\}} * 1_{\{h\}} = 1_{\{gh\}} = \delta_{gh}.$$

Thus, the Steinberg algebra of a discrete group is the group ring

$$A_R(\mathcal{G}) = \bigoplus_{g \in G} R \cdot \delta_g = R[\mathcal{G}].$$

6.3. **Steinberg bimodules of ample correspondences.** Now, we turn our attention to ample correspondences $\mathcal{X}$. We want to define a left multiplication by $A_R(\mathcal{H})$ and a right multiplication by $A_R(\mathcal{G})$ on the Steinberg module $A_R(\mathcal{X})$ of the ample correspondence $\mathcal{X}$ to turn it into a smooth $A_R(\mathcal{H})$,$A_R(\mathcal{G})$-bimodule. First, we investigate the R-module $A_R(\mathcal{X})$. By Proposition 3.20 the compact slices $\mathcal{X}^a$ form an ample base for the topology on $\mathcal{X}$. Thus, we can apply Proposition 6.3 on $\mathcal{X}$ and the ample base $\mathcal{X}^a$.

Corollary 6.15. *For an ample correspondence $\mathcal{X}$, we have that*

$$A_R(\mathcal{X}) = \langle 1_U \mid U \in \mathcal{X}^{\mathrm{a}} \rangle_R .$$

Proof. Is immediate from Proposition 6.3 and the discussion above.

Now, we define the $A_R(\mathcal{H})$,$A_R(\mathcal{G})$-bimodule structure on $A_R(\mathcal{X})$.

Definition 6.16. For an ample groupoid correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ we get two R-algebras (hence in particular they are rings) $A_R(\mathcal{H}), A_R(\mathcal{G})$ and an R-module $A_R(\mathcal{X})$ and we define an $A_R(\mathcal{H}), A_R(\mathcal{G})$-bimodule structure on $A_R(\mathcal{X})$ by

$$(\alpha * \xi)(x) := \sum_{g \in G_{s(x)}} \alpha(xg^{-1}) \cdot \xi(g) \tag{6.17}$$

$$(\zeta * \alpha)(x) := \sum_{h \in H^{r(x)}} \zeta(h) \cdot \alpha(h^{-1}x) \tag{6.18}$$

for $\zeta \in A_R(\mathcal{H})$, $\alpha \in A_R(\mathcal{X})$, $\xi \in A_R(\mathcal{G})$ and $x \in \mathcal{X}$.

Again, it is not obvious that these convolutions define functions $\mathcal{X} \to R$, as the (seemingly infinite) sum does not need to be defined in R. Furthermore, it is not obvious that the functions are in $A_R(\mathcal{X})$ and that this definition indeed gives $A_R(\mathcal{X})$ a bimodule structure. But similar to convolutions on groupoids, it is easy to check that this convolution is R-bilinear and thus this follows from the following Proposition 6.19, using that the Steinberg modules of $\mathcal{X}, \mathcal{H}$ and $\mathcal{G}$ are generated by characteristic functions of compact slices (by Corollary 6.15 and Corollary 6.7).

Proposition 6.19. *Consider an ample groupoid correspondence $\mathcal{X}: \mathcal{H} \leftarrow \mathcal{G}$ and compact slices $U \subset \mathcal{H}$, $V \subset \mathcal{X}$ and $W \subset \mathcal{G}$. Then $1_U * 1_V = 1_{UV}$, as well as $1_V * 1_W = 1_{VW}$.*

Proof. First, note that by Corollary 6.15 and Corollary 6.7 the functions $1_U, 1_V$ and 1_W are actually elements of the respective Steinberg modules. Furthermore, $UV, VW \subset \mathcal{X}$ are again compact slices by Lemma 3.21 and their characteristic maps $1_{UV}, 1_{VW}$ are indeed elements of $A_R(\mathcal{X})$. Next we want to prove the equalities $1_U * 1_V = 1_{UV}$ and $1_V * 1_W = 1_{VW}$. For $x \in \mathcal{X}$ we get

$$\begin{aligned}(1_V * 1_W)(x) &= \sum_{g \in G_{s(x)}} 1_V(xg^{-1}) \cdot 1_W(g) = \sum_{g \in (s|_W)^{-1}(s(x))} 1_V(xg^{-1}) \\ &= \begin{cases} 1_V(xg^{-1}), & \text{if } \exists g \in W \text{ with } s(g) = s(x) \\ 0, & \text{else} \end{cases} \\ &= \begin{cases} 1, & \text{if } \exists g \in W \text{ with } s(g) = s(x) \text{ and } xg^{-1} \in V \\ 0, & \text{else} \end{cases} \\ &= 1_{VW}\end{aligned}$$

as $(s|_W)^{-1}(s(x))$ is either empty or the singleton set. An analogous computation shows $1_U * 1_V = 1_{UV}$.

Hence, both left and right multiplication by convolution are well-defined, that is, the sums are finite, and the defined functions are indeed in $A_R(\mathcal{X})$. Furthermore, it is now easy to check that the left and right multiplications are compatible, that is, that $(\zeta * \alpha) * \xi = \zeta * (\alpha * \xi)$. Hence, this convolution indeed gives $A_R(\mathcal{X})$ an $A_R(\mathcal{H})$,$A_R(\mathcal{G})$-bimodule structure.

Note that this Proposition 6.19 is a more general formulation of Proposition 6.9 using that one can view $\mathcal{G}$ as the trivial groupoid correspondence $\mathcal{G}: \mathcal{G} \leftarrow \mathcal{G}$ (by Example 3.18). The proof above is inspired by the proof of [AKM22, Lemma 7.7] and the proof of Proposition 6.9, which can be found in [Ste09, Proposition 4.5].

Remark 6.20. In Example 3.18 we have discussed that the slices of the trivial correspondence $\mathcal{G}: \mathcal{G} \leftarrow \mathcal{G}$ are exactly the slices of the groupoid $\mathcal{G}$. Thus, the Steinberg algebra of $\mathcal{G}$ as a groupoid and as a correspondence are the same sets. If we take a closer look at the definition of the convolution on $A_R(\mathcal{G})$ and of the left and right action of $A_R(\mathcal{G})$ on $A_R(\mathcal{G})$, one can see that they are all equal to

$$(\xi * \eta)(g) = \sum_{xy=g} \xi(x) \cdot \eta(y).$$

Hence, the defined $A_R(\mathcal{G})$,$A_R(\mathcal{G})$-bimodule structure on $A_R(\mathcal{G})$ is by definition exactly the trivial one given by multiplication.

Now, we get to our next result, which says that this bimodule $A_R(\mathcal{X})$ is indeed a smooth bimodule and hence a 1-arrow in the bicategory Rings of smooth bimodules.

Proposition 6.21. *The above-defined bimodule structure on $A_R(\mathcal{X})$ is smooth.*

Proof. Since both $A_R(\mathcal{H})$ and $A_R(\mathcal{G})$ have local units (by Proposition 6.13), using Proposition 5.5 we only need to prove that for all $\alpha \in A_R(\mathcal{X})$ there are $\zeta \in A_R(\mathcal{H})$ and $\xi \in A_R(\mathcal{G})$ such that $\zeta * \alpha = \alpha = \alpha * \xi$. First, consider $\alpha = 1_U$ for $U \in \mathcal{X}^a$. Since U is compact and r is continuous, $r(U) \subset \mathcal{H}^0$ is a compact subset. As $\mathcal{H}^0 \subset \mathcal{H}$ open and $\mathcal{H}^a$ is a base for the topology on $\mathcal{H}$ (by Lemma 2.9 and Proposition 2.10), for all $h \in r(U)$ there is a compact slice $V_h \in \mathcal{H}^a$ with $h \in V_h \subset \mathcal{H}^0$. Thus, we get $r(U) \subset \bigcup_{h \in r(U)} V_h \subset \mathcal{H}^0$ and because of compactness we find $V_1 \ldots, V_n$ such that $r(U) \subset \bigcup_{i=1}^n V_i \subset \mathcal{H}^0$. Now, $V := \bigcup_{i=1}^n V_i \subset \mathcal{H}^0$ is a compact slice (as a finite union of compact sets, it is again compact) and since $r(U) \subset V$ we get $VU = U$. For $\alpha = \sum_{i=1}^m r_i 1_{U_i}$ we find for each U_i a compact slice $V_i \subset \mathcal{H}^0$ such that $V_i U_i = U_i$. Then $V := \bigcup_{i=1}^m V_i \subset \mathcal{H}^0$ is again a compact slice and we get $VU_i = U_i$ for all $i = 1, \ldots, m$. Thus, for $\zeta := 1_V \in \mathcal{H}^a$, we get $\zeta * \alpha = 1_V * \sum_{i=1}^m r_i 1_{U_i} = \sum_{i=1}^m r_i 1_{VU_i} = \sum_{i=1}^m r_i 1_{U_i} = \alpha$. Analogously, one can construct $\xi \in A_R(\mathcal{G})$ such that $\alpha * \xi = \alpha$. Actually, since s is an open map (by Definition 3.16 it is a local homeomorphism) this case is easier since we can just take $V := \bigcup_{i=1}^m s(U_i)$ and $\xi := 1_V$, as $s(U_i) \subset \mathcal{G}^0$ is open. Hence, $A_R(\mathcal{X})$ is indeed a smooth bimodule.

Remark 6.22. Furthermore, from the proof of Proposition 6.21, we can see that the characteristic functions $1_U \in A_R(\mathcal{X})$ not only span $A_R(\mathcal{X})$ as a R-module, but also as a right $A_R(\mathcal{G})$-module (and as a left $A_R(\mathcal{H})$-module).

6.4. **Right modules of proper correspondences.** We now want to get a better understanding of the Steinberg bimodule of a *proper* ample correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ with $\mathcal{H}, \mathcal{G}$ cocompact. We assume that $\mathcal{H}$ and $\mathcal{G}$ are cocompact, so that the Steinberg algebras $A_R(\mathcal{H})$ and $A_R(\mathcal{G})$ are unital (by Proposition 6.12). We show that for cocompact $\mathcal{H}$ and $\mathcal{G}$ and a proper ample correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ the Steinberg bimodule $A_R(\mathcal{X})$ is a finitely generated and projective (fgp) right $A_R(\mathcal{G})$-module. This is not surprising as by Theorem 3.29 the correspondence $\mathcal{X}$ is rather trivial as a right $\mathcal{G}$-module, so its Steinberg bimodule $A_R(\mathcal{X})$ should also be rather trivial as a right $A_R(\mathcal{G})$-module.

Recall that for a proper ample correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ with $\mathcal{H}, \mathcal{G}$ cocompact there are compact open subsets $K_1, \ldots, K_n \in \mathcal{G}^0$ such that the correspondence $\mathcal{X}$ is given by

$$\mathcal{X} \cong \bigsqcup_{i=1}^n r_{\mathrm{G}}^{-1}(K_i)$$

as a right $\mathcal{G}$-space by Theorem 3.29 (and the discussion right above it). Next, we want to use the following Lemma 6.23 for $\mathcal{Y} := \mathcal{G}$ and $V := K_i$.

Lemma 6.23. *Let $\mathcal{Y}\colon \mathcal{H} \leftarrow \mathcal{G}$ be an ample correspondence and $V \subset \mathcal{H}^0$ a compact open subset. Then the right $A_R(\mathcal{G})$-submodule $A_R\left(r_{\mathrm{Y}}^{-1}(V)\right)$ of $A_R(\mathcal{Y})$ is given by $1_V * A_R(\mathcal{Y})$.*

Proof. We want to show that the two submodules $A_R\left(r_{\mathrm{Y}}^{-1}(V)\right), 1_V * A_R(\mathcal{Y})$ of $A_R(\mathcal{Y})$ coincide. We use Definition (3) of the Steinberg module. Consider $1_W \in A_R\left(r_{\mathrm{Y}}^{-1}(V)\right)$ with $W \subset r_{\mathrm{Y}}^{-1}(V)$ a compact Hausdorff open subset. Then $r_{\mathrm{Y}}(W) \subset r_{\mathrm{Y}}\left(r_{\mathrm{Y}}^{-1}(V)\right) \subset V$ and hence $VW = W$ and

$$1_W = 1_{VW} = 1_V * 1_W \in 1_V * A_R(\mathcal{Y}).$$

For the other direction, take $1_V * 1_W \in 1_V * A_R(\mathcal{Y})$ with $W \subset \mathcal{Y}$ a compact Hausdorff open subset. Then $r_{\mathrm{Y}}(VW) \subset r_{\mathrm{H}}(V) = V$ and hence $VW \subset r_{\mathrm{Y}}^{-1}(V)$. Thus, we get $1_V * 1_W = 1_{VW} \in A_R\left(r_{\mathrm{Y}}^{-1}(V)\right)$.

Hence, we get $A_R\left(r_{\mathrm{G}}^{-1}(K_i)\right) = 1_{K_i} * A_R(\mathcal{G})$ and since the compact open subset $K_i \subset \mathcal{G}^0$ is a slice of $\mathcal{G}$, we get $1_{K_i} \in A_R(\mathcal{G})$. Furthermore, since $K_i \subset \mathcal{G}^0$ we have $K_i K_i = K_i$ and thus $1_{K_i} * 1_{K_i} = 1_{K_i K_i} = 1_{K_i}$ and hence 1_{K_i} is an idempotent of $A_R(\mathcal{G})$. So by Lemma 5.13 the right $A_R(\mathcal{G})$-submodule

$$A_R\left(r_{\mathrm{G}}^{-1}(K_i)\right) = 1_{K_i} * A_R(\mathcal{G}) \subset A_R(\mathcal{G})$$

is fgp.

Finally, we have all the tools we need to prove the following Theorem 6.24.

Theorem 6.24. *Consider two cocompact ample groupoids $\mathcal{H}, \mathcal{G}$ and a proper correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$. Then the right $A_R(\mathcal{G})$-module $A_R(\mathcal{X})$ is given by*

$$A_R(\mathcal{X}) \cong \bigoplus_{i=1}^{n} 1_{K_i} * A_R(\mathcal{G})$$

for compact open $K_i \subset \mathcal{G}^0$ and hence is fgp.

Proof. By Theorem 3.29 we find compact open $K_i \subset \mathcal{G}^0$ such that $\mathcal{X} \cong \bigsqcup_{i=1}^{n} r_{\mathrm{G}}^{-1}(K_i)$. Now, by Lemma 6.2 and Lemma 6.23 we have

$$A_R(\mathcal{X}) \cong A_R\left(\bigsqcup_{i=1}^{n} r_{\mathrm{G}}^{-1}(K_i)\right) = \bigoplus_{i=1}^{n} A_R\left(r_{\mathrm{G}}^{-1}(K_i)\right) = \bigoplus_{i=1}^{n} 1_{K_i} * A_R(\mathcal{G})$$

as right $A_R(\mathcal{G})$-modules. By the discussion below Lemma 6.23 the right $A_R(\mathcal{G})$-modules $A_R\left(r_{\mathrm{G}}^{-1}(K_i)\right)$ are fgp and hence their direct sum $\bigoplus_{i=1}^{n} A_R\left(r_{\mathrm{G}}^{-1}(s_{\mathrm{X}}(U_i))\right)$ is fgp as well (by Lemma 5.11). Thus, $A_R(\mathcal{X})$ is an fgp right $A_R(\mathcal{G})$-module.

7. The homomorphism to rings

In this section, we show that the definition of the Steinberg algebra of an ample groupoid and the Steinberg bimodule of an ample groupoid correspondence can be extended to a strictly unital homomorphism $A\colon \mathrm{Gr_a} \to \mathrm{Rings}$ from the bicategory of ample correspondences to the bicategory of smooth bimodules. We define the Steinberg algebra of an ample groupoid and the Steinberg bimodule of an ample correspondence in Section 6, so what is left to do is to construct an $A_R(\mathcal{H}),A_R(\mathcal{G})$-bimodule homomorphism $A(f)\colon A_R(\mathcal{X}) \to A_R(\mathcal{Y})$ for every continuous $\mathcal{H},\mathcal{G}$-equivariant map $f\colon \mathcal{X} \Rightarrow \mathcal{Y}$ and show that it is functorial. Furthermore, we need to define a multiplication map $\mu_{\mathrm{X,Y}}\colon A_R(\mathcal{X}) \otimes_{A_R(\mathrm{G})} A_R(\mathcal{Y}) \to A_R(\mathcal{X} \circ_{\mathrm{G}} \mathcal{Y})$ and, in the end, prove that all the data indeed has the properties of a strictly unital homomorphism of bicategories. Finally, previous results show that the homomorphism restricts to a strictly unital homomorphism $A\colon \mathrm{Gr_{co,proper}} \to \mathrm{Rings_{fgp}}$. We use the latter in Section 11.

Again, we fix a (commutative, unital) ring R with the discrete topology. First, we have to define the homomorphism on 2-arrows. For ample correspondences $\mathcal{X},\mathcal{Y} \in \mathrm{Gr_a}(\mathcal{G},\mathcal{H})$ and a 2-arrow $f\colon \mathcal{X} \Rightarrow \mathcal{Y}$, that is, a continuous $\mathcal{H},\mathcal{G}$-equivariant map $f\colon \mathcal{X} \to \mathcal{Y}$, we want to construct an $A_R(\mathcal{H}),A_R(\mathcal{G})$-bimodule homomorphism $A(f)\colon A_R(\mathcal{X}) \to A_R(\mathcal{Y})$. We use the construction from Proposition 6.6 and show that it has all the desired properties.

Lemma 7.1. *Consider a continuous $\mathcal{H},\mathcal{G}$-equivariant map $f\colon \mathcal{X} \to \mathcal{Y}$. Then the R-linear map*

$$A(f) := f_*\colon A_R(\mathcal{X}) \to A_R(\mathcal{Y}), \qquad \alpha \mapsto \left[y \mapsto \sum_{x\in f^{-1}(y)} \alpha(x) \right],$$

is an $A_R(\mathcal{H}),A_R(\mathcal{G})$-bimodule homomorphism that sends 1_U to $1_{f(U)}$ for all compact slices $U \in \mathcal{X}^{\mathrm{a}}$.

Proof. By Proposition 3.20 the compact slices $\mathcal{X}^a$ form an ample base for the topology on $\mathcal{X}$. By Lemma 4.1 f is a local homeomorphism and injective on $U \in \mathcal{X}^a$. Thus, we can apply Proposition 6.6 to get a well-defined R-linear map $A(f) := f_*$ that sends 1_U to $1_{f(U)}$ for $U \in \mathcal{X}^a$. So we only need to check that this map is indeed an $A_R(\mathcal{H}),A_R(\mathcal{G})$-bimodule homomorphism. We have

$$\begin{aligned} A(f)(\alpha * \xi)(y) &= \sum_{x\in f^{-1}(y)} (\alpha * \xi)(x) = \sum_{x\in f^{-1}(y)} \sum_{g\in \mathrm{G}_{s(x)}} \alpha(xg^{-1}) \cdot \xi(g) \\ &= \sum_{g\in \mathrm{G}_{s(y)}} \sum_{x\in f^{-1}(y)} \alpha(xg^{-1}) \cdot \xi(g) = \sum_{g\in \mathrm{G}_{s(y)}} A(f)(\alpha)(yg^{-1}) \cdot \xi(g) \\ &= \big(A(f)(\alpha) * \xi\big)(y) \end{aligned}$$

for $\alpha \in A_R(\mathcal{X})$, $\xi \in A_R(\mathcal{G})$ and $y \in \mathcal{Y}$. Similarly, we get $A(f)(\zeta * \alpha) = \zeta * A(f)(\alpha)$ for $\zeta \in A_R(\mathcal{H})$, $\alpha \in A_R(\mathcal{X})$.

Proposition 7.2. *For ample groupoids $\mathcal{G},\mathcal{H}$ the construction above defines a functor*

$$\begin{aligned} A_{\mathrm{G,H}}\colon \mathrm{Gr_a}(\mathcal{G},\mathcal{H}) &\to \mathrm{Rings}(A_R(\mathcal{G}),A_R(H)), \\ \mathcal{X} &\mapsto A_R(\mathcal{X}), \\ f\colon \mathcal{X} \to \mathcal{Y} &\mapsto A(f)\colon A_R(\mathcal{X}) \to A_R(\mathcal{Y}). \end{aligned}$$

Proof. The functor is well-defined, since by Proposition 6.21 $A_R(\mathcal{X})$ is indeed a smooth bimodule and by Lemma 7.1 $A(f)$ is indeed a bimodule homomorphism. Now, the identity map is sent to the identity map. Thus, we have

$$A_{\mathrm{G,H}}(1_{\mathrm{X}}) = 1_{A_R(\mathrm{X})}.$$

Additionally, for $f\colon \mathcal{Y} \to \mathcal{Z}$ and $f'\colon \mathcal{X} \to \mathcal{Y}$ we also get

$$A_{G,H}(f \circ f') = A_{G,H}(f) \circ A_{G,H}(f')$$

since $(f \circ f')_{*} = f_{*} \circ f'_{*}$ (by Proposition 6.6). Thus, $A_{G,H}$ is indeed a functor.

Secondly, we want to define the natural bimodule isomorphisms $\mu_{X,Y}$.

Definition 7.3. For ample correspondences $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$, $\mathcal{Y}\colon \mathcal{G} \leftarrow \mathcal{K}$ define the map

$$\overline{\mu_{X,Y}}\colon A_R(\mathcal{X}) \times A_R(\mathcal{Y}) \to A_R(\mathcal{X} \circ_G \mathcal{Y}),$$

$$(\alpha, \beta) \mapsto \left[[x,y] \mapsto \sum_{g \in G_{s(x)}} \alpha(xg^{-1}) \cdot \beta(gy) \right].$$

Proposition 7.4. *The above-defined map $\overline{\mu_{X,Y}}$ is well-defined and $A_R(\mathcal{G})$-balanced. Thus, it induces a unique group homomorphism*

$$\mu_{X,Y}\colon A_R(\mathcal{X}) \otimes_{A_R(G)} A_R(\mathcal{Y}) \to A_R(\mathcal{X} \circ_G \mathcal{Y}).$$

This group homomorphism $\mu_{X,Y}$ is an $A_R(\mathcal{H}),A_R(\mathcal{K})$-bimodule homomorphism and is natural in $\mathcal{X}$ and $\mathcal{Y}$.

Proof. Note first that $[x,y] \mapsto \sum_{g \in G_{s(x)}} \alpha(xg^{-1}) \cdot \beta(gy)$ does not depend on the representation of $[x,y]$, since a different representative $(x\tilde{g}^{-1}, \tilde{g}y) \in [x,y]$ only changes the order of the summands. To prove that this actually defines a function $\mathcal{X} \circ_G \mathcal{Y} \to R$ that lies in $A_R(\mathcal{X} \circ_G \mathcal{Y})$, we first see that the definition is R-linear in α and β, so it is sufficient to consider $\alpha = 1_U$ and $\beta = 1_V$ for $U \in \mathcal{X}^a, V \in \mathcal{Y}^a$ (using Corollary 6.15). Using Proposition 4.4 it is easy to check that $1_U \otimes 1_V$ is sent to 1_{UV} and $1_{UV} \in A_R(\mathcal{X} \circ_G \mathcal{Y})$. Thus, for any $\alpha \in A_R(\mathcal{X})$ and $\beta \in A_R(\mathcal{Y})$ the function $\overline{\mu_{X,Y}}(\alpha \otimes \beta)$ is indeed well-defined (that is, independent of the representative and the sum is finite and hence defined in R) and an element of $A_R(\mathcal{X} \circ_G \mathcal{Y})$.

Furthermore, it is easy to check that $\overline{\mu_{X,Y}}$ is additive in both arguments and hence by the following computation it is $A_R(\mathcal{G})$-balanced. We have

$$\begin{aligned}
\overline{\mu_{X,Y}}(\alpha * \xi, \beta) &= \sum_{g \in G_{s(x)}} (\alpha * \xi)(xg^{-1}) \cdot \beta(gy) \\
&= \sum_{g \in G_{s(x)}} \sum_{h \in G_{s(xg^{-1})}} \alpha(xg^{-1}h^{-1}) \cdot \xi(h) \cdot \beta(gy) \\
&= \sum_{h \in G_{r(g)}} \sum_{g \in G_{s(x)}} \alpha\big(x(hg)^{-1}\big) \cdot \xi(h) \cdot \beta(gy) \\
&= \sum_{h \in G^{r(k)}} \sum_{k \in G_{s(x)}} \alpha(xk^{-1}) \cdot \xi(h) \cdot \beta(h^{-1}ky) \\
&= \sum_{k \in G_{s(x)}} \alpha(xk^{-1}) \cdot (\xi * \beta)(ky) \\
&= \overline{\mu_{X,Y}}(\alpha, \xi * \beta)
\end{aligned}$$

for $\alpha \in A_R(\mathcal{X}), \xi \in A_R(\mathcal{G}), \beta \in A_R(\mathcal{Y})$ and $[x,y] \in \mathcal{X} \circ_G \mathcal{Y}$. Now, by the universal property of the tensor product, we get a unique induced group homomorphism $\mu_{X,Y}$. With the canonical $A_R(\mathcal{H}),A_R(\mathcal{K})$-bimodule structure on $A_R(\mathcal{X}) \otimes_{A_R(G)} A_R(\mathcal{Y})$, the group homomorphism $\mu_{X,Y}$ is also an $A_R(\mathcal{H}),A_R(\mathcal{K})$-bimodule homomorphism, since we have

$$\begin{aligned}
\mu_{X,Y}\big(\zeta * (\alpha \otimes \beta)\big)\big([x,y]\big) &= \mu_{X,Y}\big((\zeta * \alpha) \otimes \beta\big)\big([x,y]\big) \\
&= \sum_{g \in G_{s(x)}} (\zeta * \alpha)(xg^{-1}) \cdot \beta(gy) \\
&= \sum_{g \in G_{s(x)}} \sum_{h \in H^{r(xg^{-1})}} \zeta(h) \cdot \alpha(h^{-1}xg^{-1}) \cdot \beta(gy)
\end{aligned}$$

$$= \sum_{h \in \mathsf{H}^{r(x)}} \zeta(h) \cdot \Big(\sum_{g \in \mathsf{G}_{s(h^{-1}x)}} \alpha(h^{-1}xg^{-1}) \cdot \beta(gy) \Big)$$
$$= \sum_{h \in \mathsf{H}^{r([x,y])}} \zeta(h) \cdot \mu_{\mathsf{X},\mathsf{Y}}(\alpha \otimes \beta)\big(h^{-1}[x,y]\big)$$
$$= \big(\zeta * \mu_{\mathsf{X},\mathsf{Y}}(\alpha \otimes \beta)\big)([x,y])$$

for $\zeta \in A_R(\mathcal{H}), \alpha \in A_R(\mathcal{X}), \beta \in A_R(\mathcal{Y})$ and $[x,y] \in \mathcal{X} \circ_{\mathsf{G}} \mathcal{Y}$. Analogously, we have

$$\mu_{\mathsf{X},\mathsf{Y}}\big((\alpha \otimes \beta) * \eta\big)([x,y]) = \mu_{\mathsf{X},\mathsf{Y}}\big(\alpha \otimes (\beta * \eta)\big)([x,y])$$
$$= \sum_{g \in \mathsf{G}_{s(x)}} \alpha(xg^{-1}) \cdot (\beta * \eta)(gy)$$
$$= \sum_{g \in \mathsf{G}_{s(x)}} \alpha(xg^{-1}) \cdot \Big(\sum_{k \in \mathsf{K}_{s(gy)}} \beta(gyk^{-1}) \cdot \eta(k) \Big)$$
$$= \sum_{k \in \mathsf{K}_{s(y)}} \Big(\sum_{g \in \mathsf{G}_{s(x)}} \alpha(xg^{-1}) \cdot \beta(gyk^{-1}) \Big) \cdot \eta(k)$$
$$= \sum_{k \in \mathsf{K}_{s(y)}} \mu_{\mathsf{X},\mathsf{Y}}(\alpha \otimes \beta)\big([x,y]k^{-1}\big) \cdot \eta(k)$$
$$= \big(\mu_{\mathsf{X},\mathsf{Y}}(\alpha \otimes \beta) * \eta\big)([x,y])$$

for $\alpha \in A_R(\mathcal{X}), \beta \in A_R(\mathcal{Y}), \eta \in A_R(\mathcal{K})$, and $[x,y] \in \mathcal{X} \circ_{\mathsf{G}} \mathcal{Y}$.

For the naturality we need to prove that for ample correspondences $\mathcal{X}, \mathcal{X}' : \mathcal{H} \leftarrow \mathcal{G}$ and $\mathcal{Y}, \mathcal{Y}' : \mathcal{G} \leftarrow \mathcal{K}$ with 2-arrows $f : \mathcal{X} \Rightarrow \mathcal{X}'$ and $f' : \mathcal{Y} \Rightarrow \mathcal{Y}'$ the diagram

$$\begin{array}{ccc} A_R(\mathcal{X}) \otimes_{A_R(\mathsf{G})} A_R(\mathcal{Y}) & \xrightarrow[\cong]{\mu_{\mathsf{X},\mathsf{Y}}} & A_R(\mathcal{X} \circ_{\mathsf{G}} \mathcal{Y}) \\ \downarrow{\scriptstyle A(f) \otimes_{A_R(\mathsf{G})} A(f')} & & \downarrow{\scriptstyle A(f \circ_{\mathsf{G}} f')} \\ A_R(\mathcal{X}') \otimes_{A_R(\mathsf{G})} A_R(\mathcal{Y}') & \xrightarrow[\cong]{\mu_{\mathsf{X}',\mathsf{Y}'}} & A_R(\mathcal{X}' \circ_{\mathsf{G}} \mathcal{Y}') \end{array}$$

commutes. We have

$$\big(A(f \circ_{\mathsf{G}} f') \circ \mu_{\mathsf{X},\mathsf{Y}}\big)(\alpha \otimes \beta)\big([x',y']\big)$$
$$= \sum_{[x,y] \in (f \circ f')^{-1}([x',y'])} \mu_{\mathsf{X},\mathsf{Y}}(\alpha \otimes \beta)\big([x,y]\big)$$
$$= \sum_{[x,y] \in (f \circ f')^{-1}([x',y'])} \sum_{g \in \mathsf{G}_{s(x)}} \alpha(xg^{-1}) \cdot \beta(gy)$$
$$= \sum_{\substack{g \in \mathsf{G}_{s(x')} \\ [x,y] \in (f \circ f')^{-1}([x',y'])}} \alpha(xg^{-1}) \cdot \beta(gy)$$
$$= \sum_{\substack{g \in \mathsf{G}_{s(x')} \\ x \in f^{-1}(x') \\ y \in f'^{-1}(y')}} \alpha(xg^{-1}) \cdot \beta(gy)$$
$$= \sum_{g \in \mathsf{G}_{s(x')}} \Big(\sum_{x \in f^{-1}(x'g^{-1})} \alpha(x) \Big) \cdot \Big(\sum_{y \in f'^{-1}(gy')} \beta(y) \Big)$$
$$= \sum_{g \in \mathsf{G}_{s(x')}} A(f)(\alpha)(x'g^{-1}) \cdot A(f')(\beta)(gy')$$
$$= \mu_{\mathsf{X}',\mathsf{Y}'}\big(A(f)(\alpha) \otimes A(f')(\beta)\big)\big([x',y']\big)$$
$$= \Big(\mu_{\mathsf{X}',\mathsf{Y}'} \circ \big(A(f) \otimes A(f')\big)\Big)(\alpha \otimes \beta)\big([x',y']\big)$$

for $\alpha \in A_R(\mathcal{X}), \beta \in A_R(\mathcal{Y})$ and $[x',y'] \in \mathcal{X}' \circ_{\mathsf{G}} \mathcal{Y}'$.

The following result is inspired by [Mil23, Proposition 2.9].

Lemma 7.5. *The group homomorphism*

$$\mu_{X,Y}: A_R(\mathcal{X}) \otimes_{A_R(G)} A_R(\mathcal{Y}) \to A_R(\mathcal{X} \circ_G \mathcal{Y})$$

is an isomorphism.

Proof. We want to define a map

$$A_R(\mathcal{X} \circ_G \mathcal{Y}) \to A_R(\mathcal{X}) \otimes_{A_R(G)} A_R(\mathcal{Y})$$

that is an inverse to $\mu_{X,Y}$. By Proposition 4.4 the set $\mathcal{B}_{X \circ_G Y}$ is an ample base for the topology of $\mathcal{X} \circ_G \mathcal{Y}$ and thus we can apply Proposition 6.4 to get that the Steinberg module $A_R(\mathcal{X} \circ_G \mathcal{Y})$ is given by the quotient of the direct sum

$$\bigoplus_{UV \in B_{X \circ_G Y}} R \cdot 1_{UV}$$

by

$$\langle 1_{UV} - 1_{U_1V_1} - 1_{U_2V_2} \mid U_1V_1, U_2V_2, UV = U_1V_1 \sqcup U_2V_2 \in \mathcal{B}_{X \circ_G Y} \rangle$$

as an R-module. Now, at first we define the R-module homomorphism

$$\bigoplus_{UV \in B_{X \circ_G Y}} R \cdot 1_{UV} \to A_R(\mathcal{X}) \otimes_{A_R(G)} A_R(\mathcal{Y}), \qquad 1_{UV} \mapsto 1_U \otimes 1_V$$

and check that it is well-defined. Take $U_1, U_2 \in \mathcal{X}^a$ and $V_1, V_2 \in \mathcal{Y}^a$ with $s(U_i) \supset r(V_i)$ for $i = 1, 2$ such that $U_1V_1 = U_2V_2$. Then we get the compact slice $W := \langle U_1 \mid U_2 \rangle \subset \mathcal{G}$ (by Lemma 3.21) and it is easy to check that we have $V_1 = WV_2$, $U_1W = U_2s(W)$ and $s(W)V_2 = V_2$. Thus, we get

$$\begin{aligned} 1_{U_1} \otimes 1_{V_1} &= 1_{U_1} \otimes 1_{WV_2} = 1_{U_1} \otimes 1_W 1_{V_2} = 1_{U_1}1_W \otimes 1_{V_2} \\ &= 1_{U_1W} \otimes 1_{V_2} = 1_{U_2s(W)} \otimes 1_{V_2} = 1_{U_2}1_{s(W)} \otimes 1_{V_2} \\ &= 1_{U_2} \otimes 1_{s(W)}1_{V_2} = 1_{U_2} \otimes 1_{s(W)V_2} = 1_{U_2} \otimes 1_{V_2} \end{aligned}$$

and hence the map is well-defined. Next, we check that

$$\langle 1_{UV} - 1_{U_1V_1} - 1_{U_2V_2} \mid U_1V_1, U_2V_2, UV = U_1V_1 \sqcup U_2V_2 \in \mathcal{B}_{X \circ_G Y} \rangle$$

is in the kernel of this R-module map. Take $U_1V_1, U_2V_2, UV = U_1V_1 \sqcup U_2V_2 \in \mathcal{B}_{X \circ_G Y}$, then, without loss of generality (see the end of the proof of Lemma 4.3), we have $U = U_1 = U_2$ and $V = V_1 \sqcup V_2$ for $V_1, V_2 \in \mathcal{Y}^a$ such that $UV = UV_1 \sqcup UV_2$. So we get

$$1_{UV} \mapsto 1_U \otimes 1_V = 1_U \otimes 1_{V_1} + 1_U \otimes 1_{V_2}.$$

Thus, the map descends to a well-defined R-module homomorphism

$$A_R(\mathcal{X} \circ_G \mathcal{Y}) \to A_R(\mathcal{X}) \otimes_{A_R(G)} A_R(\mathcal{Y})$$

that sends $1_{UV} \mapsto 1_U \otimes 1_V$ and is hence inverse to $\mu_{X,Y}$.

Finally, we can combine all the discussed data above to a strictly unital homomorphism $\mathsf{Gr_a} \to \mathsf{Rings}$.

Theorem 7.6. *We can combine the constructions above to a strictly unital homomorphism*

$$A: \mathsf{Gr_a} \to \mathsf{Rings}$$

given by the data:

- *a map on objects given by $\mathcal{G} \mapsto A_R(\mathcal{G})$ sending an ample groupoid $\mathcal{G}$ to its Steinberg algebra $A_R(\mathcal{G})$;*

- *for* $\mathcal{G},\mathcal{H} \in \mathrm{Gr_a}$ *the functor*

$$
\begin{aligned}
A_{\mathrm{G,H}}\colon \mathrm{Gr_a}(\mathcal{G},\mathcal{H}) \to \mathrm{Rings}\big(A_R(\mathcal{G}), A_R(H)\big),\\
\mathcal{X} \mapsto A_R(\mathcal{X}),\\
f\colon \mathcal{X}\to\mathcal{Y} \mapsto A(f)\colon A_R(\mathcal{X}) \to A_R(\mathcal{Y}),
\end{aligned}
$$

that sends an ample correspondence $\mathcal{X}$ *to its Steinberg bimodule* $A_R(\mathcal{X})$ *and a continuous equivariant map* f *to the bimodule homomorphism* $A(f)$*; and*
- *for ample correspondences* $\mathcal{X}\colon\mathcal{H}\leftarrow\mathcal{G}$, $\mathcal{Y}\colon\mathcal{G}\leftarrow\mathcal{K}$ *natural (in* $\mathcal{X}$ *and* $\mathcal{Y}$*) bimodule isomorphisms*

$$\mu_{\mathrm{X,Y}}\colon A_R(\mathcal{X}) \otimes_{A_R(\mathrm{G})} A_R(\mathcal{Y}) \to A_R(\mathcal{X}\circ_{\mathrm{G}}\mathcal{Y}).$$

Proof. The homomorphism is well-defined on objects, because of Proposition 6.13 and Proposition 5.4. The functor on 1- and 2-arrows is defined and handled in Proposition 7.2. The definition of $\mu_{\mathrm{X,Y}}$ can be found in Definition 7.3 and Proposition 7.4. In Proposition 7.4 and Lemma 7.5, we show that it has all the necessary properties. In Remark 6.20, we discuss that our homomorphism is indeed strictly unital.

Thus, all that is left to prove is that the three required diagrams commute. For ample correspondences $\mathcal{X}\colon\mathcal{H}\leftarrow\mathcal{G}$, $\mathcal{Y}\colon\mathcal{G}\leftarrow\mathcal{K}$ and $\mathcal{Z}\colon\mathcal{K}\leftarrow\mathcal{E}$ the diagram

$$
\begin{array}{ccc}
\big(A_R(\mathcal{X})\otimes_{A_R(\mathrm{G})} A_R(\mathcal{Y})\big)\otimes_{A_R(\mathrm{K})} A_R(\mathcal{Z}) & \xrightarrow[\cong]{\mu_{\mathrm{X,Y}}\otimes\mathrm{id}} & A_R(\mathcal{X}\circ_{\mathrm{G}}\mathcal{Y})\otimes_{A_R(\mathrm{K})} A_R(\mathcal{Z})\\
\cong\big\downarrow \text{assoc} & & \cong\big\downarrow \mu_{\mathrm{X}\circ_{\mathrm{G}}\mathrm{Y,Z}}\\
A_R(\mathcal{X})\otimes_{A_R(\mathrm{G})}\big(A_R(\mathcal{Y})\otimes_{A_R(\mathrm{K})} A_R(\mathcal{Z})\big) & & A_R((\mathcal{X}\circ_{\mathrm{G}}\mathcal{Y})\circ_{\mathrm{K}}\mathcal{Z})\\
\cong\big\downarrow \mathrm{id}\otimes\mu_{\mathrm{Y,Z}} & & \cong\big\downarrow A(\mathrm{assoc})\\
A_R(\mathcal{X})\otimes_{A_R(\mathrm{G})} A_R(\mathcal{Y}\circ_{\mathrm{K}}\mathcal{Z}) & \xrightarrow[\cong]{\mu_{\mathrm{X,Y}\circ_{\mathrm{K}}\mathrm{Z}}} & A_R(\mathcal{X}\circ_{\mathrm{G}}(\mathcal{Y}\circ_{\mathrm{K}}\mathcal{Z}))
\end{array}
$$

commutes, because for $\alpha\in A_R(\mathcal{X}), \beta\in A_R(\mathcal{Y}), \gamma\in A_R(\mathcal{Z})$ and $x\in\mathcal{X}, y\in\mathcal{Y}, z\in\mathcal{Z}$ going counterclockwise, we get

$$
\begin{aligned}
&(\mu_{\mathrm{X,Y}\circ_{\mathrm{K}}\mathrm{Z}}\circ \mathrm{id}\otimes\mu_{\mathrm{Y,Z}}\circ\mathrm{assoc})\big((\alpha\otimes\beta)\otimes\gamma\big)\big([x,[y,z]]\big)\\
&\qquad= \mu_{\mathrm{X,Y}\circ_{\mathrm{K}}\mathrm{Z}}\big(\alpha,\mu_{\mathrm{Y,Z}}(\beta\otimes\gamma)\big)\big([x,[y,z]]\big)\\
&\qquad= \sum_{g\in\mathrm{G}_{s(x)}} \alpha(xg^{-1})\cdot\mu_{\mathrm{Y,Z}}(\beta\otimes\gamma)\big(g[y,z]\big)\\
&\qquad= \sum_{g\in\mathrm{G}_{s(x)}} \alpha(xg^{-1})\cdot\sum_{k\in\mathrm{K}_{s(y)}}\beta(gyk^{-1})\cdot\gamma(kz)\\
&\qquad= \sum_{g\in\mathrm{G}_{s(x)}}\sum_{k\in\mathrm{K}_{s(y)}} \alpha(xg^{-1})\cdot\beta(gyk^{-1})\cdot\gamma(kz).
\end{aligned}
$$

And going clockwise, we get

$$
\begin{aligned}
&\big(A(\mathrm{assoc})\circ\mu_{\mathrm{X}\circ_{\mathrm{G}}\mathrm{Y,Z}}\circ\mu_{\mathrm{X,Y}}\otimes\mathrm{id}\big)\big((\alpha\otimes\beta)\otimes\gamma\big)\big([x,[y,z]]\big)\\
&\qquad= \mu_{\mathrm{X}\circ_{\mathrm{G}}\mathrm{Y,Z}}\big(\mu_{\mathrm{X,Y}}(\alpha,\beta),\gamma\big)\big([[x,y],z]\big)\\
&\qquad= \sum_{k\in\mathrm{K}_{s([x,y])}} \mu_{\mathrm{X,Y}}(\alpha,\beta)\big([x,y]k^{-1}\big)\cdot\gamma(kz)\\
&\qquad= \sum_{k\in\mathrm{K}_{s([x,y])}}\sum_{g\in\mathrm{G}_{s(x)}} \alpha(xg^{-1})\cdot\beta(gyk^{-1})\cdot\gamma(kz)\\
&\qquad= \sum_{g\in\mathrm{G}_{s(x)}}\sum_{k\in\mathrm{K}_{s(x)}} \alpha(xg^{-1})\cdot\beta(gyk^{-1})\cdot\gamma(kz).
\end{aligned}
$$

Hence, we have the same result and thus the maps are the same and the diagram commutes. For an ample correspondence $\mathcal{X}\colon\mathcal{H}\leftarrow\mathcal{G}$ the diagram

$$\begin{array}{ccc} A_R(\mathcal{X}) \otimes_{A_R(\mathcal{G})} A_R(\mathcal{G}) & & \\ {\scriptstyle \mathrm{mult}}\big\downarrow \square & \overset{\square}{\underset{\square}{}} \searrow^{\mu_{\mathcal{X},\mathcal{G}}} & \\ A_R(\mathcal{X}) & \xleftarrow[A(r_{\mathcal{X}})]{} & A_R(\mathcal{X}\circ_{\mathcal{G}}\mathcal{G}) \end{array}$$

commutes, since for $\alpha\in A_R(\mathcal{X}),\xi\in A_R(\mathcal{G})$ and $x\in\mathcal{X}$, we have

$$\begin{aligned}\big(A(\mathrm{r}_{\mathcal{X}})\circ\mu_{\mathcal{X},\mathcal{G}}\big)(\alpha\otimes\xi)(x) &= \mu_{\mathcal{X},\mathcal{G}}(\alpha,\xi)\Big(\big[x,s(x)\big]\Big)\\ &= \sum_{g\in\mathcal{G}_{s(x)}}\alpha(xg^{-1})\cdot\xi\big(gs(x)\big)\\ &= \sum_{g\in\mathcal{G}_{s(x)}}\alpha(xg^{-1})\cdot\xi(g)\\ &= (\alpha*\xi)(x)\\ &= \mathrm{mult}(\alpha\otimes\xi)(x).\end{aligned}$$

Analogously, the diagram

$$\begin{array}{ccc} A_R(\mathcal{H}) \otimes_{A_R(\mathcal{H})} A_R(\mathcal{X}) & & \\ {\scriptstyle \mathrm{mult}}\big\downarrow \square & \overset{\square}{\underset{\square}{}} \searrow^{\mu_{\mathcal{H},\mathcal{X}}} & \\ A_R(\mathcal{X}) & \xleftarrow[A(l_{\mathcal{X}})]{} & A_R(\mathcal{H}\circ_{\mathcal{H}}\mathcal{X}) \end{array}$$

commutes. Hence, all the required diagrams indeed commute, turning A into a strictly unital homomorphism.

Remark 7.7. Restricting the domain bicategory $\mathsf{Gr_a}$ to the subbicategory $\mathsf{Gr_{co,proper}}$ of cocompact ample groupoids and proper correspondences leads to a strictly unital homomorphism

$$A\colon\mathsf{Gr_{co,proper}}\to\mathsf{Rings_{fgp}},$$

since for a cocompact ample groupoid $\mathcal{G}$, the Steinberg algebra $A_R(\mathcal{G})$ is unital (by Proposition 6.12) and for a proper ample correspondence $\mathcal{X}\colon\mathcal{H}\leftarrow\mathcal{G}$ between cocompact groupoids $\mathcal{H},\mathcal{G}$ the Steinberg bimodule is fgp (by Theorem 6.24).

8. Filtered colimits

In this section, we take a little excursion and establish the necessary knowledge on *filtered* colimits in the categories Top, Ring, AbGroup, Set and R-Mod for the fixed (commutative and unital) ring R, that we need in the next sections. The knowledgeable reader may only skim over or even skip this section.

Filtered colimits form a particularly manageable class of colimits since they behave well with the forgetful functors Ring → Set, Ring → AbGroup, as well as AbGroup → Set. In general, forgetful functors like these are only right adjoints and hence behave well with limits, but not with arbitrary colimits (for example they do not preserve coproducts). That is why we want to restrict ourselves to *filtered* colimits, which these functors actually preserve and create. This means that we can explicitly construct filtered colimits in Set and we can define a canonical abelian group and unital ring structure on them to turn them into filtered colimits in AbGroup and Ring. Furthermore, we can define a canonical topology on the filtered colimit in Set to get the filtered colimit in Top. This is less surprising, since this actually works with all colimits in Top, as the forgetful functor U:Top → Set is also a left adjoint and hence preserves all colimits. The same goes for the forgetful functor U:R-Mod → AbGroup, and hence a filtered colimit in R-Mod is given by the filtered colimit in AbGroup with a canonical R-module structure. We deal with the two more believable cases last and in less detail

So now, first, what is a filtered colimit? A filtered colimit is a colimit of a diagram over a *filtered category*. A filtered category can be viewed as the generalization of a filtered preorder. A preorder[4] $(X,\leq)$ is called *filtered*, if for any two elements $x,y \in X$ there is an upper bound, that is, an element $u \in X$ such that $x \leq u$ and $y \leq u$. For example, the preorder $(\mathbb{N},\leq)$ is filtered. This notion of being *filtered* can be generalized to categories.

Definition 8.1 ([AM15, Definition 3.6]). A category $\mathcal{J}$ is called *filtered*, if it is non-empty and

- (F1) for two objects x,y in $\mathcal{J}$, there is an object z in $\mathcal{J}$ and arrows $f \in \mathcal{J}(x,z)$, $g \in \mathcal{J}(y,z)$; and
- (F2) for two arrows $f,g \in \mathcal{J}(x,y)$, there is an object z in $\mathcal{J}$ and an arrow $k \in \mathcal{J}(y,z)$ with $kf = kg$.

Definition 8.2. For a diagram $F:\mathcal{C} \to \mathcal{D}$ over a filtered category $\mathcal{C}$ that admits a colimit, we denote its colimit by $\varinjlim F$ and call it a *filtered colimit*.

As mentioned above, the category induced by a filtered preorder (for example, $(\mathbb{N},\leq)$) is an important example of a filtered category. Now, the colimit of a diagram over $(\mathbb{N},\leq)$ is called an *inductive limit* and is very well understood. This is also where we borrow our notation for a filtered colimit from, as $\varinjlim F$ is the standard notation for an inductive limit. The colimit of a diagram over a filtered category can be viewed as a generalization of an inductive limit and behaves similarly. If we have a diagram over a *countable* filtered category, the colimit actually is given by an inductive limit (see Lemma 8.11 and Lemma 8.12).

From now on we fix a small filtered category $\mathcal{C}$. Let $\mathcal{D}$ be one of the categories Ring, AbGroup or Set. We define $\mathcal{D}$ as a placeholder for all three of these categories to cover all three cases in one go. Let $F:\mathcal{C} \to \mathcal{D}$ be a diagram over $\mathcal{C}$ in $\mathcal{D}$. Now, we want to construct a colimit of F in $\mathcal{D}$.

Definition 8.3. Define the set

$$\mathcal{O}_{\square} := \bigsqcup_{c \square \mathrm{C}^0} F(c)$$

[4] A *preorder* is a reflexive and transitive binary relation.

and the equivalence relation generated by $(x, c) \sim \big(Ff(x), d\big)$ for all $c, d \in \mathcal{C}^0$, $x \in F(c)$ and $f: c \to d \in \mathcal{C}$. Let $\mathcal{O}_{\square}$ be the set of equivalence classes with elements denoted as $[x, c] \in \mathcal{O}_{\square}$. We get canonical maps $\iota_c: F(c) \to \mathcal{O}_{\square}$, $x \mapsto [x, c]$.

In case $\mathcal{D} = \mathtt{AbGroup}$, that is, $\big(F(c), +\big)$ is an abelian group for every $c \in \mathcal{C}^0$, we define an abelian group structure on $\mathcal{O}_{\square}$ by

$$[x, c] + [y, d] := \big[Ff(x) + Fg(y), z\big]$$

for $c, d \in \mathcal{C}^0$, $x \in F(c)$, $y \in F(d)$ and $z \in \mathcal{C}^0$, $f: c \to z$, $g: d \to z$ (given by (F1)). In case $\mathcal{D} = \mathtt{Ring}$, that is, $\big(F(c), +, \cdot\big)$ is a unital ring for every $c \in \mathcal{C}^0$, we define a unital ring structure on the set $\mathcal{O}_{\square}$, where addition is defined as above and multiplication is given by

$$[x, c] \cdot [y, d] := \big[Ff(x) \cdot Fg(y), z\big]$$

for $c, d \in \mathcal{C}^0, x \in F(c), y \in F(d)$ and $z \in \mathcal{C}^0$, $f: c \to z, g: d \to z$ (given by (F1)).

Remark 8.4. Note that the equivalence relation $\sim$ *generated* by $(x, c) \sim \big(Ff(x), d\big)$ is given by the relation defined as

$$(x, c) \sim (y, d) :\Leftrightarrow \exists f: c \to e, g: d \to e \text{ such that } Ff(x) = Fg(y)$$

for all $x \in F(c), y \in F(d)$. Because of transitivity, any equivalence relation generated by $(x, c) \sim \big(Ff(x), d\big)$ must contain these relations, and hence it is sufficient to check that this relation indeed is an equivalence relation. Now, reflexivity and symmetry follow by definition, and (F1) and (F2) (from Definition 8.1) together imply that the defined relation is transitive.

Lemma 8.5. *For $\mathcal{D} = \mathtt{AbGroup}$ ($\mathcal{D} = \mathtt{Ring}$, resp.) the above-defined addition (and multiplication, resp.) is well-defined and satisfies all the abelian group axioms (unital ring axioms, resp.) turning $\mathcal{O}_{\square}$ into an abelian group (unital ring, resp.). The canonical maps $\iota_c: F(c) \to \mathcal{O}_{\square}, x \mapsto [x, c]$ are group homomorphism (unital ring homomorphism, resp.) and define a cone $\iota: F \Rightarrow \mathcal{O}_{\square}$.*

Proof. First, we prove that the operations on $\mathcal{O}_{\square}$ are well-defined. We denote the operation as $*$ to prove the case $* := +$ and $* := \cdot$ simultaneously. This works since the definition of both operations is

$$[x, c] * [y, d] := \big[Ff(x) * Fg(y), z\big].$$

for $c, d \in \mathcal{C}^0$, $x \in F(c)$, $y \in F(d)$ and $z \in \mathcal{C}^0$, $f: c \to z$, $g: d \to z$ (given by (F1) in Definition 8.1). Now, we take different representations $\big[Fa_1(x), c'\big] = [x, c]$ and $\big[Fa_2(y), d'\big] = [y, d]$ given by $a_1: c \to c', a_2: d \to d' \in \mathcal{C}$ and by (F1) we get $z' \in \mathcal{C}^0$ and $f': c' \to z', g': d' \to z'$. Now, again by (F1) we find $z'' \in \mathcal{C}^0$ and $f'': z \to z'', g'': z' \to z''$. Hence, we have two parallel maps

$$c \underset{f'' \circ f}{\overset{g'' \circ f' \circ a_1}{\rightrightarrows}} z''$$

and by (F2) we find $n \in \mathcal{C}^0$ and $k_1: z'' \to n$ such that $k_1 \circ g'' \circ f' \circ a_1 = k_1 \circ f'' \circ f$. Now, we have two parallel maps

$$d \underset{k_1 \circ g'' \circ g' \circ a_2}{\overset{k_1 \circ f'' \circ g}{\rightrightarrows}} n$$

and hence by (F2) we find $m \in \mathcal{C}^0$ and $k_2: n \to m$ such that

$$k_2 \circ k_1 \circ f'' \circ g = k_2 \circ k_1 \circ g'' \circ g' \circ a_2.$$

Now, with $k := k_2 \circ k_1$ we get that

$$[x, c] * [y, d] = \big[Ff(x) * Fg(y), z\big]$$

$$
\begin{aligned}
&= \Big[F(k \circ f'')\big(Ff(x) * Fg(y)\big), m\Big] \\
&= \big[F(k \circ f'' \circ f)(x) * F(k \circ f'' \circ g)(y), m\big] \\
&= \big[F(k \circ g'' \circ f' \circ a_1)(x) * F(k \circ g'' \circ g' \circ a_2)(y), m\big] \\
&= \Big[F(k \circ g'')\big(F(f' \circ a_1)(x) * F(g' \circ a_2)(y)\big), m\Big] \\
&= \big[F(f' \circ a_1)(x) * F(g' \circ a_2)(y), z'\big] \\
&= \big[F(a_1)(x), c'\big] * \big[F(a_2)(y), d'\big]
\end{aligned}
$$

using that $F(k \circ f'')$ and $F(k \circ g'')$ preserve the $*$ structure and that F is functorial. Hence, $*$ is well-defined.

Secondly, given a finite number of elements in $\mathcal{O}_{\square}$ using (F1) repeatedly, we can choose representations that are all in the same $F(c)$ for some suitable $c \in \mathcal{C}^0$. Hence, $\mathcal{O}_{\square}$ inherits all the abelian group (unital ring, resp.) axioms from $F(c)$. In the unital ring case, the unit of $\mathcal{O}_{\square}$ is given by $[1, c]$ for any $c \in \mathcal{C}^0$.

Finally, by the construction of the abelian group (unital ring, resp.) structure on $\mathcal{O}_{\square}$ the canonical maps $\iota_c: F(c) \to \mathcal{O}_{\square}$ are group homomorphisms (unital ring homomorphism, resp.). Furthermore, they are natural in $f: c \to d \in \mathcal{C}$ as $\big(\iota_d \circ Ff\big)(x) = \big[Ff(x), d\big] = [x, c] = \iota_c(x)$ for all $x \in F(c)$.

In other words, in any case for $\mathcal{D}$ the constructed $\mathcal{O}_{\square}$ is an object in $\mathcal{D}$ and $\iota: F \to \mathcal{O}_{\square}$ is a cone. Now, we want to prove that they indeed form a colimit of the diagram F, that is, that $\iota: F \to \mathcal{O}_{\square}$ is universal.

Proposition 8.6. *The object $\mathcal{O}_{\square} \in \mathcal{D}$ and the cone $\iota: F \Rightarrow \mathcal{O}_{\square}$ form a colimit of the diagram F in the category $\mathcal{D}$, that is, we have $\varinjlim F = \mathcal{O}_{\square}$.*

Proof. Consider an object $S \in \mathcal{D}$ and a cone under F with nadir S called $\alpha: F \Rightarrow S$. Now, α is given by morphisms $\alpha_c: F(c) \to S$ (that is, maps/group homomorphism/unital ring homomorphism, respectively, depending on $\mathcal{D}$) that are natural in c, that is, $\alpha_c = \alpha_d \circ Ff$ for all $f: c \to d \in \mathcal{C}$. Now, we can define the map

$$\tilde{\alpha}: \mathcal{O}_{\square} \to S, \qquad [x, c] \mapsto \alpha_c(x).$$

This map is well-defined by the naturality of the α_c. Furthermore, if we have an operation $*$ on objects of $\mathcal{D}$ (that is, in case $\mathcal{D} = \mathtt{AbGroup}$ or $\mathcal{D} = \mathtt{Ring}$) then

$$
\begin{aligned}
\tilde{\alpha}\big([x, c] * [y, d]\big) &= \tilde{\alpha}\Big(\big[Ff(x) * Fg(y), z\big]\Big) = \alpha_z\big(Ff(x) * Fg(y)\big) \\
&= \alpha_z\big(Ff(x)\big) * \alpha_z\big(Fg(y)\big) = \tilde{\alpha}\Big(\big[Ff(x), z\big]\Big) * \tilde{\alpha}\Big(\big[Fg(y), z\big]\Big) \\
&= \tilde{\alpha}\big([x, c]\big) * \tilde{\alpha}\big([y, d]\big)
\end{aligned}
$$

for $c, d \in \mathcal{C}^0, x \in F(c), y \in F(d)$ and $z \in \mathcal{C}^0, f: c \to z, g: d \to z$ (given by (F1) in Definition 8.1). Additionally, in case $\mathcal{D} = \mathtt{Ring}$ we have

$$\tilde{\alpha}\big([1, c]\big) = \alpha_c(1) = 1 \in S.$$

Hence (for all cases of $\mathcal{D}$) $\tilde{\alpha}$ is indeed a morphism in $\mathcal{D}$.

Now, by construction $\tilde{\alpha}$ is the unique morphism such that $\alpha_c = \tilde{\alpha} \circ \iota_c$ for all $c \in \mathcal{C}^0$.

Finally, since the construction of the colimits in $\mathtt{Set}$, $\mathtt{AbGroup}$, and $\mathtt{Ring}$ are the same, we get the following Corollary 8.7.

Corollary 8.7. *The forgetful functors*

$$
\begin{aligned}
&U: \mathtt{Ring} \to \mathtt{AbGroup}, \\
&U: \mathtt{AbGroup} \to \mathtt{Set}, \ \textit{and}
\end{aligned}
$$

$$U\colon \mathtt{Ring} \to \mathtt{Set}$$

create and preserve filtered colimits.

Furthermore, the previous results can be extended to the category of topological spaces and continuous maps $\mathtt{Top}$. If we consider the case $\mathcal{D} = \mathtt{Top}$ and define the topology on $\mathcal{O}_{\Box}$ to be the quotient topology of the disjoint union topology of the $F(c)$, then by definition, the maps $\iota_c\colon F(c) \to \mathcal{O}_{\Box}$ are continuous and for any topological space S a map $\tilde{\alpha}\colon \mathcal{O}_{\Box} \to S$ is continuous if and only if $\alpha_c = \tilde{\alpha} \circ \iota_c$ is continuous. Hence, they also form a colimit in the category of topological spaces $\mathtt{Top}$.

Corollary 8.8. *The topological space $\mathcal{O}_{\Box}$ with the canonical topology and the cone $\iota\colon F \Rightarrow \mathcal{O}_{\Box}$ form a colimit of the diagram F in the category* $\mathtt{Top}$*.*

The same is true for the category of R-modules and R-module homomorphisms $R\text{-}\mathtt{Mod}$ for the fixed (commutative and unital) ring R. In the case that $\mathcal{D} = R\text{-}\mathtt{Mod}$ the abelian groups $F(c)$ have an R-module structure and the morphisms Ff are R-module homomorphisms. So we can define an R-module structure on $\mathcal{O}_{\Box}$ by $r \cdot [x, c] := [rx, c]$, which is well-defined as $Ff(rx) = rFf(x)$. Then by definition, the group homomorphisms $\iota_c\colon F(c) \to \mathcal{O}_{\Box}$ are R-module homomorphisms and for any cone given by R-module homomorphisms $\alpha_c\colon F(c) \to S$ the unique map $\tilde{\alpha}\colon \mathcal{O}_{\Box} \to S$ is an R-module homomorphism. Hence, $\mathcal{O}_{\Box}$ also forms a colimit in the category of R-modules $R\text{-}\mathtt{Mod}$.

Corollary 8.9. *The abelian group $\mathcal{O}_{\Box}$ with the canonical R-module structure and the cone $\iota\colon F \Rightarrow \mathcal{O}_{\Box}$ form a colimit of the diagram F in the category $R\text{-}\mathtt{Mod}$.*

Finally, we collect some results that give some insight into the relation between filtered colimits and inductive limits. For this, we introduce the notion of a *final* functor, which is a functor we can precompose our diagram with, to change the domain of the diagram but preserve the colimit.

Definition 8.10. For two categories $\mathcal{I}, \mathcal{J}$, a functor $F\colon \mathcal{I} \to \mathcal{J}$ is called *final* if for all $j \in \mathcal{J}$ the comma categories $j \downarrow F$ are non-empty and connected (that is, there is a finite zigzag of arrows between any two objects).

Lemma 8.11. *Given a final functor $L\colon \mathcal{I} \to \mathcal{J}$ and a functor $F\colon \mathcal{J} \to \mathcal{C}$ such that the colimit of FL exists, then the colimit of F exists and is canonically isomorphic to the colimit of FL.*

Proof. See [Mac71, IX 3) Theorem 1].

Lemma 8.12 ([AM15, Lemma 3.12])**.** *If $\mathcal{J}$ is a countable, filtered category, then there is a final functor $(\mathbb{N}, \leq) \to J$.*

Proof. See [AM15, Lemma 3.12].

Hence, given a diagram over a *countable* filtered category in a category where inductive limits exist (for example, $\mathtt{Top}$, $R\text{-}\mathtt{Mod}$, $\mathtt{Ring}$, $\mathtt{AbGroup}$ or $\mathtt{Set}$), the colimit of this diagram exists and is given by an inductive limit.

9. Ore diagrams of bimodules and covariance rings

In this section, we want to introduce certain diagrams in Rings and construct a strong covariance ring which is a bicategorical limit for these diagrams. The theory of diagrams and their covariance rings we want to build on is developed in [Mey22a] for the bicategory $\mathsf{Rings_u}$ of unital rings and bimodules. So, we need to restrict ourselves to this full subbicategory $\mathsf{Rings_u} \subset \mathsf{Rings}$. In [Mey22a] diagrams over arbitrary categories in $\mathsf{Rings_u}$, and lax and strong covariance rings are introduced. It is shown that assembling the data from such a diagram into one universal ring gives the lax covariance ring. So, we can explicitly compute *lax* covariance rings. For diagrams in the subbicategory $\mathsf{Rings_{fgp}} \subset \mathsf{Rings_u}$ of fgp bimodules, the strong covariance ring is given by the Cohn localization of the lax covariance ring at specific maps. Now, the Cohn localization of a ring is a rather abstract object and not very explicit. To work with the strong covariance ring in Section 11, we need a more explicit version of the strong covariance ring. In Section 11, we are only interested in diagrams over Ore monoids, that is, small categories with only one object and a certain cancellation condition on the morphisms. We call these diagrams *Ore diagrams*. Then, for an Ore diagram $\mathcal{F}$ in $\mathsf{Rings_{fgp}}$, we construct the strong covariance ring explicitly. We do this by first constructing a related lax diagram $\mathcal{O}$ in $\mathsf{Rings_u}$ and showing that the strong covariance ring of $\mathcal{F}$ is given by the lax covariance ring of $\mathcal{O}$, which we can compute explicitly.

9.1. Diagrams over monoids and covariance rings.

We start by recalling the notion of a monoid and a (lax) diagram over a monoid in the bicategory $\mathsf{Rings_u}$.

Definition 9.1. A *monoid* is a small category with exactly one object. In other words, it is a set P equipped with an associative binary operation and an identity element 1.

From now on, we fix a monoid P for this subsection. Sometimes we might also view this category P as a strict bicategory with only identity 2-arrows and still call it P. Now, we define a (lax) diagram over a monoid P in $\mathsf{Rings_u}$. Since we are only interested in (lax) diagrams over monoids in $\mathsf{Rings_u}$ in this section, we sometimes just write "(lax) diagrams" and mean "(lax) diagrams over monoids in $\mathsf{Rings_u}$".

Definition 9.2. A *lax diagram in* $\mathsf{Rings_u}$ *over the monoid* P is a strictly unital morphism $\mathcal{F}: P \to \mathsf{Rings_u}$, that is, it is described by the data $\mathcal{F} = (P, F_p, \mu_{p,q})$ with

- a unital ring $F_1 \in \mathsf{Rings_u^0}$;
- for every $p \in P$ an F_1,F_1-bimodule F_p (for $p = 1$ this is the trivial F_1,F_1-bimodule F_1);
- for every $p, q \in P$ an F_1,F_1-bimodule homomorphism $\mu_{p,q}: F_p \otimes_{F_1} F_q \to F_{pq}$;

such that $\mu_{1,q}$ is the left uniter l_{F_q}, $\mu_{p,1}$ is the right uniter r_{F_p} and the diagram

$$\begin{array}{ccc} F_p \otimes_{F_1} F_q \otimes_{F_1} F_t & \xrightarrow{\mathrm{id}\otimes\mu_{q,t}} & F_p \otimes_{F_1} F_{qt} \\ \Big\downarrow{\scriptstyle \mu_{p,q}\otimes\mathrm{id}} & & \Big\downarrow{\scriptstyle \mu_{p,qt}} \\ F_{pq} \otimes_{F_1} F_t & \xrightarrow{\mu_{pq,t}} & F_{pqt} \end{array}$$

commutes for all $p, q, t \in P$.

If, additionally, the $\mu_{p,q}$ are isomorphisms (that is, if $\mathcal{F}$ is a homomorphism) we call $\mathcal{F}$ a *diagram in* $\mathsf{Rings_u}$.[5]

Definition 9.3. A lax diagram $\mathcal{F}$ is called fgp, if for all $p \in P$ the F_1,F_1-bimodules F_p are fgp, that is, if the right F_1-modules F_p are finitely generated and projective.

[5] Sometimes we call a diagram a *strong* diagram to distinguish it from a *lax* diagram. Note that not every lax diagram is a diagram, but every diagram is a lax diagram.

Note that a (lax) diagram in the bicategory $\mathsf{Rings_u}$ is analogously defined to a diagram in a category. Now, in category theory, we are interested in the limit of a diagram $\mathcal{F}$, which is a representing object of the cone functor, where a cone over $\mathcal{F}$ with summit D is a natural transformation $D \Rightarrow \mathcal{F}$. Similarly, a (lax) limit in a bicategory is defined by considering (lax) cones $D \Rightarrow \mathcal{F}$ and some universal object representing the (lax) cone morphism. Note that (lax) limits in bicategories are only defined up to equivalence, which in the bicategory of bimodules $\mathsf{Rings_u}$ is *Morita equivalence*.[6] To get a more rigid definition of a bicategorical (lax) limit in $\mathsf{Rings_u}$, we introduce (lax) covariance rings. We define a ***(lax) covariance ring*** of a lax diagram $\mathcal{F}$ to be the universal unital ring representing particularly manageable cones, called ***(lax) covariant representations***. The (lax) covariant ring is actually defined up to unital ring isomorphisms, and by Proposition 9.12, it is also a (lax) limit for the diagram $\mathcal{F}$.

We start by defining a (lax) covariant representation of $\mathcal{F}$ on a unital ring D.

Definition 9.4. For a lax diagram in $\mathsf{Rings_u}$ given by $\mathcal{F} = (P, F_p, \mu_{p,q})$ and a unital ring $D \in \mathsf{Rings_u^0}$, a *lax covariant representation of* $\mathcal{F}$ on D is described by the data $\nu = (\tilde{\nu}_1, \nu_p)$ with

- a left F_1-action on D turning it into an F_1,D-bimodule (with the trivial right D-action on D), that is, a ring homomorphism $\tilde{\nu}_1 \colon F_1 \to \mathrm{End}_{-,D}(D)$;
- for every $p \in P$ an F_1,D-bimodule homomorphism $\nu_p \colon F_p \otimes_{F_1} D \to D$;

such that ν_1 is the left uniter l_{F_1} and the diagram

$$\begin{array}{ccc} F_p \otimes_{F_1} F_q \otimes_{F_1} D & \xrightarrow{\mathrm{id}\square\nu_q} & F_p \otimes_{F_1} D \\ \big\downarrow{\scriptstyle \mu_{p,q}\square\mathrm{id}} & & \big\downarrow{\scriptstyle \nu_p} \\ F_{pq} \otimes_{F_1} D & \xrightarrow{\nu_{pq}} & D \end{array}$$

commutes.

If the ν_p are isomorphisms, we call ν a *strong covariant representation of* $\mathcal{F}$ on D; or just *covariant representation of* $\mathcal{F}$ on D.

Now, we want to translate this definition into a different form, using the tensor-hom adjunction given by Theorem 9.6. We denote the left F_1-action on D multiplicatively.

Remark 9.5. It is easy to check, that for two unital rings F_1, D and an F_1,D-bimodule V the abelian group $\mathrm{End}_{-,D}(V)$ has an F_1,F_1-bimodule structure, given by

$$(a \cdot f)(v) := af(v)$$
$$(f \cdot a)(v) := f(av)$$

for all $f \in \mathrm{End}_{-,D}(V)$ and $a \in F_1$.

Theorem 9.6 (Tensor-hom adjunction). *For two unital rings F_1, D an F_1,F_1-bimodule F_p and an F_1,D-bimodule V the map*

$$\mathrm{Hom}_{F_1,D}(F_p \otimes_{F_1} V, V) \to \mathrm{Hom}_{F_1,F_1}\big(F_p, \mathrm{End}_{-,D}(V)\big), \qquad f \mapsto \tilde{f}$$

given by $\tilde{f}(x)(v) = f(x \otimes v)$, is an isomorphism of groups.

Proof. See [Mac63, Corollary 3.2 on p.145].

Now, in our definition of a (lax) covariant representation, we impose $\nu_1 = l_{F_1}$ or, in other words, ν_1 and $\tilde{\nu}_1$ are adjoint to one another. Thus, we do not need to include $\tilde{\nu}_1$ in the data of a (lax) covariant representation ν, and from now on, we just write $\nu = (\nu_p)$.

[6] For more details on bicategorical (lax) limits in the bicategory $\mathsf{Rings_u}$ see [Mey22a].

In general, using the tensor-hom adjunction, the F_1,D-bimodule homomorphisms $\nu_p\colon F_p \otimes_{F_1} D \to D$ correspond exactly to their adjoint F_1,F_1-bimodule homomorphisms $\tilde{\nu}_p\colon F_p \to \mathrm{End}_{-,D}(D)$, where the F_1,F_1-bimodule structure on $\mathrm{End}_{-,D}(D)$ is given by Remark 9.5.

Furthermore, we have an isomorphism $\mathrm{End}_{-,F_1}(D) \cong D$ of unital rings and F_1,F_1-bimodules, where the F_1,F_1-bimodule structure on D is given by taking $\tilde{\nu}_1$ and multiplying from the left or right, respectively. Thus, the F_1,D-bimodule homomorphisms $\nu_p\colon F_p \otimes_{F_1} D \to D$ correspond exactly to F_1,F_1-bimodule homomorphisms $\overline{\nu}_p\colon F_p \to D$ for all $p \in P$, where $\overline{\nu_1}\colon F_p \to D$ is a ring homomorphism that induces the F_1,F_1-bimodule structure on D.

Proposition 9.7. *A lax covariant representation $\nu = (\nu_p)$ of a lax diagram $\mathcal{F}$ on a unital ring D is given exactly by maps $\overline{\nu}_p\colon F_p \to D$ for all $p \in P$ such that*

- *$\overline{\nu_1}$ is a ring homomorphism;*
- *for all $p \in P$ the maps $\overline{\nu}_p$ are group homomorphisms;*
- *for all $x \in F_p$, $y \in F_q$ we have $\overline{\nu_{pq}}\big(\mu_{p,q}(x \otimes y)\big) = \overline{\nu}_p(x)\overline{\nu}_p(y)$.*

Proof. The discussion above explains how we get the maps $\overline{\nu}_p$ and their respective homomorphism properties. What is left to translate is the commutativity of the diagram and that the maps $\overline{\nu}_p$ are F_1,F_1-bimodule homomorphism. It is easy to check that the commutativity of the diagram means exactly that

$$\overline{\nu_{pq}}\big(\mu_{p,q}(x \otimes y)\big) = \overline{\nu}_p(x)\overline{\nu}_p(y)$$

for all $x \in F_p$, $y \in F_q$. By construction this already implies that the maps $\overline{\nu}_p$ are F_1,F_1-bimodule homomorphisms (where the F_1,F_1-bimodule structure on D is the obvious structure induced by $\overline{\nu_1}$).

Remark 9.8. Now, using this notation, a lax covariant representation $\nu = (\overline{\nu}_p)$ is strong if and only if the induced maps

$$\nu_p\colon F_p \otimes_{F_1} D \to D, \qquad x \otimes d \mapsto \overline{\nu}_p(x)d,$$

are bijective.

Thus, the three maps

- $\nu_p\colon F_p \otimes_{F_1} D \to D$;
- $\tilde{\nu}_p\colon F_p \to \mathrm{End}_{-,D}(D)$;
- $\overline{\nu}_p\colon F_p \to D$;

all contain the same data and hence describe the same object. We try to distinguish between the three different versions by using the notation ν_p, $\tilde{\nu}_p$ and $\overline{\nu}_p$.

Now, we want to define the covariance ring as the representing object of a fitting functor from the category of unital rings `Ring` to the category of sets `Set`.

Definition 9.9. For a lax diagram $\mathcal{F}$ and a unital ring D, we define the set of all covariant representations $\mathrm{CovRep}(D,\mathcal{F})$ of $\mathcal{F}$ on D and turn them into a functor

$$\begin{aligned} \mathrm{CovRep}(-,\mathcal{F})\colon \mathtt{Ring} &\to \mathtt{Set}, \\ D &\mapsto \mathrm{CovRep}(D,\mathcal{F}), \\ f\colon D \to D' &\mapsto f_{\square}\colon \mathrm{CovRep}(D,\mathcal{F}) \to \mathrm{CovRep}(D',\mathcal{F}), \end{aligned}$$

where $f_{\square}$ maps $\overline{\nu}_p$ to $f \circ \overline{\nu}_p$ for all $p \in P$.

Definition 9.10. For a lax diagram $\mathcal{F}$, we call a unital ring $\Omega \in$ `Ring` that represents the functor $\mathrm{CovRep}(-,\mathcal{F})$ a *strong covariance ring of $\mathcal{F}$*; or just *covariance ring of $\mathcal{F}$*.

Note that covariance rings are representing objects in the category of unital rings and hence unique up to unital isomorphisms (by the Yoneda Lemma, see [Mac71, p. 61]).

Remark 9.11. Similarly, one can define the *lax covariance ring* of $\mathcal{F}$ as the representing object of the functor sending a ring D to all *lax* covariant representations of $\mathcal{F}$ on D. The lax covariance ring is given by the following construction:

Given a lax diagram $\mathcal{F} = (P, F_p, \mu_{p,q})$ we can take the direct sum of all the F_1,F_1-bimodules

$$F := \bigoplus_{p \in P} F_p$$

to get an abelian group. For $p, q \in P$ and $a \in F_p$, $b \in F_q$ we define the *multiplication*

$$a \cdot b := \mu_{p,q}(a \otimes b) \in F_{pq}$$

and extend it linearly to F, that is, for $f_p \in F_p$, $e_q \in F_q$ (and almost all of the f_p and e_q are equal to zero, respectively)

$$\sum_{p \in P} f_p \cdot \sum_{q \in P} e_q := \sum_{p,q \in P} f_p \cdot e_q \in F.$$

Now, because of the first conditions in Definition 9.2 this multiplication has a unit $1 \in F_1 \subset F$ and because of the commutative diagram it is associative. Since the $\mu_{p,q}$ are F_1,F_1-bimodule maps, the defined multiplication is distributive with respect to the abelian group structure on F. Hence, F is a (not necessarily commutative) unital *P-graded ring*.

This defines the lax covariance ring since using Proposition 9.7 a lax covariant representation is given by abelian group homomorphisms $\overline{\nu}_p\colon F_p \to D$, which we can assemble into the group homomorphism

$$\overline{\nu}\colon \bigoplus_{p \in P} F_p \to D.$$

The extra conditions on the $(\overline{\nu}_p)_{p \in P}$ translate exactly into $\overline{\nu}$ being a unital ring homomorphism. Thus, one can check that we indeed get a natural isomorphism

$$\mathrm{CovRep}_{\mathsf{lax}}(-, \mathcal{F}) \cong \mathtt{Ring}\Big(\bigoplus_{p \in P} F_p, -\Big)$$

and hence the covariance ring is given by $\bigoplus_{p \in P} F_p$. For a detailed proof, see [Mey22a, Proposition 4.6.7].

Now, both constructions actually give us a (lax) bicategorical limit.

Proposition 9.12 (compare [Mey22a, Proposition 4.7.15]). *Let $\mathcal{F}\colon P \to \mathrm{Rings}_{\mathrm{u}}$ be a diagram. A lax covariance ring of $\mathcal{F}$ is also a lax limit of this diagram, and a covariance ring of $\mathcal{F}$ is also a limit.*

Proof. See [Mey22a, Proposition 4.7.15].

Hence, to find a (lax) limit for a (lax) diagram $\mathcal{F} = (P, F_p, \mu_{p,q})$, it is sufficient to find a (lax) covariance ring. The construction of the lax covariance ring is immediate. Constructing the strong covariance ring of a (lax) diagram can be more difficult. First, we need to get a better understanding of what it means for a lax covariant representation $(\overline{\nu}_p\colon F_p \to D)_{p \in P}$ of $\mathcal{F}$ on some unital ring D to be strong. Denote by $S := \bigoplus_{p \in P} F_p$ the lax covariance ring of $\mathcal{F}$. We define the right S-module homomorphisms

$$\Psi_p\colon F_p \otimes_{F_1} S \to S, \qquad x_p \otimes (x_{p_i})_{i=1}^n \mapsto \big(\mu_{p,p_i}(x_p \otimes x_{p_i})\big)_{i=1}^n$$

for each $p \in P$. Now, according to the following Lemma 9.13 the lax covariant representation $(\nu_p)_{p \in P}$ is strong if and only if the maps $\Psi_p \otimes_S \mathrm{id}_D$ are invertible for all $p \in P$.

Lemma 9.13. *For each $p \in P$, the map ν_p is invertible if and only if $\Psi_p \otimes_S \mathrm{id}_D$ is invertible.*

Proof. The diagram

$$\begin{array}{ccc} F_p \otimes_{F_1} D & \xrightarrow{\ \nu_p\ } & D \\ \uparrow \cong & & \uparrow \cong \\ (F_p \otimes_{F_1} S) \otimes_S D & \xrightarrow[\Psi_p \otimes \mathrm{id}]{} & S \otimes_S D \end{array}$$

commutes, since the multiplication of S on D is given by the unital ring homomorphism $\nu: S \to D$.

Thus, for each $p \in P$ the map ν_p is invertible if and only if the map $\Psi_p \otimes_S \mathrm{id}_D$ is invertible.

In [Mey22a] it is shown that for fgp diagrams[7] the strong covariance ring exists and is given by the *Cohn localization* of the lax covariance ring at the maps Ψ_p (see [Mey22a, Proposition 4.6.13]). Now, the Cohn localization of a ring is a rather abstract object and not very explicit. In Section 11 we need an explicit version of a covariance ring for fgp diagrams over well-behaved monoids, namely Ore monoids. So in the next subsection, we explicitly construct a covariance ring for fgp Ore diagrams.

9.2. **Covariance ring of fgp Ore diagrams.** In this subsection, we want to explicitly construct a covariance ring $\mathcal{O}_\mathsf{F}$ for an fgp Ore diagram $\mathcal{F}$, that is, a diagram over an Ore monoid in the subbicategory $\mathsf{Rings}_{\mathsf{fgp}} \subset \mathsf{Rings}_{\mathsf{u}}$ of unital rings and fgp bimodules. Note that this covariance ring is a Cohn localization of the lax covariance ring, so our construction might be useful as an explicit construction of a Cohn localization for certain rings. We start by introducing what it means for a monoid P to be *Ore*.

Definition 9.14 (compare [AM15, Definition 3.7]). For a monoid P, the two properties

- (O1) For all $x_1, x_2 \in P$, there are $y_1, y_2 \in P$ such that $x_1 y_1 = x_2 y_2$.
- (O2) For all $x, y_1, y_2 \in P$ such that $x y_1 = x y_2$, there is a $z \in P$ such that $y_1 z = y_2 z$.

are called the *Ore conditions*. We call P an *Ore monoid*, if it has these two properties.

Note that the Ore conditions are some sort of cancellation conditions on the monoid.

Example 9.15. Any group is an Ore monoid and any commutative monoid is an Ore monoid. For more examples of Ore monoids see [AM15, Examples 3.8–3.11].

We impose these conditions because, first, there is a convenient construction of a group completion for an Ore monoid (see Definition 9.16), which we are going to use. Secondly, these conditions are equivalent to the coslice category $\mathcal{C}_P := * \downarrow P^{\mathrm{op}}$ (see Definition 9.17) being filtered, which we need to get well-behaved colimits of diagrams over these categories (in Definition 9.23 and Definition 9.29).

So from now on, we fix an Ore monoid P and an fgp Ore diagram $\mathcal{F} = (P, F_p, \mu_{p,q})$, that is, a diagram $\mathcal{F}: P \to \mathsf{Rings}_{\mathsf{u}}$ over an Ore monoid P such that the bimodules F_p are fgp.

[7] In [Mey22a] the notion of a diagram is kept more general. A diagram is defined to be a morphism over a category with finitely many objects.

9.2.1. *The strategy.* Our construction can be separated into the following steps:

Step 1: We construct a fitting group G (namely the group completion of P, see Definition 9.16) and an induced lax diagram $\mathcal{O}$ over G in $\mathsf{Rings_u}$ given by $\mathcal{O}_g$ for all $g \in G$ (see Corollary 9.37).

Step 2: We prove that the covariant representations functor $\mathrm{CovRep}(-, \mathcal{F})$ of $\mathcal{F}$ and the lax covariant representations functor $\mathrm{CovRep}_{\mathsf{lax}}(-, \mathcal{O})$ of $\mathcal{O}$ are naturally isomorphic (Theorem 9.48).

Step 3: We conclude by the Yoneda Lemma ([Mac71, p. 61]) that since the functors are naturally isomorphic their representing objects are isomorphic as well, and thus the covariance ring $\mathcal{O}_{\mathsf{F}}$ of $\mathcal{F}$ is given by the lax covariance ring of $\mathcal{O}$, which by Remark 9.11 is explicitly given by

$$\mathcal{O}_{\mathsf{F}} \cong \bigoplus_{g \in G} \mathcal{O}_g.$$

Step 1: Constructing a related lax diagram over the group completion. First, we need to define the group completion G of the Ore monoid P.

Definition 9.16. For an Ore monoid P define the *group completion* G of P as the set of equivalence classes

$$G := P \times P \,/_{\sim},$$

where we define $(p_1, p_2) \sim (q_1, q_2)$ if there are $t_1, t_2 \in P$ with $p_1 t_1 = q_1 t_2$ and $p_2 t_1 = q_2 t_2$. We denote an element of G represented by (p_1, p_2) as $p_1 p_2^{-1} \in G$. The group operation in G is given by

$$p_1 p_2^{-1} \cdot q_1 q_2^{-1} := (p_1 t_1)(q_2 t_2)^{-1}$$

where $t_1, t_2 \in P$ are such that $p_2 t_1 = q_1 t_2$ (given by (O1)) and the neutral element of the group is $e := 11^{-1} \in G$ the equivalence class of $(1, 1)$.

For more details on this group construction and proof that this is indeed a group, see [AM15, after Definition 3.6]. The canonical monoid homomorphism $P \to G$, $p \mapsto p1^{-1}$ is not necessarily injective, but we still sometimes write "$p \in G$" and mean the element $p1^{-1} \in G$ for $p \in P$.

Next, we want to construct a lax diagram $\mathcal{O} = (G, \mathcal{O}_g, \omega_{g,h})$, that is, a strictly unital morphism $\mathcal{O}\colon G \to \mathsf{Rings_u}$.

We start by constructing a unital ring $\mathcal{O}_e$ as the filtered colimit of a functor we need to define. Afterward, we analogously define the abelian groups $\mathcal{O}_g$ for any $g \in G$. To justify our notation, we prove that for $g := e$, we have $\mathcal{O}_g \cong \mathcal{O}_e$ as abelian groups. Finally, we show that they can be assembled into a strictly unital morphism $\mathcal{O}\colon G \to \mathsf{Rings_u}$, that is, a lax diagram over G in $\mathsf{Rings_u}$.

First, we construct a functor to $\mathtt{Ring}$ whose colimit we define as $\mathcal{O}_e$. We introduce the domain category.

Definition 9.17. For a monoid P define the associated coslice category $\mathcal{C}_P := * \downarrow P^{\mathrm{op}}$ with

- P as the set of objects;
- $P \times P$ as the set of arrows, where $(p, q)\colon p \to pq$ for $p, q \in P$; and
- composition is defined as

$$(pq, t) \cdot (p, q) := (p, qt)$$

  for $p, q, t \in P$.

Note that the unit arrows are given by $(p, 1)$ for all $p \in P$.

The category $\mathcal{C}_P$ is closely related to the monoid P. If we unpack the definition of the category $\mathcal{C}_P$ being filtered, it is easy to check that the conditions (F1) and

(F2) directly translate into the *Ore conditions* (O1) and (O2). Since we assumed P to be Ore, we get that $\mathcal{C}_P$ is filtered.

Now, for morphisms $(p,q) \in \mathcal{C}_P$, we want to define unital ring homomorphisms $\phi_{p,q}$.

Definition 9.18. For $p, q \in P$ define the maps

$$\phi_{p,q}\colon \mathrm{End}_{-,F_1}(F_p) \to \mathrm{End}_{-,F_1}(F_{pq}), \qquad T \mapsto \mu_{p,q} \circ (T \otimes_{F_1} \mathrm{id}_{F_q}) \circ \mu_{p,q}^{-1}.$$

Remark 9.19. It is easy to check, that for a unital ring F_1 and a right F_1-module M the abelian group $\mathrm{End}_{-,F_1}(M)$ is a unital ring, where multiplication is given by composition.

Applying the Remark 9.19 to $M = F_p$ and $M = F_{pq}$ we see that both the domain and the codomain of $\phi_{p,q}$ are unital rings, and hence we can formulate the following Lemma 9.20.

Lemma 9.20. *The maps $\phi_{p,q}$ defined above are unital ring homomorphisms.*

Proof. For $\phi_{p,q}$ the additivity is immediate from the additivity of $\mu_{p,q}$, $\mu_{p,q}^{-1}$ and $- \otimes \mathrm{id}$. For the multiplicativity we have

$$\begin{aligned}
\phi_{p,q}(S \circ T) &= \mu_{p,q} \circ (S \circ T \otimes_{F_1} \mathrm{id}_{F_q}) \circ \mu_{p,q}^{-1} \\
&= \mu_{p,q} \circ (S \otimes_{F_1} \mathrm{id}_{F_q}) \circ (T \otimes_{F_1} \mathrm{id}_{F_q}) \circ \mu_{p,q}^{-1} \\
&= \mu_{p,q} \circ (S \otimes_{F_1} \mathrm{id}_{F_q}) \circ \mu_{p,q}^{-1} \circ \mu_{p,q} \circ (T \otimes_{F_1} \mathrm{id}_{F_q}) \circ \mu_{p,q}^{-1} \\
&= \phi_{p,q}(S) \circ \phi_{p,q}(T)
\end{aligned}$$

for all $S, T \in \mathrm{End}_{F_1,F_1}(F_p)$. Finally, we have $\phi_{p,q}(\mathrm{id}) = \mathrm{id}$ and hence $\phi_{p,q}$ is indeed a unital ring homomorphism.

Next, we need that the $\phi_{p,q}$ respect the concatenation structure on $\mathcal{C}_P$.

Lemma 9.21. *For all $p, q, t \in P$ we have*

(1) $\phi_{p,qt} = \phi_{pq,t} \circ \phi_{p,q}$; *and*
(2) $\phi_{p,1} = \mathrm{id}_{\mathrm{End}_{-,F_1}(F_p)}$.

Proof. The first statement is equivalent to that the diagram

$$\begin{array}{ccccccc}
F_{pq} \otimes_R F_t & \xleftarrow[\square]{\mu_{p,q} \square_R \mathrm{id}} & F_p \otimes_R F_q \otimes_R F_t & \xleftarrow{T \square \mathrm{id} \square \mathrm{id}} & F_p \otimes_R F_q \otimes_R F_t & \xrightarrow[\square]{\mu_{p,q} \square_R \mathrm{id}} & F_{pq} \otimes_R F_t \\
\square \big\downarrow \mu_{pq,t} & & \square \big\downarrow \mathrm{id} \square_R \mu_{q,t} & & \square \big\downarrow \mathrm{id} \square_R \mu_{q,t} & & \square \big\downarrow \mu_{pq,t} \\
F_{pqt} & \xleftarrow[\square]{\mu_{p,qt}} & F_p \otimes_R F_{qt} & \xleftarrow{T \square \mathrm{id}} & F_p \otimes_R F_{qt} & \xrightarrow[\square]{\mu_{p,qt}} & F_{pqt}
\end{array}$$

commutes for all $T \in \mathrm{End}_{-,F_1}(F_p)$. Now, this diagram commutes, since the left and right square commute by Definition 9.2 and the middle square commutes because tensor is a bifunctor.

The second statement is equivalent to the naturality of $\mu_{p,1} = r_{F_p}$, the right uniter, in T.

Now, by Remark 9.19 the sets $\mathrm{End}_{-,F_1}(F_p)$ are unital rings and by Lemma 9.20 the maps $\phi_{p,q}$ are unital ring homomorphisms. Furthermore, by Lemma 9.21 the maps $\phi_{p,q}$ behave in a functorial way. Hence, we can define the following functor.

Definition 9.22. We define a functor from the filtered category $\mathcal{C}_P$ to the category $\mathtt{Ring}$ of unital rings and unital ring homomorphisms via

$$\begin{aligned}
E_{\mathrm{F}}\colon \mathcal{C}_P &\to \mathtt{Ring}, \\
p &\mapsto \mathrm{End}_{-,F_1}(F_p), \\
(p,q) &\mapsto \phi_{p,q}.
\end{aligned}$$

Next, we want to define $\mathcal{O}_e := \operatorname{colim} E_{\mathsf{F}}$ to be the colimit of this functor. Since P is Ore, the domain category $\mathcal{C}_P$ is filtered and hence the colimit of E_{F} is a filtered colimit $\mathcal{O}_e := \varinjlim E_{\mathsf{F}}$ in $\mathtt{Ring}$. Thus, we can use our results from Section 8. In the case that P is countable (or equivalently $\mathcal{C}_P$ is countable) Lemma 8.12 gives us the existence of a final functor and Lemma 8.11 gives us that the colimit of E_{F} is given as the inductive limit of the composition with the final functor. In the general case, we need to refer to Definition 8.3 and Proposition 8.6.

Let us recall the explicit construction of the unital colimit ring $\mathcal{O}_e$ and universal cone $\iota\colon E_{\mathsf{F}} \Rightarrow \mathcal{O}_e$ for this filtered diagram E_{F}.

Definition 9.23. Define the set

$$\mathcal{O}_{\sqcup,e} := \bigsqcup_{p\in P} \operatorname{End}_{-,F_1}(F_p)$$

and the equivalence relation generated by

$$(x,p) \sim \big(\phi_{p,q}(x), pq\big)$$

for all $p,q \in P$ and $x \in \operatorname{End}_{-,F_1}(F_p)$. Let $\mathcal{O}_e$ be the set of equivalence classes with elements denoted as $[x,p] \in \mathcal{O}_e$. Define a ring structure on $\mathcal{O}_e$ via

$$[x,p] + [y,q] := \big[\phi_{p,t}(x) + \phi_{q,s}(y), pt\big],$$
$$[x,p]\cdot[y,q] := \big[\phi_{p,t}(x) \circ \phi_{q,s}(y), pt\big]$$

for $p,q \in P$, $x \in \operatorname{End}_{-,F_1}(F_p)$, $y \in \operatorname{End}_{-,F_1}(F_q)$ and $t,s \in P$ with $pt = qs$ (given by (O1)). Furthermore, we have canonical unital ring homomorphisms

$$\iota_p\colon \operatorname{End}_{-,F_1}(F_p) \to \mathcal{O}_e, \qquad x \mapsto [x,p],$$

that assemble into a cone $\iota\colon E_{\mathsf{F}} \Rightarrow \mathcal{O}_e$.

Now, applying Proposition 8.6 to the functor E_{F} gives us the following Corollary 9.24.

Corollary 9.24. *The unital ring $\mathcal{O}_e$ and the cone $\iota\colon E_{\mathsf{F}} \Rightarrow \mathcal{O}_e$ form a colimit for the functor E_{F}.*

In other words, for any unital ring $S \in \mathtt{Ring}$ and cone $\alpha\colon E_{\mathsf{F}} \Rightarrow S$, there is a unique unital ring homomorphism $\tilde{\alpha}\colon \mathcal{O}_e \to S$ such that $\alpha_p = \tilde{\alpha} \circ \iota_p$ for all $p \in P$.

Next, we want to generalize the construction of the ring $\mathcal{O}_e$ to abelian groups $\mathcal{O}_g$ for all $g \in G$. To justify our notation, we want that $\mathcal{O}_g \cong \mathcal{O}_e$ as an abelian group if $g := e \in G$ is the neutral element, which is not obvious by construction but proven in Lemma 9.32.

First, we want to define a generalization $\mathcal{C}_P^g$ of the category $\mathcal{C}_P$. We do not quite get $\mathcal{C}_P^e = \mathcal{C}_P$ as one might wish, but we get Lemma 9.31, which is sufficient so that $\mathcal{O}_g \cong \mathcal{O}_e$.

Definition 9.25 ([AM15, Definition 3.14])**.** Fix an element $g \in G$ and define the set of representatives

$$R_g := \big\{(p_1,p_2) \in P \times P \mid p_1 p_2^{-1} = g \in G\big\}.$$

Define the associated category $\mathcal{C}_P^g$ with

- R_g as the set of objects;
- $R_g \times P$ as the set of arrows, where $(p_1,p_2,q)\colon (p_1,p_2) \to (p_1 q, p_2 q)$; and
- composition is defined as

  $$(p_1 q, p_2 q, t)\cdot(p_1,p_2,q) = (p_1,p_2,qt).$$

  for $p_1,p_2,q,t \in P$.

It is easy to check that $\mathcal{C}_P^g$ are indeed categories for all $g \in G$. Furthermore, if P is an Ore monoid, then the categories $\mathcal{C}_P^g$ are filtered by [AM15, Lemma 3.15].

For morphisms $(p_1, p_2, q) \in \mathcal{C}_P^g$, we want to define abelian group homomorphisms $\phi_{p_1,p_2,q}$.

Definition 9.26. For $(p_1, p_2) \in R_g$ and $q \in P$ define the maps

$$\phi_{p_1,p_2,q}\colon \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1}) \to \mathrm{Hom}_{-,F_1}(F_{p_2q}, F_{p_1q}), \quad T \mapsto \mu_{p_1,q} \circ (T \otimes_{F_1} \mathrm{id}_{F_q}) \circ \mu_{p_2,q}^{-1}.$$

Lemma 9.27. *For $(p_1, p_2) \in R_g$ the maps $\phi_{p_1,p_2,q}$ are group homomorphisms. Furthermore, we have $\phi_{p_1q,p_2q,t} \circ \phi_{p_1,p_2,q} = \phi_{p_1,p_2,qt}$ and $\phi_{p_1,p_2,1} = \mathrm{id}_{\mathrm{Hom}_{-,F_1}(F_{p_2},F_{p_1})}$.*

Proof. The proof is analogous to the proofs of Lemma 9.20 and Lemma 9.21.

Hence, the constructions combine into a functor $H_{\mathrm{F},g}$ analogously to E_{F}.

Definition 9.28. For $g \in G$ we define a functor from the filtered category $\mathcal{C}_P^g$ to the category $\mathtt{AbGroup}$ of abelian groups and group homomorphisms via

$$\begin{aligned} H_{\mathrm{F},g}\colon \mathcal{C}_P^g &\to \mathtt{AbGroup}, \\ (p_1, p_2) &\mapsto \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1}), \\ (p_1, p_2, q) &\mapsto \phi_{p_1,p_2,q}. \end{aligned}$$

As before, we can explicitly construct a colimit for the functor $H_{\mathrm{F},g}$ by applying our results from Section 8 to this explicit case. Let us recall how the colimit group $\mathcal{O}_g$ and universal cone $\iota_g\colon H_{\mathrm{F},g} \Rightarrow \mathcal{O}_g$ are defined.

Definition 9.29. Define the set

$$\mathcal{O}_{\sqcup,g} := \bigsqcup_{(p_1,p_2)\in R_g} \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1})$$

and the equivalence relation generated by

$$\big(x, (p_1, p_2)\big) \sim \big(\phi_{p_1,p_2,q}(x), (p_1q, p_2q)\big)$$

for all $(p_1, p_2) \in R_g$, $q \in P$ and $x \in \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1})$. Let $\mathcal{O}_g$ be the set of equivalence classes with elements denoted as $[x, (p_1, p_2)] \in \mathcal{O}_g$. Define an abelian group structure on $\mathcal{O}_g$ via

$$\big[x, (p_1, p_2)\big] + \big[y, (q_1, q_2)\big] := \big[\phi_{p_1,p_2,t}(x) + \phi_{q_1,q_2,s}(y), (p_1t, p_2t)\big]$$

for $(p_1, p_2), (q_1, q_2) \in R_g$, $x \in \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1})$, $y \in \mathrm{Hom}_{-,F_1}(F_{q_2}, F_{q_1})$ and $t, s \in P$ such that $p_1t = q_1s$ and $p_2t = q_2s$ (given by $p_1p_2^{-1} = g = q_1q_2^{-1}$).

Furthermore, we have canonical group homomorphisms

$$\iota_{p_1,p_2}\colon \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1}) \to \mathcal{O}_g$$

for all $(p_1, p_2) \in R_g$ that assemble into a cone $\iota_g\colon H_{\mathrm{F},g} \Rightarrow \mathcal{O}_g$.

Now, applying Proposition 8.6 to the functor $H_{\mathrm{F},g}$ gives us the following Corollary 9.30.

Corollary 9.30. *The abelian group $\mathcal{O}_g$ and the cone $\iota_g\colon H_{\mathrm{F},g} \Rightarrow \mathcal{O}_g$ form a colimit for the functor $H_{\mathrm{F},g}$.*

Finally, we want to argue why, if $g := e \in G$ is the neutral element, we get $\mathcal{O}_g \cong \mathcal{O}_e$ as abelian groups.

Lemma 9.31. *The functor*

$$\begin{aligned} d\colon \mathcal{C}_P &\to \mathcal{C}_P^e, \\ p &\mapsto (p, p), \\ (p, q) &\mapsto (p, p, q), \end{aligned}$$

is final.

Proof. It is easy to check that this is indeed a functor. For finality, we note first that for $(p_1, p_2) \in R_e$ the comma category $(p_1, p_2) \downarrow d$ has as objects $t \in P$ such that $p_1 t = p_2 t$ and arrows $t \rightsquigarrow s$ are given by $q \in P$ such that $tq = s$. Now, $(p_1, p_2) \in R_e$ is equivalent to the existence of a $t \in P$ such that $p_1 t = p_2 t$. Hence, t is an object in the comma category, that is, it is non-empty. To show it is connected we take $t, s \in (p_1, p_2) \downarrow d$ and by (O1) we find $x_1, x_2 \in P$ such that $tx_1 = sx_2$. Hence, we get the zigzag of arrows

$$t \rightsquigarrow tx_1 = sx_2 \leftsquigarrow s$$

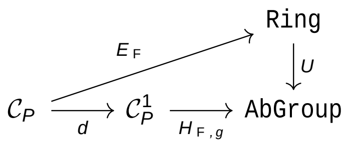
in $(p_1, p_2) \downarrow d$ connecting t and s.

Lemma 9.32. *Let $g := e \in G$ be the neutral element in G. Then the abelian group $\mathcal{O}_g$ is isomorphic to $\mathcal{O}_e$ (as abelian groups).*

Proof. Consider the forgetful functor $U\colon \mathtt{Ring} \to \mathtt{AbGroup}$ and $g := e \in G$, the neutral element. By Corollary 9.24 and Corollary 8.7 the colimit of

$$U \circ E_{\mathrm{F}} \colon \mathcal{C}_P \to \mathtt{Ring} \to \mathtt{AbGroup}$$

is given by $U(\mathcal{O}_e)$, that is, by $\mathcal{O}_e$ viewed as an abelian group. Furthermore, the diagram



commutes and thus $U(\mathcal{O}_e)$ is the colimit of $H_{\mathrm{F},g} \circ d$. By Lemma 9.31, the functor d is final, and hence by Lemma 8.11 the colimit of $H_{\mathrm{F},g}$ is canonically isomorphic to $U(\mathcal{O}_e)$. Finally, by Corollary 9.30 the colimit of $H_{\mathrm{F},g}$ is $\mathcal{O}_g$, hence we have a canonical isomorphism $\mathcal{O}_g \cong U(\mathcal{O}_e)$ of abelian groups.

Next, we construct an $\mathcal{O}_e$,$\mathcal{O}_e$-bimodule structure on the $\mathcal{O}_g$ and $\mathcal{O}_e$,$\mathcal{O}_e$-bimodule homomorphisms $\omega_{g,h}\colon \mathcal{O}_g \otimes_{\mathcal{O}_e} \mathcal{O}_h \to \mathcal{O}_{gh}$ for all $g, h \in G$.

Definition 9.33. Consider $g, h \in G$ represented by $g = p_1 p_2^{-1}$, $h = q_1 q_2^{-1}$ and $t_1, t_2 \in P$ such that $p_2 t_1 = q_1 t_2$, and hence $gh = (p_1 t_1)(q_2 t_2)^{-1}$. We define the map

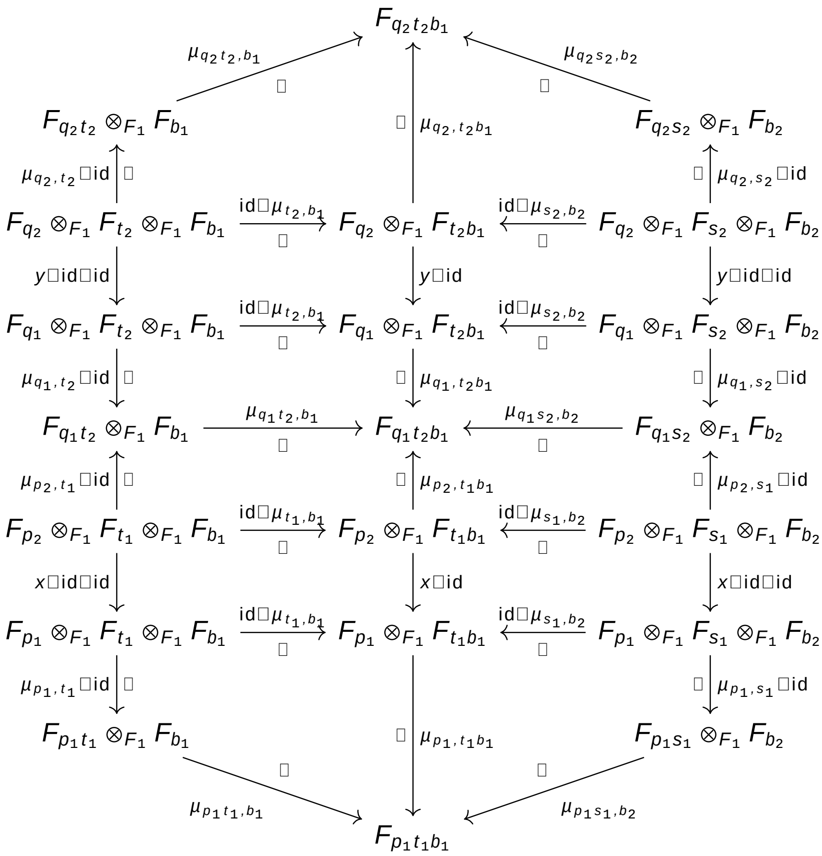
$$w_{g,h}\colon \mathcal{O}_g \times \mathcal{O}_h \to \mathcal{O}_{gh},$$
$$\big([x, (p_1, p_2)], [y, (q_1, q_2)]\big) \mapsto \big[\phi_{p_1,p_2,t_1}(x) \circ \phi_{q_1,q_2,t_2}(y), (p_1 t_1, q_2 t_2)\big].$$

Lemma 9.34. *The map $w_{g,h}$ is well-defined.*

Proof. For the definition we need to choose representatives $(p_1, p_2) \in R_g$, $(q_1, q_2) \in R_h$ for $g, h \in G$ and $t_1, t_2 \in P$ such that $p_2 t_1 = q_1 t_2$. First, we check the independence of the choice of t_1, t_2. So take t_1, t_2 as above and let $s_1, s_2 \in P$ with $p_2 s_1 = q_1 s_2$. Now, by (O1) we find $x_1, x_2 \in P$ with $t_1 x_1 = s_1 x_2$. Hence, we get $q_1 t_2 x_1 = p_2 t_1 x_1 = p_2 s_1 x_2 = q_1 s_2 x_2$ and by (O2) we find $n \in P$ such that $t_2 x_1 n = s_2 x_2 n$. Hence, with $b_1 := x_1 n$ and $b_2 := x_2 n$, we get $t_1 b_1 = s_1 b_2$ and $t_2 b_1 = s_2 b_2$. What is left to prove is that

$$\phi_{p_1 t_1, q_2 t_2, b_1}\big(\phi_{p_1,p_2,t_1}(x) \circ \phi_{q_1,q_2,t_2}(y)\big) = \phi_{p_1 s_1, q_2 s_2, b_2}\big(\phi_{p_1,p_2,s_1}(x) \circ \phi_{q_1,q_2,s_2}(y)\big),$$

which is equivalent to that the diagram



commutes. The commutativity of the small squares is either obvious or by Definition 9.2. Thus, the map is indeed independent of the choices of t_1, t_2.

Next, we want to check that the map is independent of the choices of the representatives of $[x, (p_1, p_2)]$ and $[y, (q_1, q_2)]$. Take $[x, (p_1, p_2)] = [x', (p_1', p_2')]$ and $[y, (q_1, q_2)] = [y', (q_1', q_2')]$ then there are $n, n' \in P$ such that $(p_1 n, p_2 n) = (p_1' n', p_2' n')$ and $\phi_{p_1, p_2, n}(x) = \phi_{p_1', p_2', n'}(x')$. Similarly, by definition we find $m, m' \in P$ such that $(q_1 m, q_2 m) = (q_1' m', q_2' m')$ and $\phi_{q_1, q_2, m}(y) = \phi_{q_1', q_2', m'}(y')$. Next, we take $t_1, t_2 \in P$ such that $p_2 n t_1 = q_1 m t_2$. Hence, we also have $p_2' n' t_1 = q_1' m' t_2$. So we get

$$
\begin{aligned}
\phi_{p_1, p_2, n t_1}(x) \circ \phi_{q_1, q_2, m t_2}(y) &= \phi_{p_1 n, p_2 n, t_1}\big(\phi_{p_1, p_2, n}(x)\big) \circ \phi_{q_1 m, q_2 m, t_2}\big(\phi_{q_1, q_2, m}(y)\big) \\
&= \phi_{p_1' n', p_2' n', t_1}\big(\phi_{p_1', p_2', n'}(x')\big) \circ \phi_{q_1' m', q_2' m', t_2}\big(\phi_{q_1', q_2', m'}(y')\big) \\
&= \phi_{p_1', p_2', n' t_1}(x') \circ \phi_{q_1', q_2', m' t_2}(y')
\end{aligned}
$$

using Lemma 9.27 and hence the map is well-defined.

Now, in particular, we get maps $w_{e,g} : \mathcal{O}_e \times \mathcal{O}_g \to \mathcal{O}_g$ and $w_{g,e} : \mathcal{O}_g \times \mathcal{O}_e \to \mathcal{O}_g$ for all $g \in G$ and it is easy to check that this defines an $\mathcal{O}_e$,$\mathcal{O}_e$-bimodule structure on $\mathcal{O}_g$ for every $g \in G$ (it basically boils down to checking that concatenation of maps is well-behaved).

Lemma 9.35. *The map $w_{g,h}$ is $\mathcal{O}_e$-balanced for all $g, h \in G$, and hence induces a unique $\mathcal{O}_e$,$\mathcal{O}_e$-bimodule homomorphism $\omega_{g,h} : \mathcal{O}_g \otimes_{\mathcal{O}_e} \mathcal{O}_h \to \mathcal{O}_{gh}$. Furthermore, $\omega_{e,g}$*

and $\omega_{g,e}$ are given by left and right multiplication and the diagram

$$
\begin{array}{ccc}
\mathcal{O}_g \otimes_{\mathcal{O}_e} \mathcal{O}_h \otimes_{\mathcal{O}_e} \mathcal{O}_k & \xrightarrow{\omega_{g,h} \otimes \mathrm{id}} & \mathcal{O}_{gh} \otimes_{\mathcal{O}_e} \mathcal{O}_k \\
\downarrow{\scriptstyle \mathrm{id} \otimes \omega_{h,k}} & & \downarrow{\scriptstyle \omega_{gh,k}} \\
\mathcal{O}_g \otimes_{\mathcal{O}_e} \mathcal{O}_{hk} & \xrightarrow{\omega_{g,hk}} & \mathcal{O}_{ghk}
\end{array}
$$

commutes for all $g, h, k \in G$.

Proof. For the additivity in both arguments, we note that since the map is independent of the choice of representative, we can, without loss of generality, assume that the two summands are in the same $\mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1})$. Then as $\phi_{p_1,p_2,t}$ is additive by Lemma 9.27 and composition distributes over addition, the additivity of $w_{g,h}$ follows for each argument.

For $g, h, k \in G$ and $\tilde{x} \in \mathcal{O}_g$, $\tilde{y} \in \mathcal{O}_h$ and $\tilde{z} \in \mathcal{O}_k$ we can choose representatives such that these elements are given by $\tilde{x} = [x, (p_1, p_2)]$, $\tilde{y} = [y, (p_2, p_3)]$ and $\tilde{z} = [z, (p_3, p_4)]$. Then the equality

$$w_{gh,k}\big(w_{g,h}(\tilde{x}, \tilde{y}), \tilde{z}\big) = w_{g,hk}\big(\tilde{x}, w_{h,k}(\tilde{y}, \tilde{z})\big) \tag{9.36}$$

is immediate from the associativity of composition. Now, (9.36) with $h := e$ already implies that $w_{g,h}$ is $\mathcal{O}_e$-balanced and compatible with the left and right multiplication by $\mathcal{O}_e$. Hence, we get the unique induced $\mathcal{O}_e$,$\mathcal{O}_e$-bimodule homomorphism $\omega_{g,h}$. We have defined the left and right multiplication to be exactly $\omega_{e,g}$ and $\omega_{g,e}$. Finally, the commutativity of the diagram follows from (9.36).

Finally, we can assemble all the above results. Since the group G is also a monoid we can view it as a category with one object and G as its set of morphisms. In the following, we view this category G as a strict bicategory with only identity 2-arrows.

Corollary 9.37. *The data*

$$\mathcal{O}: G \to \mathsf{Rings}_{\mathsf{u}},$$
$$g \mapsto \mathcal{O}_g$$

with the $\mathcal{O}_e$,$\mathcal{O}_e$-bimodule homomorphisms $\omega_{g,h}: \mathcal{O}_g \otimes_{\mathcal{O}_e} \mathcal{O}_h \to \mathcal{O}_{gh}$ for all $g, h \in G$ assembles into a strictly unital morphism of bicategories, that is, a lax diagram in $\mathsf{Rings}_{\mathsf{u}}$ over the group G.

Proof. By Definition 9.23 the object $\mathcal{O}_e$ is indeed a unital ring and by the discussion above Lemma 9.35 the $\mathcal{O}_g$ are indeed $\mathcal{O}_e$,$\mathcal{O}_e$-bimodules. Furthermore, because of Lemma 9.35 the $\omega_{g,h}$ are indeed $\mathcal{O}_e$,$\mathcal{O}_e$-bimodule homomorphisms and the necessary diagrams commute.

As discussed in Remark 9.11, we can define a G-graded ring $\mathcal{O}_{\mathsf{F}} := \bigoplus_{g \in G} \mathcal{O}_g$ induced by the diagram $\mathcal{O}$ that is the lax covariance ring of $\mathcal{O}$.

Definition 9.38. Define $\mathcal{O}_{\mathsf{F}}$ to be the G-graded ring

$$\mathcal{O}_{\mathsf{F}} := \bigoplus_{g \in G} \mathcal{O}_g = \bigoplus_{g \in G} \varinjlim_{(p_1,p_2) \in R_g} \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1}),$$

where multiplication is defined as

$$a \cdot b := \omega_{g,h}(a \otimes b) \in \mathcal{O}_{gh}$$

for $g, h \in G$, $a \in \mathcal{O}_g$, $b \in \mathcal{O}_h$ and extended distributively to $\mathcal{O}_{\mathsf{F}}$.

Since the maps $\omega_{g,h}: \mathcal{O}_g \otimes_{\mathcal{O}_e} \mathcal{O}_h \to \mathcal{O}_{gh}$ are not necessarily invertible, our lax diagram $\mathcal{O}$ is not a *strong* diagram. But the following technical lemma implies Proposition 9.40 saying that some $\omega_{g,h}$ actually are invertible. Note that the

$\mathrm{End}_{-,F_1}(F_{p_1})$-module structure on the homomorphism sets is given by pre- and postcomposition.

Lemma 9.39. *For $p \in P$, $(p_1, p_2) \in R_g$, the concatenation map*

$$I_{p,p_1,p_2} \colon \mathrm{Hom}_{-,F_1}(F_{p_1}, F_{pp_1}) \otimes_{\mathrm{End}_{-,F_1}(F_{p_1})} \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1}) \to \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{pp_1}),$$
$$f_1 \otimes f_2 \mapsto f_1 \circ f_2,$$

is an isomorphism of abelian groups, with

$$\phi_{pp_1,p_2,t}\big(I_{p,p_1,p_2}(f_1 \otimes f_2)\big) = I_{p,p_1t,p_2t}\big(\phi_{pp_1,p_1,t}(f_1) \otimes \phi_{p_1,p_2,t}(f_2)\big)$$

for all $f_1 \in \mathrm{Hom}_{-,F_1}(F_{p_1}, F_{pp_1})$, $f_2 \in \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1})$ and $t \in P$.

Proof. Using that $\mu_{p,q} \colon F_p \otimes_{F_1} F_q \to F_{pq}$ is an isomorphism (according to Definition 9.2), Theorem 5.10 and a canonical isomorphism, we get a chain of isomorphisms of abelian groups

$$\begin{array}{cc}
\mathrm{Hom}_{-,F_1}(F_{p_1}, F_{pp_1}) \otimes_{\mathrm{End}_{-,F_1}(F_{p_1})} \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1}), & \mu_{p,p_1}(x \otimes x_1)\varphi_1 \otimes y_1\varphi_2 \\
\downarrow \square & \uparrow \\
(F_{pp_1} \otimes_{F_1} F_{p_1}^{\square}) \otimes_{F_{p_1} \square_{F_1} F_{p_1}^{\square}} (F_{p_1} \otimes_{F_1} F_{p_2}^{\square}), & \big(\mu_{p,p_1}(x \otimes x_1) \otimes \varphi_1\big) \otimes \big(y_1 \otimes \varphi_2\big) \\
\uparrow \square & \uparrow \\
\big(F_p \otimes_{F_1} (F_{p_1} \otimes_{F_1} F_{p_1}^{\square})\big) \otimes_{F_{p_1} \square_{F_1} F_{p_1}^{\square}} (F_{p_1} \otimes_{F_1} F_{p_2}^{\square}), & \big(x \otimes (x_1 \otimes \varphi_1)\big) \otimes \big(y_1 \otimes \varphi_2\big) \\
\downarrow \square & \downarrow \\
F_p \otimes_{F_1} F_{p_1} \otimes_{F_1} F_{p_2}^{\square}, & x \otimes x_1\varphi_1(y_1) \otimes \varphi_2 \\
\downarrow \square & \downarrow \\
F_{pp_1} \otimes_{F_1} F_{p_2}^{\square}, & \mu_{p,p_1}\big(x \otimes x_1\varphi_1(y_1)\big) \otimes \varphi_2 \\
\downarrow \square & \downarrow \\
\mathrm{Hom}_{-,F_1}(F_{p_2}, F_{pp_1}), & \mu_{p,p_1}\big(x \otimes x_1\varphi_1(y_1)\big)\varphi_2
\end{array}$$

with $x \in F_p$, $x_1 \in F_{p_1}$, $\varphi_1 \in F_{p_1}^{\square}$, $y_1 \in F_{p_1}$ and $\varphi_2 \in F_{p_2}^{\square}$. For $f_1 = \mu_{p,p_1}(x \otimes x_1)\varphi_1$, $f_2 = y_1\varphi_2$ the chain of isomorphisms sends $f_1 \otimes f_2$ to

$$\mu_{p,p_1}\big(x \otimes x_1\varphi_1(y_1)\big)\varphi_2 = \mu_{p,p_1}(x \otimes x_1)\varphi_1(y_1\varphi_2) = f_1 \circ f_2.$$

Now, arbitrary f_1, f_2 are given by finite sums of these and thus are also mapped to their concatenation. Hence, the chain of isomorphisms is exactly the map I_{p,p_1,p_2} as defined above. Furthermore, we have

$$\begin{aligned}
\phi_{pp_1,p_2,t}\big(I_{p,p_1,p_2}(f_1 \otimes f_2)\big) &= \phi_{pp_1,p_2,t}(f_1 \circ f_2) \\
&= \mu_{pp_1,t} \circ \big((f_1 \circ f_2) \otimes_{F_1} \mathrm{id}_{F_t}\big) \circ \mu_{p_2,t}^{-1} \\
&= \mu_{pp_1,t} \circ (f_1 \otimes_{F_1} \mathrm{id}_{F_t}) \circ \mu_{p_1,t}^{-1} \circ \mu_{p_1,t} \circ (f_2 \otimes_{F_1} \mathrm{id}_{F_t}) \circ \mu_{p_2,t}^{-1} \\
&= \phi_{pp_1,p_1,t}(f_1) \circ \phi_{p_1,p_2,t}(f_2) \\
&= I_{p,p_1t,p_2t}\big(\phi_{pp_1,p_1,t}(f_1) \otimes \phi_{p_1,p_2,t}(f_2)\big)
\end{aligned}$$

for $f_1 \in \mathrm{Hom}_{-,F_1}(F_{p_1}, F_{pp_1})$, $f_2 \in \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1})$ and $t \in P$.

Proposition 9.40. *For all $p \in P$ (with the notation $p := p1^{-1} \in G$) and $g \in G$ the $\mathcal{O}_e$,$\mathcal{O}_e$-bimodule homomorphisms $\omega_{p,g} \colon \mathcal{O}_p \otimes_{\mathcal{O}_e} \mathcal{O}_g \to \mathcal{O}_{pg}$ are isomorphisms.*

Proof. The proof is basically just that the isomorphism I_{p,p_1,p_2} from Lemma 9.39 descends to the filtered colimits. We first prove surjectivity and then injectivity by reducing it to the fact that I_{p,p_1,p_2} is surjective and injective.

Take $g = g_1g_2^{-1}$, then $p = (pg_1)g_1^{-1}$ and $pg = (pg_1)g_2^{-1}$. Take $y \in \mathcal{O}_{pg}$. Then there is $(p_1, p_2) \in R_{pg}$ and $f \in \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1})$ such that $y = \big[f, (p_1, p_2)\big]$. Since

$p_1p_2^{-1} = pg = (pg_1)g_2^{-1}$, there are $t_1, t_2 \in P$ such that $p_1t_1 = pg_1t_2$ and $p_2t_1 = g_2t_2$. Thus, after rechoosing the representative of y, we can, without loss of generality, assume that $y = [f, (pp_1, p_2)]$ for $g = p_1p_2^{-1}$ and $f \in \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{pp_1})$. Since I_{p,p_1,p_2} is surjective (see Lemma 9.39) we find $\varphi_i \in \mathrm{Hom}_{-,F_1}(F_{p_1}, F_{pp_1})$ and $\psi_i \in \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1})$ such that $I_{p,p_1,p_2}(\sum_{i=1}^n \varphi_i \otimes \psi_i) = f$. Hence,

$$f = I_{p,p_1,p_2}\Big(\sum_{i=1}^n \varphi_i \otimes \psi_i\Big) = \sum_{i=1}^n \varphi_i \circ \psi_i.$$

Now, $x := \sum_{i=1}^n [\varphi_i, (pp_1, p_1)] \otimes [\psi_i, (p_1, p_2)] \in \mathcal{O}_p \otimes_{\mathrm{O}_e} \mathcal{O}_g$ and

$$\begin{aligned}
\omega_{p,g}(x) &= \omega_{p,g}\Big(\sum_{i=1}^n [\varphi_i, (pp_1, p_1)] \otimes [\psi_i, (p_1, p_2)]\Big)\\
&= \sum_{i=1}^n \omega_{p,g}\Big([\varphi_i, (pp_1, p_1)] \otimes [\psi_i, (p_1, p_2)]\Big)\\
&= \sum_{i=1}^n [\varphi_i \circ \psi, (pp_1, p_2)]\\
&= \Big[\sum_{i=1}^n \varphi_i \circ \psi, (pp_1, p_2)\Big]\\
&= [f, (pp_1, p_2)]\\
&= y
\end{aligned}$$

and thus $\omega_{p,g}$ is surjective.

For injectivity, we start with $x, y \in \mathcal{O}_p \otimes_{\mathrm{O}_e} \mathcal{O}_g$ such that $\omega_{p,g}(x) = \omega_{p,g}(y)$. Now, they are given by $x = \sum_{i=1}^n x_i \otimes x_i'$ and $y = \sum_{j=1}^m y_j \otimes y_j'$ for $x_i, y_j \in \mathcal{O}_p$ and $x_i', y_j' \in \mathcal{O}_g$. So for these finitely many elements $x_i, y_j \in \mathcal{O}_p$ we can choose representations that all lie in the same $\mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1})$, with $p_1p_2^{-1} = p1^{-1}$ after lifting these, we can, without loss of generality, assume that they lie in the same $\mathrm{Hom}_{-,F_1}(F_t, F_{pt})$. Similarly, for the finitely many $x_i', y_j' \in \mathcal{O}_g$, we find representations that are all in the same $\mathrm{Hom}_{-,F_1}(F_{g_2}, F_{g_1})$ with $g = g_1g_2^{-1}$. Now, we find $t_1, t_2 \in P$ such that $tt_1 = g_1t_2$. Hence, after lifting them again, we can, without loss of generality, assume that we find $(p_1, p_2) \in R_g$ and representations $x_i = [\varphi_i, (pp_1, p_1)]$, $y_j = [\psi_j, (pp_1, p_1)]$, $x_i' = [\varphi_i', (p_1, p_2)]$ and $y_j' = [\psi_j', (p_1, p_2)]$. Then,

$$\begin{aligned}
\omega_{p,g}(x) &= \omega_{p,g}\Big(\sum_{i=1}^n x_i \otimes x_i'\Big)\\
&= \sum_{i=1}^n \omega_{p,g}\Big([\varphi_i, (pp_1, p_1)] \otimes [\varphi_i', (p_1, p_2)]\Big)\\
&= \sum_{i=1}^n [\varphi_i \circ \varphi_i', (pp_1, p_2)]\\
&= \sum_{i=1}^n [I_{p,p_1,p_2}(\varphi_i \otimes \varphi_i'), (pp_1, p_2)]\\
&= \Big[I_{p,p_1,p_2}\Big(\sum_{i=1}^n \varphi_i \otimes \varphi_i'\Big), (pp_1, p_2)\Big]
\end{aligned}$$

and analogously $\omega_{p,g}(y) = [I_{p,p_1,p_2}(\sum_{j=1}^m \psi_j \otimes \psi_j'), (pp_1, p_2)]$. Thus $\omega_{p,g}(x) = \omega_{p,g}(y)$ implies

$$\Big[I_{p,p_1,p_2}\Big(\sum_{i=1}^n \varphi_i \otimes \varphi_i'\Big), (pp_1, p_2)\Big] = \Big[I_{p,p_1,p_2}\Big(\sum_{j=1}^m, \psi_j \otimes \psi_j'\Big), (pp_1, p_2)\Big]$$

and by definition of the equivalence relation (see Remark 8.4) and after using (O2), there is a $t \in P$ such that

$$\phi_{pp_1,p_2,t}\left(I_{p,p_1,p_2}\Big(\sum_{i=1}^{n}\varphi_i \otimes \varphi_i'\Big)\right) = \phi_{pp_1,p_2,t}\left(I_{p,p_1,p_2}\Big(\sum_{j=1}^{m}\psi_j \otimes \psi_j'\Big)\right).$$

Next we can apply the second part of Lemma 9.39 to get

$$I_{p,p_1t,p_2t}\Big(\sum_{i=1}^{n}\phi_{pp_1,p_1,t}(\varphi_i)\otimes\phi_{p_1,p_2,t}(\varphi_i')\Big) = I_{p,p_1t,p_2t}\Big(\sum_{j=1}^{m}\phi_{pp_1,p_1,t}(\psi_j)\otimes\phi_{p_1,p_2,t}(\psi_j')\Big)$$

and thus

$$\sum_{i=1}^{n}\phi_{pp_1,p_1,t}(\varphi_i)\otimes\phi_{p_1,p_2,t}(\varphi_i') = \sum_{j=1}^{m}\phi_{pp_1,p_1,t}(\psi_j)\otimes\phi_{p_1,p_2,t}(\psi_j')$$

as I_{p,p_1t,p_2t} is injective (by Lemma 9.39). Hence, their images under the well-defined map

$$\mathrm{Hom}_{-,F_1}(F_{p_1},F_{pp_1})\otimes_{\mathrm{End}_{-,F_1}(F_{p_1})}\mathrm{Hom}_{-,F_1}(F_{p_2},F_{p_1}) \to \mathcal{O}_p\otimes_{\mathcal{O}_e}\mathcal{O}_g,$$
$$a\otimes b \mapsto \big[a,(pp_1,p)\big]\otimes\big[b,(p_1,p_2)\big]$$

are equal. Now, the images are also given by x and y and thus $x = y$. Hence, the map is injective.

Now, this gives us some information about the map Ψ_p (from Lemma 9.13) for $p \in P$.

Corollary 9.41. *For $p \in P$ (with the notation $p := p1^{-1} \in G$) the map*

$$\Psi_p\colon \mathcal{O}_p\otimes_{\mathcal{O}_e}\bigoplus_{g\in G}\mathcal{O}_g \to \bigoplus_{g\in G}\mathcal{O}_g$$

is an isomorphism of abelian groups. Furthermore, for any lax covariant representation $(\overline{\alpha_g}\colon \mathcal{O}_g \to D)_{p\in P}$ of $\mathcal{O}$ on a unital ring D the maps

$$\alpha_p\colon \mathcal{O}_p\otimes_{\mathcal{O}_e} D \to D, \qquad x\otimes d \mapsto \overline{\alpha_g}(x)d,$$

are isomorphisms.

Proof. By Proposition 9.40 the maps $\omega_{p,g}\colon \mathcal{O}_p\otimes_{\mathcal{O}_e}\mathcal{O}_g \to \mathcal{O}_{pg}$ are invertible for all $g \in G$, and hence the direct sum of all these maps is also invertible. After using that direct sums and tensor products commute and that G is a group, the direct sum of the invertible maps $\omega_{p,g}$ is exactly the map $\Psi_p\colon \mathcal{O}_p\otimes_{\mathcal{O}_e}\bigoplus_{g\in G}\mathcal{O}_g \to \bigoplus_{g\in G}\mathcal{O}_g$ as defined in Lemma 9.13. Thus, Ψ_p is invertible.

Now, since Ψ_p is invertible, $\Psi_p\otimes\mathrm{id}$ is invertible and hence, by Lemma 9.13, the map α_p is invertible.

Step 2: Natural isomorphism between the (lax) covariant representations. Next, we prove that there are natural isomorphisms between the covariant representations of $\mathcal{F}$ and the lax covariant representations of $\mathcal{O}$. We do this by explicitly constructing a map between them and proving that it is bijective and natural.

We start with a technical result, which we need for this construction. Note that the canonical ring homomorphism $F_1 \cong \mathrm{End}_{-,F_1}(F_1) \to \mathcal{O}_e$ defines a left F_1-module structure on $\mathcal{O}_e$ that is given by $a\cdot[f,t] := [a\cdot f(-),t]$ for all $a \in F_1$, $t \in P$, $f \in \mathrm{End}_{-,F_1}(F_1)$.

Lemma 9.42. *For all $p \in P$ (with the notation $p := p1^{-1} \in G$) there is a well-defined F_1-balanced map*

$$F_p\times\mathcal{O}_e \to \mathcal{O}_p, \qquad (x,[f,t]) \mapsto \big[\mu_{p,t}(x\otimes f(-)),(pt,t)\big],$$

that descends to an isomorphism

$$\kappa_p\colon F_p\otimes_{F_1}\mathcal{O}_e \to \mathcal{O}_p, \qquad x\otimes[f,t] \mapsto \big[\mu_{p,t}(x\otimes f(-)),(pt,t)\big],$$

of abelian groups.

Proof. For $x \in F_p$, $t \in P$, $f \in \mathrm{End}_{-,F_1}(F_1)$, the map $\mu_{p,t}\big(x \otimes f(-)\big)$ is indeed a right F_1-module homomorphism since f and $\mu_{p,t}$ are. We also need to show that $\big(x, [f, t]\big)$ is mapped to the same element in $\mathcal{O}_p$, as $\Big(x, \big[\phi_{t,q}(f), tq\big]\Big)$ is for all $q \in P$. Now, $\Big(x, \big[\phi_{t,q}(f), tq\big]\Big)$ is mapped to $\Big[\mu_{p,tq}\big(x \otimes \phi_{t,q}(f)(-)\big), (ptq, tq)\Big]$, which is represented by the map that sends an element $z \in F_{tq}$ that is given by $z = \mu_{t,q}(a \otimes b)$ for $a \in F_t$, $b \in F_q$, to

$$\mu_{p,tq}\big(x \otimes \phi_{t,q}(f)(z)\big) = \mu_{p,tq}\Big(x \otimes \mu_{t,q}\big(f(a) \otimes b\big)\Big).$$

Now, $\big(x, [f, t]\big)$ is sent to

$$\Big[\mu_{p,t}\big(x \otimes f(-)\big), (pt, t)\Big] = \bigg[\phi_{pt,t,q}\Big(\mu_{p,t}\big(x \otimes f(-)\big)\Big), (ptq, tq)\bigg],$$

which is represented by a map that sends $z = \mu_{t,q}(a \otimes b)$ to

$$\mu_{pt,q}\Big(\mu_{p,t}\big(x \otimes f(a)\big) \otimes b\Big).$$

According to the commutative square in Definition 9.2, the two terms are equal and, since every element in F_{tq} is given by a finite sum of elements of the form $\mu_{t,q}(a \otimes b)$, the maps coincide. Thus, the map is indeed well-defined.

The map is F_1-balanced since the tensor product and $\mu_{p,t}$ are additive, and since the tensor product behaves well with multiplication of F_1 in the middle. Hence, it descends to the tensor product as a homomorphism κ_p of abelian groups.

What is left to show is that κ_p is an isomorphism. Note that the proof is quite similar to the proof of Proposition 9.40. We again first define an isomorphism similar to I_{p,p_1,p_2} in Lemma 9.39. For this, we make repeated use of Theorem 5.10. For all $t \in P$, the theorem gives us a chain of isomorphisms of abelian groups

$$\begin{array}{ccccccc} F_p \otimes_{F_1} \mathrm{Hom}_{-,F_1}(F_t, F_t) & \overset{\square}{\longleftarrow} & F_p \otimes_{F_1} F_t \otimes_{F_1} F_t^{\square} & \overset{\square}{\longrightarrow} & F_{pt} \otimes_{F_1} F_t^{\square} & \overset{\square}{\longrightarrow} & \mathrm{Hom}_{-,F_1}(F_t, F_{pt}), \\ x \otimes y\psi(-) & \longleftarrow\!\!\shortmid & x \otimes y \otimes \psi & \longmapsto & \mu_{p,t}(x \otimes y) \otimes \psi & \longmapsto & \mu_{p,t}(x \otimes y)\psi(-), \end{array}$$

which we call k_t. Hence, for any $f \in \mathrm{Hom}_{-,F_1}(F_t, F_t)$, we find $y_i \in F_t, \psi_i \in F_t^{\square}$ such that $f = \sum_{i=1}^n y\psi$ and the chain of isomorphisms sends

$$x \otimes f \mapsto \sum_{i=1}^n \mu_{p,t}(x \otimes y_i)\psi_i(-) = \mu_{p,t}\Big(x \otimes \sum_{i=1}^n y_i\psi_i(-)\Big) = \mu_{p,t}\big(x \otimes f(-)\big).$$

Note that $\kappa_p\big(x \otimes [f]\big) = \big[k_t(x \otimes f), (pt, t)\big]$ for all $f \in \mathrm{End}_{-,F_1}(F_t)$.

Next, we want to prove that κ_p is bijective using the fact that k_t is. For injectivity we take $\tilde{x}, \tilde{y} \in F_p \otimes_{F_1} \mathcal{O}_e$ with $\kappa_p(\tilde{x}) = \kappa_p(\tilde{y})$. Now, $\tilde{x}$ and $\tilde{y}$ are given by $\tilde{x} = \sum_{i=1}^n x_i \otimes [f_i]$ and $\tilde{y} = \sum_{j=1}^m y_j \otimes [h_i]$ with $n, m \in \mathbb{N}$, $x_i, y_j \in F_p$ and $[f_i], [h_j] \in \mathcal{O}_e$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Now, without loss of generality, we have $f_i, h_j \in \mathrm{End}_{-,F_1}(F_t)$ for some $t \in P$. Denote $x = \sum_{i=1}^n x_i \otimes f_i$ and $y = \sum_{j=1}^m y_j \otimes h_i$ then

$$\big[k_t(x), (pt, t)\big] = \kappa_p(x) = \kappa_p(y) = \big[k_t(y), (pt, t)\big]$$

and hence there is a $q \in P$ such that $\phi_{pt,t,q}\big(k_t(x)\big) = \phi_{pt,t,q}\big(k_t(y)\big)$. The calculations we did for well-definedness give that

$$\phi_{pt,t,q}\big(k_t(x)\big) = \sum_{i=1}^n \phi_{pt,t,q}\big(k_t(x_i \otimes f_i)\big) = \sum_{i=1}^n k_{tq}\big(x_i \otimes \phi_{t,q}(f_i)\big) = k_{tq}\Big(\sum_{i=1}^n x_i \otimes \phi_{t,q}(f_i)\Big)$$

Hence

$$k_{tq}\Big(\sum_{i=1}^{n} x_i \otimes \phi_{t,q}(f_i)\Big) = k_{tq}\Big(\sum_{j=1}^{m} y_j \otimes \phi_{t,q}(h_j)\Big)$$

and as k_{tq} is injective we get

$$\sum_{i=1}^{n} x_i \otimes \phi_{t,q}(f_i) = \sum_{j=1}^{m} y_j \otimes \phi_{t,q}(h_j)$$

and hence their images under the well-defined map

$$F_p \otimes_{F_1} \mathrm{End}_{-,F_1}(F_{tq}) \to F_p \otimes_{F_1} \mathcal{O}_e \qquad a \otimes b \mapsto a \otimes [b, tq]$$

are equal as well. Now, the images are given by $\tilde{x}$ and $\tilde{y}$. Hence, $\tilde{x} = \tilde{y}$ and the map is injective.

For surjectivity we take $(p_1, p_2) \in R_p$ and $[f, (p_1, p_2)] \in \mathcal{O}_p$. Now, there is a $t' \in P$ such that $p_1 t' = p p_2 t'$, and hence with $t = p_2 t'$ we get $p_1 p_2^{-1} = ptt^{-1}$. So, without loss of generality, we can start with $[f, (pt, t)] \in \mathcal{O}_p$ for $f \in \mathrm{Hom}_{-,F_1}(F_t, F_{pt})$. As k_t is surjective there is an

$$x = \sum_i x_i \otimes f_i \in F_p \otimes_{F_1} \mathrm{Hom}_{-,F_1}(F_t, F_t)$$

with $k_t(x) = f$. Hence, $\tilde{x} = \sum_i x_i \otimes [f_i, t] \in F_p \otimes_{F_1} \mathcal{O}_e$ and

$$\kappa_p(\tilde{x}) = \big[k_t(x), (pt, t)\big] = \big[f, (pt, t)\big]$$

so κ_p is indeed surjective.

We want to define a map $\beta_D\colon \mathrm{CovRep}(D, \mathcal{F}) \to \mathrm{CovRep}_{\mathrm{lax}}(D, \mathcal{O})$. We fix a unital ring D and a covariant representation $\nu = (\nu_p)$ of $\mathcal{F}$ on D (using Proposition 9.7). We now construct a lax covariant representation $(\Theta_g)_{g \in G}$ of $\mathcal{O}$ on D.

Definition 9.43. For $p \in P$ define the maps

$$\vartheta_p\colon \mathrm{End}_{-,F_1}(F_p) \to \mathrm{End}_{-,D}(D), \qquad T \mapsto \nu_p \circ (T \otimes_{F_1} \mathrm{id}_D) \circ \nu_p^{-1},$$

and for $(p_1, p_2) \in R_g$ define the maps

$$\vartheta_{p_1,p_2}\colon \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1}) \to \mathrm{End}_{-,D}(D), \qquad T \mapsto \nu_{p_1} \circ (T \otimes_{F_1} \mathrm{id}_D) \circ \nu_{p_2}^{-1}.$$

Lemma 9.44. *The maps ϑ_p are unital ring homomorphisms and the maps ϑ_{p_1,p_2} are group homomorphisms. Furthermore, for all $p, q \in P$ and $(p_1, p_2) \in R_g$ we have*

(1) $\vartheta_{pq} \circ \phi_{p,q} = \vartheta_p$; *and*
(2) $\vartheta_{p_1 q, p_2 q} \circ \phi_{p_1,p_2,q} = \vartheta_{p_1,p_2}$.

Hence, the ϑ_p can be combined into a cone $\vartheta_e\colon E_{\mathsf{F}} \Rightarrow \mathrm{End}_{-,D}(D)$ and the ϑ_{p_1,p_2} can be combined into a cone $\vartheta_g\colon H_{\mathsf{F},g} \Rightarrow \mathrm{End}_{-,D}(D)$.

Proof. The proof that the maps are homomorphisms is completely analogous to the proof of Lemma 9.20. The second equality is exactly that the diagram

$$\begin{array}{ccccccc}
F_{p_2 q} \otimes_{F_1} D & \xleftarrow[\cong]{\mu_{p_2,q} \otimes \mathrm{id}} & F_{p_2} \otimes_{F_1} F_q \otimes_D D & \xleftarrow{T \otimes \mathrm{id} \otimes \mathrm{id}} & F_{p_1} \otimes_{F_1} F_q \otimes_D D & \xrightarrow[\cong]{\mu_{p_1,q} \otimes \mathrm{id}} & F_{p_1 q} \otimes_{F_1} D \\
\cong\downarrow \nu_{p_2 q} & & \cong\downarrow \mathrm{id} \otimes \nu_q & & \cong\downarrow \mathrm{id} \otimes \nu_q & & \cong\downarrow \nu_{p_1 q} \\
D & \xleftarrow[\nu_{p_2}]{\cong} & F_{p_2} \otimes_{F_1} D & \xleftarrow[T \otimes \mathrm{id}]{} & F_{p_1} \otimes_{F_1} D & \xrightarrow[\nu_{p_1}]{\cong} & D
\end{array}$$

commutes. Now, this diagram commutes, since the left and right squares commute by Definition 9.4 and the middle square commutes by simple calculation.

The first equality follows from the second one, since we have $\phi_{p,q} = \phi_{p,p,q}$ and $\vartheta_p = \vartheta_{p,p}$ as maps.

Finally, since all the maps are homomorphisms in their respective categories and the equalities say exactly that they are natural, they indeed define cones.

To justify our ambiguous notation, we need to argue why for $g := e$ the cones ϑ_e and ϑ_g are "equal". Using the notation from Lemma 9.32 it is easy to check that

$$U\vartheta_e = \vartheta_g d,$$

that is, they are equal up to whiskering with the canonical functors.

Now, by Corollary 9.24 the unital ring $\mathcal{O}_e$ is a colimit of E_{F} and hence it factors through any cone. This of course also holds for the cones ϑ_e.

Corollary 9.45. *There is a unique unital ring homomorphism* $\Theta_e \colon \mathcal{O}_e \to \mathrm{End}_{-,D}(D)$ *such that the diagram*

$$\begin{array}{ccc} \mathrm{End}_{-,F_1}(F_p) & & \\ \downarrow{\scriptstyle \iota_p} & \searrow^{\vartheta_p} & \\ \mathcal{O}_e & \xrightarrow{\Theta_e} & \mathrm{End}_{-,D}(D) \end{array}$$

commutes for all $p \in P$.

Similarly, by Corollary 9.30 for all $g \in G$ the abelian group $\mathcal{O}_g$ is a colimit of $H_{\mathsf{F},g}$ and hence it factors through any cone. Again, this also holds for the cones ϑ_g.

Corollary 9.46. *There is a unique group homomorphism* $\Theta_g \colon \mathcal{O}_g \to \mathrm{End}_{-,D}(D)$ *such that the diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1}) & & \\ \downarrow{\scriptstyle \iota_{p_1,p_2}} & \searrow^{\vartheta_{p_1,p_2}} & \\ \mathcal{O}_g & \xrightarrow{\Theta_g} & \mathrm{End}_{-,D}(D) \end{array}$$

commutes for all $(p_1, p_2) \in R_g$.

Note that if $g := e \in G$ is the neutral element, we actually have an equality of maps $\Theta_g = \Theta_e$ after composing with the canonical isomorphisms.

Now, we use the characterization of covariant representations of $\mathcal{O}$ on D from Proposition 9.7. By abuse of notation, we write Θ_g for the map $\mathcal{O}_g \to D$, as well as the map $\mathcal{O}_g \to \mathrm{End}_{-,D}(D)$, using that $\mathrm{End}_{-,D}(D) \cong D$ as unital rings.

Proposition 9.47. *The data*

$$\Theta_e \colon \mathcal{O}_e \to D;$$
$$\Theta_g \colon \mathcal{O}_g \to D;$$

defines a lax covariant representation $(\Theta_g)_{g\in G}$ *of* $\mathcal{O}$ *on* D.

Proof. By construction, Θ_e is a unital ring homomorphism and the maps Θ_g are group homomorphisms for all $g \in G$. We need to prove that

$$\Theta_{gh}\big(\omega_{g,h}(x \otimes y)\big) = \Theta_g(x)\Theta_h(y)$$

for all $g, h \in G$ and $x \in \mathcal{O}_g$, $y \in \mathcal{O}_h$. For $x = \big[a, (p_1, p_2)\big] \in \mathcal{O}_g$, $y = \big[b, (q_1, q_2)\big] \in \mathcal{O}_h$, we find $t_1, t_2 \in P$ such that $p_2 t_1 = q_1 t_2$ (by (O1)). Hence, we have new representatives $x = \big[\phi_{p_1,p_2,t_1}(a), (p_1 t_1, p_2 t_1)\big]$ and $y = \big[\phi_{q_1,q_2,t_2}(b), (q_1 t_2, q_2 t_2)\big]$. Thus, without loss of generality, take $(p_1, p_2) \in R_g$, $(p_2, p_3) \in R_h$ and representing objects $a \in \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1})$, $b \in \mathrm{Hom}_{-,F_1}(F_{p_3}, F_{p_2})$ with $x = \big[a, (p_1, p_2)\big]$ and $y = \big[b, (p_2, p_3)\big]$. Then after identifying the unital rings $\mathrm{End}_{-,D}(D) \cong D$, we get

$$\begin{aligned} \Theta_{gh}\big(\omega_{g,h}(x \otimes y)\big) &= \Theta_{gh}\Big(\big[a \circ b, (p_1, p_3)\big]\Big) = (\Theta_{gh} \circ \iota_{p_1,p_3})(a \circ b) \\ &= \vartheta_{p_1,p_3}(a \circ b) = v_{p_1} \circ \big((a \circ b) \otimes \mathrm{id}_D\big) \circ v_{p_3}^{-1} \\ &= v_{p_1} \circ (a \otimes \mathrm{id}_D) \circ v_{p_2}^{-1} \circ v_{p_2} \circ (b \otimes \mathrm{id}_D) \circ v_{p_3}^{-1} \end{aligned}$$

$$= \vartheta_{p_1,p_2}(a) \circ \vartheta_{p_2,p_3}(b) = (\Theta_g \circ \iota_{p_1,p_2})(a) \circ (\Theta_h \circ \iota_{p_2,p_3})(b)$$
$$= \Theta_g(x)\Theta_h(y).$$

Hence, this indeed is a lax covariant representation.

Thus, we can now assign a lax covariant representation of $\mathcal{O}$ to every covariant representation of $\mathcal{F}$. So we only need to establish that this assignment is bijective and natural.

Theorem 9.48. *Consider an fgp Ore diagram $\mathcal{F} = (P, F_p, \mu_{p,q})$, that is, a strictly unital homomorphism $\mathcal{F}: P \to \mathsf{Rings}_{\mathrm{fgp}}$ over an Ore monoid P. Let $\mathcal{O} = (G, \mathcal{O}_g, \omega_{g,h})$ be the associated lax diagram in $\mathsf{Rings}_{\mathrm{u}}$ we constructed above (see Corollary 9.37). Then the map*

$$\beta_D : \mathrm{CovRep}(D, \mathcal{F}) \to \mathrm{CovRep}_{\mathrm{lax}}(D, \mathcal{O}), \qquad (\nu_p) \mapsto (\Theta_g),$$

where (Θ_g) is the induced lax covariant representation (see Proposition 9.47), defines a natural isomorphism $\beta: \mathrm{CovRep}(-, \mathcal{F}) \Rightarrow \mathrm{CovRep}_{\mathrm{lax}}(-, \mathcal{O})$.

Proof. By Proposition 9.47 the map is indeed well-defined, so we only need to prove that it is bijective and natural in D. We start with the injectivity.

Let (ϑ_p) and (ϑ'_p) be two covariant representations of $\mathcal{F}$, such that the induced lax covariant representations (Θ_g) and (Θ'_g) of $\mathcal{O}$ are equal. Then

$$\vartheta_{p_1,p_2} = \Theta_g \circ \iota_{p_1,p_2} = \Theta'_g \circ \iota_{p_1,p_2} = \vartheta'_{p_1,p_2}$$

are equal for all $g \in G$ and $(p_1, p_2) \in R_g$. By precomposing with the canonical group isomorphism (for $p = 1$ even ring isomorphism) $F_p \cong \mathrm{Hom}_{-,F_1}(F_1, F_p)$ we get $\vartheta_{p,1} = \tilde{\nu}_p$. Finally, $\tilde{\nu}_p = \vartheta_{p,1} = \vartheta'_{p,1} = \tilde{\nu}'_p$ and hence $\nu_p = \nu'_p$.

For surjectivity, we start with a lax covariant representation α of $\mathcal{O}$ on D. By Proposition 9.7 a lax covariant representation is given by group homomorphisms $\overline{\alpha_g}: \mathcal{O}_g \to D$ for all $g \in G$ such that $\overline{\alpha_e}: \mathcal{O}_e \to D$ is a ring homomorphism and for all $x \in \mathcal{O}_g$, $y \in \mathcal{O}_h$ we have $\overline{\alpha_{gh}}\big(\omega_{g,h}(x \otimes y)\big) = \overline{\alpha_g}(x)\overline{\alpha_h}(y)$. We now define

$$\overline{\nu}_p : F_p \xrightarrow{\;\square\;} \mathrm{Hom}_{-,F_1}(F_1, F_p) \xrightarrow{\;\iota_{p,1}\;} \mathcal{O}_p \xrightarrow{\;\overline{\alpha_p}\;} D$$

for all $p \in P$ (with the notation $p := p1^{-1} \in G$). First, $\overline{\nu}_p$ are group homomorphisms as concatenations of group homomorphisms. Furthermore, for $p = 1$, the map $\overline{\nu_1}$ is actually given by a concatenation of unital ring homomorphisms and thus is also a unital ring homomorphism. Next, we want to prove that

$$\overline{\nu_{pq}}\big(\mu_{p,q}(x \otimes y)\big) = \overline{\nu}_p(x)\overline{\nu}_p(y)$$

for all $x \in F_p$, $y \in F_q$. Note that for $x \in F_p$ we get the associated right F_1-module homomorphism $\mathrm{mult}_x : F_1 \to F_p$, $a \mapsto xa$, and analogously for $y \in F_q$, $\mu_{p,q}(x \otimes y) \in F_{pq}$. Now, $\phi_{p,1,q}(\mathrm{mult}_x) = \mu_{p,q}(x \otimes -)$ and hence

$$\big[\mathrm{mult}_x, (p, 1)\big] = \big[\phi_{p,1,q}(\mathrm{mult}_x), (pq, q)\big] = \big[\mu_{p,q}(x \otimes -), (pq, q)\big].$$

Thus, we get

$$\begin{aligned}\big[\mathrm{mult}_{\mu_{p,q}(x \square y)}, (pq, 1)\big] &= \big[\mu_{p,q}(x \otimes -) \circ \mathrm{mult}_y, (pq, 1)\big]\\ &= \omega_{p,q}\Big(\big[\mu_{p,q}(x \otimes -), (pq, q)\big] \otimes \big[\mathrm{mult}_y, (q, 1)\big]\Big)\\ &= \omega_{p,q}\Big(\big[\mathrm{mult}_x, (p, 1)\big] \otimes \big[\mathrm{mult}_y, (q, 1)\big]\Big).\end{aligned}$$

Thus, the equality

$$\begin{aligned}\overline{\nu_{pq}}\big(\mu_{p,q}(x \otimes y)\big) &= (\overline{\alpha_{pq}} \circ \iota_{pq,1})(\mathrm{mult}_{\mu_{p,q}(x \square y)})\\ &= \overline{\alpha_{pq}}\Big(\big[\mathrm{mult}_{\mu_{p,q}(x \square y)}, (pq, 1)\big]\Big)\end{aligned}$$

$$
\begin{aligned}
&= \overline{\alpha_{pq}}\Big(\omega_{p,q}\big(\big[\mathrm{mult}_x,(p,1)\big]\otimes\big[\mathrm{mult}_y,(q,1)\big]\big)\Big)\\
&= \overline{\alpha_p}\big(\big[\mathrm{mult}_x,(p,1)\big]\big)\overline{\alpha_q}\big(\big[\mathrm{mult}_y,(q,1)\big]\big)\\
&= \overline{\nu}_p(x)\overline{\nu}_p(y)
\end{aligned}
$$

holds. Hence, the defined $\nu = (\overline{\nu}_p)$ is indeed a lax covariant representation of $\mathcal{F}$ on D. Next, we want to prove that it is a strong covariant representation, that is, that $\nu_p\colon F_p \otimes_{F_1} D \to D$ is an isomorphism for all $p \in P$. The diagram

$$
\begin{array}{ccc}
F_p \otimes_{F_1} D & \xrightarrow{\quad \nu_p \quad} & D\\
\cong\big\downarrow & & \big\uparrow \cong\ \alpha_p\\
(F_p \otimes_{F_1} \mathcal{O}_e) \otimes_{\mathcal{O}_e} D & \xrightarrow[\kappa_p \otimes \mathrm{id}]{\cong} & \mathcal{O}_p \otimes_{\mathcal{O}_e} D
\end{array}
$$

commutes, as $\mu_{p,1}\big(x \otimes \mathrm{id}_{F_1}(-)\big) = \mathrm{mult}_x$. The left map is a canonical isomorphism, the bottom map is an isomorphism by Lemma 9.42 and the right map α_p is an isomorphism by Corollary 9.41. Thus, also ν_p is an isomorphism. Hence, $\nu = (\overline{\nu}_p)$ is a covariant representation of $\mathcal{F}$ on D.

Finally, we need to prove that β_D maps ν to α, that is, that $\Theta_g = \overline{\alpha_g}$ for all $g \in G$. If we identify $D \cong \mathrm{End}_{-,D}(D)$ we need to show that for $(p_1,p_2) \in R_g$ and $T \in \mathrm{Hom}_{-,F_1}(F_{p_2},F_{p_1})$ we have

$$
\nu_{p_1} \circ (T \otimes_{F_1} \mathrm{id}_D) \circ \nu_{p_2}^{-1} = \mathrm{mult}_{\overline{\alpha_g}([T,(p_1,p_2)])}.
$$

This is exactly that the diagram

$$
\begin{array}{ccc}
F_{p_2} \otimes_{F_1} D & \xrightarrow[\cong]{\nu_{p_2}} & D\\
T \otimes \mathrm{id}\big\downarrow & & \big\downarrow \mathrm{mult}_{\overline{\alpha_g}([T,(p_1,p_2)])}\\
F_{p_1} \otimes_{F_1} D & \xrightarrow[\cong]{\nu_{p_1}} & D
\end{array}
$$

commutes. The diagram indeed commutes, since we have

$$
\begin{aligned}
\big(\nu_{p_1} \circ (T \otimes_{F_1} \mathrm{id}_D)\big)(x \otimes d) &= \overline{\alpha_{p_1}}\big(\big[\mathrm{mult}_{Tx},(p_1,1)\big]\big)d\\
&= \overline{\alpha_{p_1}}\Big(\omega_{p_1p_2^{-1},p_2}\big(\big[T,(p_1,p_2)\big]\otimes\big[\mathrm{mult}_x,(p_2,1)\big]\big)\Big)d\\
&= \overline{\alpha_g}\big(\big[T,(p_1,p_2)\big]\big)\overline{\alpha_{p_2}}\big(\big[\mathrm{mult}_x,(p_2,1)\big]\big)d\\
&= (\mathrm{mult}_{\overline{\alpha_g}([T,(p_1,p_2)])} \circ \nu_{p_2})(x \otimes d)
\end{aligned}
$$

for all $x \in F_{p_2}, d \in D$. Hence, β_D is also surjective and thus a bijection.

Finally, we want to prove that β_D is natural in D. Take a unital ring homomorphism $f\colon D \to S$. If we unpack the definitions, what we need to prove boils down to

$$
f\Big(\big(\nu_{p_1} \circ (T \otimes \mathrm{id}) \circ \nu_{p_2}^{-1}\big)(1)\Big) = \big(f_*(\nu_{p_1}) \circ (T \otimes \mathrm{id}) \circ f_*(\nu_{p_2})\big)(1)
$$

for $g \in G, (p_1,p_2) \in R_g$, and $T \in \mathrm{Hom}_{-,F_1}(F_{p_2},F_{p_1})$. So if we take $x := \sum_{i=1}^n x_i \otimes d_i$ in $F_{p_2} \otimes_{F_1} D$ such that $\nu_{p_2}(x) = 1$, then $f_*(\nu_{p_2})\big(\sum_{i=1}^n x_i \otimes f(d_i)\big) = 1$ and hence

$$
\begin{aligned}
f\Big(\big(\nu_{p_1} \circ (T \otimes \mathrm{id}) \circ \nu_{p_2}^{-1}\big)(1)\Big) &= f\left(\big(\nu_{p_1} \circ (T \otimes \mathrm{id})\big)\Big(\sum_{i=1}^n x_i \otimes d_i\Big)\right)\\
&= \sum_{i=1}^n f\Big(\overline{\nu_{p_1}}(T(x_i))d_i\Big)
\end{aligned}
$$

$$= \sum_{i=1}^{n} f\Big(\overline{\nu_{p_1}}(T(x_i))\Big) f(d_i)$$

$$= f_*(\nu_{p_1})\Big(\sum_{i=1}^{n} T(x_i) \otimes f(d_i)\Big)$$

$$= \big(f_*(\nu_{p_1}) \circ (T \otimes \mathrm{id}) \circ f_*(\nu_{p_2})\big)(1).$$

So, β is indeed a natural isomorphism.

Step 3: The conclusion. In the end, we can draw our desired conclusion. From Theorem 9.48 it is immediate that the lax covariance ring $\mathcal{O}_\mathsf{F}$ (as defined in Definition 9.38) is a covariance ring of $\mathcal{F}$.

Corollary 9.49. *For an fgp Ore diagram $\mathcal{F} = (P, F_p, \mu_{p,q})$, the covariance ring of $\mathcal{F}$ is given by the lax covariance ring $\mathcal{O}_\mathsf{F}$ of the lax diagram $\mathcal{O}$ in $\mathsf{Rings_u}$, that is, by*

$$\mathcal{O}_\mathsf{F} = \bigoplus_{g \in G} \mathcal{O}_g = \bigoplus_{g \in G} \varinjlim_{(p_1,p_2) \in R_g} \mathrm{Hom}_{-,F_1}(F_{p_2}, F_{p_1}),$$

where multiplication is given by concatenation.

Proof. By Theorem 9.48 we have a natural isomorphism of functors

$$\mathrm{CovRep}(-, \mathcal{F}) \cong \mathrm{CovRep}_{\mathsf{lax}}(-, \mathcal{O}).$$

By Remark 9.11 the unital ring $\mathcal{O}_\mathsf{F}$ as constructed in Definition 9.38 is the representing object of the functor $\mathrm{CovRep}_{\mathsf{lax}}(-, \mathcal{O})$. The multiplication is given by the maps $\omega_{g,h}$, which are defined to be the concatenation of the equivalence classes of right F_1-module homomorphisms. By the Yoneda Lemma (see [Mac71, p. 61]) the unital ring $\mathcal{O}_\mathsf{F}$ is then also the representing object of $\mathrm{CovRep}(-, \mathcal{F})$ and hence the strong covariance ring of $\mathcal{F}$.

10. Ore diagrams of ample correspondences and groupoid models

In this section, we introduce diagrams in $\mathsf{Gr_a}$ over monoids P and their groupoid models. We recall the explicit construction of a groupoid model for tight Ore diagrams in $\mathsf{Gr_a}$, as done in [Alb15] and [Mey22b]. We follow closely the definitions and results from [Alb15] and [Mey22b]. A tight Ore diagram in $\mathsf{Gr_a}$ is analogous to an fgp Ore diagram in $\mathsf{Rings_u}$, and a groupoid model is analogous to a covariance ring. The construction of the groupoid model of a tight Ore diagram in $\mathsf{Gr_a}$ is also quite similar to the construction of a covariance ring of an fgp Ore diagram in $\mathsf{Rings_u}$ in Subsection 9.2. The similarity of the constructions leads to Section 11, where we prove that the Steinberg algebra of the groupoid model is isomorphic to the covariance ring of the induced fgp Ore diagram of bimodules. For a more detailed and general review of diagrams and groupoid models in GR, we refer to [Mey22b]. For a review of Ore diagrams and a detailed construction of the groupoid model of a tight Ore diagram in $\mathsf{Gr_{inj}}$, we refer to [Alb15].

10.1. Diagrams of ample correspondences and groupoid models.

We fix a monoid P. We start by defining a (proper/tight) diagram in $\mathsf{Gr_a}$.

Definition 10.1 (compare [Mey22b, Proposition 3.1]). A *diagram in* $\mathsf{Gr_a}$ is a *strictly unital homomorphism* $P \to \mathsf{Gr_a}$ over P, that is, it is described by the data $\mathsf{X} = (P, \mathcal{G}, \mathcal{X}_p, \mu_{p,q})$ with

- an ample groupoid $\mathcal{G}$;
- ample correspondences $\mathcal{X}_p\colon \mathcal{G} \leftarrow \mathcal{G}$ for all $p \in P$;
- isomorphisms of correspondences $\mu_{p,q}\colon \mathcal{X}_p \circ_{\mathsf{G}} \mathcal{X}_q \xrightarrow{\sim} \mathcal{X}_{pq}$ for all $p, q \in P$;

such that

(1) $\mathcal{X}_1$ for the unit $1 \in P$ is the identity correspondence $\mathcal{G}$ on $\mathcal{G}$;
(2) $\mu_{p,1}\colon \mathcal{X}_p \circ_{\mathsf{G}} \mathcal{G} \xrightarrow{\sim} \mathcal{X}_p$ and $\mu_{1,p}\colon \mathcal{G} \circ_{\mathsf{G}} \mathcal{X}_p \xrightarrow{\sim} \mathcal{X}_p$ for $p \in P$ are the canonical left and right uniters $\mathrm{l}_{\mathsf{X}_p}, \mathrm{r}_{\mathsf{X}_p}$ described in Lemma 4.5;
(3) for all $p, q, t \in P$, the diagram of isomorphisms

$$\begin{array}{ccc} (\mathcal{X}_p \circ_{\mathsf{G}} \mathcal{X}_q) \circ_{\mathsf{G}} \mathcal{X}_t & \xRightarrow{\mu_{p,q} \circ_{\mathsf{G}} \mathrm{id}_{\mathsf{X}_t}} & \mathcal{X}_{pq} \circ_{\mathsf{G}} \mathcal{X}_t \\ \Big\Downarrow \text{associator} & & \Big\Downarrow \mu_{pq,t} \\ & & \mathcal{X}_{pqt} \\ & & \Big\Uparrow \mu_{p,qt} \\ \mathcal{X}_p \circ_{\mathsf{G}} (\mathcal{X}_q \circ_{\mathsf{G}} \mathcal{X}_t) & \xRightarrow[\mathrm{id}_{\mathsf{X}_p} \circ_{\mathsf{G}} \mu_{q,t}]{} & \mathcal{X}_p \circ_{\mathsf{G}} \mathcal{X}_{qt} \end{array} \tag{10.2}$$

commutes.

If all the ample correspondences $\mathcal{X}_p$ are tight (proper, resp.), we call the diagram X *tight* (*proper*, resp.).

Next, we assemble all the relevant definitions to define a groupoid model. From now on, we fix a diagram $\mathsf{X} = (P, \mathcal{G}, \mathcal{X}_p, \mu_{p,q})$ in $\mathsf{Gr_a}$.

Definition 10.3 (compare [Mey22b, Definition 4.5]). Let Y be a topological space. An X*-action* on Y consists of $\alpha = (\alpha_p, r)$ with

- a continuous map $r\colon Y \to \mathcal{G}^0$;
- open, continuous, surjective maps $\alpha_p\colon \mathcal{X}_p \times_{s,\mathcal{G}^0,r} Y \to Y$ for $p \in P$, denoted multiplicatively as $\alpha_p(\gamma, y) = \gamma \cdot y$;

such that

(10.3.1) $r(\gamma_2 \cdot y) = r(\gamma_2)$ and $\gamma_1 \cdot (\gamma_2 \cdot y) = (\gamma_1 \cdot \gamma_2) \cdot y$ for $p, q \in P$, $\gamma_1 \in \mathcal{X}_p$, $\gamma_2 \in \mathcal{X}_q$, and $y \in Y$ with $s(\gamma_1) = r(\gamma_2)$, $s(\gamma_2) = r(y)$;

(10.3.2) if $\gamma \cdot y = \gamma' \cdot y'$ for $\gamma, \gamma' \in \mathcal{X}_p$, $y, y' \in Y$, there is $\eta \in \mathcal{G}$ with $\gamma' = \gamma \cdot \eta$ and $y = \eta \cdot y'$.

Note that since $\mathcal{G} = \mathcal{X}_1$ the multiplication map α_1 is a left $\mathcal{G}$-action on Y (see [Mey22b, Lemma 4.6]).

Definition 10.4 (compare [Mey22b, Definition 4.8]). A continuous map $\phi: Y \to Y'$ between two topological spaces with X-actions is X-*equivariant* if $r\big(\phi(y)\big) = r(y)$ and $\phi(\gamma \cdot y) = \gamma \cdot \phi(y)$ for all $p \in P$, $y \in Y$ and $\gamma \in \mathcal{X}_p$ with $s(\gamma) = r(y)$.

Definition 10.5 (compare [Mey22b, Definition 4.13]). A *groupoid model for* X-*actions* is an ample groupoid $\mathcal{U}$ with natural bijections between the sets of $\mathcal{U}$-actions and X-actions on Y for all spaces Y.

We call these bijections natural if a continuous map $Y \to Y'$ is $\mathcal{U}$-equivariant if and only if it is X-equivariant. These bijections for all spaces Y can be combined into an isomorphism between the categories of $\mathcal{U}$-actions and X-actions.

Furthermore, groupoid models are unique up to isomorphism.

Proposition 10.6 (compare [Mey22b, Proposition 4.16]). *Let $\mathcal{U}$ and $\mathcal{U}'$ be two groupoid models for X-actions. There is a unique groupoid isomorphism $\mathcal{U} \cong \mathcal{U}'$ that is compatible with the equivalence between actions of $\mathcal{U}$, $\mathcal{U}'$ and X.*

Proof. See [Mey22b, Proposition 4.16].

Finally, a groupoid model of a diagram X is indeed a bicategory theoretical limit of the diagram.

Theorem 10.7 (compare [Mey22b, Theorem 10.6]). *A groupoid model $\mathcal{U}$ for a diagram $\mathsf{X}: P \to \mathsf{Gr_a}$ is also a limit for X in $\mathsf{Gr_a}$.*

Proof. See [Mey22b, Theorem 10.6].

10.2. **The groupoid model of a tight Ore diagram.** We want to recall the construction of a groupoid model for a tight Ore diagram. We follow closely the construction in [Alb15] and [Mey22b], where this is done for tight Ore diagrams in $\mathsf{Gr_{inj}}$ and for tight diagrams of Ore shape[8] in GR, respectively. One might note the similarity of the following construction with our construction of the covariance ring $\mathcal{O}_\mathsf{F}$ in Section 9.

We start by fixing an Ore monoid P and a tight Ore diagram $\mathsf{X} = (P, \mathcal{G}, \mathcal{X}_p, \mu_{p,q})$ in $\mathsf{Gr_a}$. Recall from Section 9, that we can define the group completion G of P (see Definition 9.16), and for each $g \in G$ the set

$$R_g := \{(p_1, p_2) \in \mathsf{G} \mid p_1 p_2^{-1} = g \in \mathsf{G}\},$$

and the filtered category $\mathcal{C}_\mathsf{P}^g$ with R_g as the set of objects and $R_g \times P$ as the set of arrows (see Definition 9.25). Next, we fix some $g \in G$ and want to build a functor from the filtered category $\mathcal{C}_\mathsf{P}^g$ to the category of topological spaces Top.

Definition 10.8. For $(p_1, p_2) \in R_g$ define the topological space

$$\mathcal{X}_{p_1} \circ_\mathsf{G} \mathcal{X}_{p_2}^{\square} := \mathcal{X}_{p_1} \times_{s, \mathsf{G}^0, s} \mathcal{X}_{p_2} /_\sim$$

to be the quotient space of $\mathcal{X}_{p_1} \times_{s,\mathsf{G}^0,s} \mathcal{X}_{p_2}$ by the equivalence relation $(x, y) \sim (xg, yg)$ for all $g \in \mathcal{G}$ with $s(x) = s(y) = r(g)$. We denote its elements by $[x, y] \in \mathcal{X}_{p_1} \circ_\mathsf{G} \mathcal{X}_{p_2}^{\square}$.

[8] A diagram of Ore shape is a diagram over a small category that satisfies certain right Ore conditions. This is a generalization of an Ore diagram.

Remark 10.9. For a groupoid $\mathcal{G}$ and right $\mathcal{G}$-space $\mathcal{X}$, we can define the left $\mathcal{G}$-space $\mathcal{X}^{\square}$ by taking $r := s$ as an anchor map, and

$$\mathrm{mult}\colon \mathcal{G} \times_{s,G^0,r} \mathcal{X}^{\square} \to \mathcal{X}^{\square}, \qquad (g,x) \mapsto xg^{-1},$$

as the multiplication map. Now, if the right $\mathcal{G}$-space $\mathcal{X}$ is basic, $\mathcal{X}/\mathcal{G}$ is Hausdorff or s is a local homeomorphism then the left $\mathcal{G}$-space $\mathcal{X}^{\square}$ has the same properties, respectively. This justifies the notation in Definition 10.8 and explains the similarity to the definition of a composition of correspondences from Section 4. Furthermore, this shows that $\mathcal{X}_{p_1} \circ_G \mathcal{X}_{p_2}^{\square}$ is given as the orbit space of a basic right $\mathcal{G}$-action, and hence the orbit space projection $\mathcal{X}_{p_1} \times_{s,G^0,s} \mathcal{X}_{p_2} \to \mathcal{X}_{p_1} \circ_G \mathcal{X}_{p_2}^{\square}$ is a surjective local homeomorphism (by Lemma 3.14).

We can now find an ample base for the topology of $\mathcal{X}_{p_1} \circ_G \mathcal{X}_{p_2}^{\square}$.

Corollary 10.10. *For all $U \in \mathcal{X}_{p_1}^a$, $V \in \mathcal{X}_{p_2}^a$ with $s(U) \supset s(V)$ the set of all*

$$UV := \{[x,y] \mid (x,y) \in U \times_{s,G^0,s} V\} \subset \mathcal{X}_{p_1} \circ_G \mathcal{X}_{p_2}^{\square}$$

is an ample base for the topology of $\mathcal{X}_{p_1} \circ_G \mathcal{X}_{p_2}^{\square}$. We denote this base by $\mathcal{B}_{X_{p_1} \circ_G X_{p_2}^{\square}}$.

Proof. This is immediate from Lemma 4.3.

Next, we want to define a continuous map for every $(p_1, p_2 q) \in R_g \times P$.

Definition 10.11. For $(p_1,p_2) \in R_g$ and $q \in P$ define the map

$$\alpha_{p_1,p_2}^q\colon \mathcal{X}_{p_1} \circ_G \mathcal{X}_{p_2}^{\square} \to \mathcal{X}_{p_1 q} \circ_G \mathcal{X}_{p_2 q}^{\square}, \qquad [x,y] \mapsto [xz, yz],$$

where $z \in \mathcal{X}_q$ is an element such that $s(x) = s(y) = r(z)$ and with the notation $xz := \mu_{p_1,q}([x,z]) \in \mathcal{X}_{p_1 q}$.

Since $\mathcal{X}_g$ is tight, the map $r\colon \mathcal{X}/\mathcal{G} \to \mathcal{G}^0$ is a homeomorphism and hence the element $z \in \mathcal{X}_q$ is unique up to right multiplication by some $g \in \mathcal{G}$.

Lemma 10.12 (compare [Alb15, Lemma 3.6])**.** *The above-defined map α_{p_1,p_2}^q is well-defined, a local homeomorphism and injective on all $UV \in \mathcal{B}_{X_{p_1} \circ_G X_{p_2}^{\square}}$. Furthermore,*

$$\alpha_{p_1 q, p_2 q}^t \circ \alpha_{p_1,p_2}^q = \alpha_{p_1,p_2}^{qt}$$

and $\alpha_{p_1,p_2}^1 = \mathrm{id}_{X_{p_1} \circ_G X_{p_2}^{\square}}$ for all $(p_1,p_2) \in R_g$ and $t, q \in P$.

Proof. See [Alb15, Lemma 3.6]. Note that in the proof it is shown that α_{p_1,p_2}^q is injective on all $UV \in \mathcal{B}_{X_{p_1} \circ_G X_{p_2}^{\square}}$.

Hence, the data above defines a functor.

Definition 10.13. For $g \in G$, we define a functor from the filtered category $\mathcal{C}_P^g$ to the category of topological spaces (denoted as Top) via

$$\begin{aligned} H_{X,g}\colon \mathcal{C}_P^g &\to \mathsf{Top} \\ (p_1,p_2) &\mapsto \mathcal{X}_{p_1} \circ_G \mathcal{X}_{p_2}^{\square} \\ (p_1,p_2,q) &\mapsto \alpha_{p_1,p_2}^q. \end{aligned}$$

Now, we can take the colimit of this functor. Since $\mathcal{C}_P^g$ is filtered, we can apply our results from Section 8. Applying Corollary 8.8 gives us an explicit construction of the colimit.

Definition 10.14. Define the topological space $\mathcal{H}_g$ to be the colimit of the functor $H_{X,g}$, that is, it is given by the set

$$\mathcal{H}_g := \varinjlim_{(p_1,p_2)\square R_g} \mathcal{X}_{p_1} \circ_G \mathcal{X}_{p_2}^{\square} = \bigsqcup_{(p_1,p_2)\square R_g} \mathcal{X}_{p_1} \circ_G \mathcal{X}_{p_2}^{\square} /_{\sim}$$

where the equivalence relation is generated by $[x_1, x_2] \sim \alpha^q_{p_1,p_2}([x_1, x_2])$ for all $[x_1, x_2] \in \mathcal{X}_{p_1} \circ_G \mathcal{X}^{\square}_{p_2}$ and $(p_1, p_2) \in R_g, t \in P$, with the canonical topology.

Lemma 10.15 (compare [Alb15, Lemma 3.9])**.** *The canonical maps*

$$\lambda_{p_1,p_2}\colon \mathcal{X}_{p_1} \circ_G \mathcal{X}^{\square}_{p_2} \to \mathcal{H}_g, \text{ and}$$
$$\lambda\colon \bigsqcup_{(p_1,p_2)\square R_g} \mathcal{X}_{p_1} \circ_G \mathcal{X}^{\square}_{p_2} \to \mathcal{H}_g$$

are local homeomorphisms. The first map is injective on all $UV \in \mathcal{B}_{X_{p_1} \circ_G X^{\square}_{p_2}}$.

Proof. For the first map see [Alb15, Lemma 3.9]. From this, it is immediate that the second map is a local homeomorphism, due to the topology of the disjoint union. Now, the maps λ_{p_1,p_2} are injective on all $UV \in \mathcal{B}_{X_{p_1} \circ_G X^{\square}_{p_2}}$ because the maps $\alpha^q_{p_1,p_2}$ are (by Lemma 10.12).

Next, we want to find an ample base for this topological space.

Proposition 10.16. *The set*

$$\mathcal{B}_{H_g} := \left\{\lambda_{p_1,p_2}(U) \mid U \in \mathcal{B}_{X_{p_1} \circ_G X^{\square}_{p_2}}, (p_1, p_2) \in R_g\right\}$$

is an ample base for the topology on $\mathcal{H}_g$.

Proof. Consider $U \in \mathcal{B}_{X_{p_1} \circ_G X^{\square}_{p_2}}$. By Corollary 10.10, U is a compact Hausdorff open subset. As λ_{p_1,p_2} is a local homeomorphism and injective on U, we get that $\lambda_{p_1,p_2}(U) \subset \mathcal{H}_g$ is a compact Hausdorff open subset. The set defines a base, since for $h \in \mathcal{H}_g$ and an open $h \in W \subset \mathcal{H}_g$, we find $(p_1, p_2) \in R_g$ and $x \in \mathcal{X}_{p_1} \circ_G \mathcal{X}^{\square}_{p_2}$ such that $\lambda_{p_1,p_2}(x) = h$. Then $x \in \lambda^{-1}_{p_1,p_2}(W) \subset \mathcal{X}_{p_1} \circ_G \mathcal{X}^{\square}_{p_2}$ open and hence we find $U \in \mathcal{B}_{X_{p_1} \circ_G X^{\square}_{p_2}}$ such that $x \in U \subset \lambda^{-1}_{p_1,p_2}(W) \subset \mathcal{X}_{p_1} \circ_G \mathcal{X}^{\square}_{p_2}$. Thus, we get $h = \lambda_{p_1,p_2}(x) \in \lambda_{p_1,p_2}(U) \subset W$ and $\mathcal{B}_{H_g}$ indeed defines a compact Hausdorff base of $\mathcal{H}_g$.

Finally, we want to check that the base is stable under taking compact open subsets. Consider $U \in \mathcal{B}_{X_{p_1} \circ_G X^{\square}_{p_2}}$ and a compact open subset $W \subset \lambda_{p_1,p_2}(U)$. Since $\lambda_{p_1,p_2}|_U\colon U \to \lambda_{p_1,p_2}(U)$ defines a homeomorphism, the compact subset W is homeomorphic via $\lambda_{p_1,p_2}|_U$ to a compact open subset $V \subset U$. As $\mathcal{B}_{X_{p_1} \circ_G X^{\square}_{p_2}}$ is stable under taking compact open subsets (by Corollary 10.10), we get $V \in \mathcal{B}_{X_{p_1} \circ_G X^{\square}_{p_2}}$ and hence $W = \lambda_{p_1,p_2}(V) \in \mathcal{B}_{H_g}$.

The following Lemma 10.17 gives that for $\lambda_{p_1,p_2}(U) \in \mathcal{B}_{H_g}$ the $U \in \mathcal{B}_{X_{p_1} \circ_G X^{\square}_{p_2}}$ is unique up to lifting along $\alpha^t_{p_1,p_2}$.

Lemma 10.17. *Consider $(p_1, p_2), (q_1, q_2) \in R_g$ and $U_1 \in \mathcal{X}_{p_1} \circ_G \mathcal{X}^{\square}_{p_2}$, $U_2 \in \mathcal{B}_{X_{q_1} \circ_G X^{\square}_{q_2}}$ such that $\lambda_{p_1,p_2}(U_1) = \lambda_{q_1,q_2}(U_2)$, then we find $t_1, t_2 \in P$ such that $(p_1 t_1, p_2 t_1) = (q_1 t_2, q_2 t_2)$ and $\alpha^{t_1}_{p_1,p_2}(U_1) = \alpha^{t_2}_{q_1,q_2}(U_2)$.*

Proof. First, we find $t_1, t_2 \in P$ such that $(p_1 t_1, p_2 t_1) = (q_1 t_2, q_2 t_2)$. Then $U_1' := \alpha^{t_1}_{p_1,p_2}(U_1)$ and $U_2' := \alpha^{t_2}_{p_1,p_2}(U_2)$ are two compact Hausdorff open subsets (since the maps α are local homeomorphisms that are injective on U_1, U_2 by Lemma 10.12) of the same space $\mathcal{X}_{p_1 t_1} \circ_G \mathcal{X}^{\square}_{p_2 t_1} = \mathcal{X}_{q_1 t_2} \circ_G \mathcal{X}^{\square}_{q_2 t_2}$. Now,

$$\begin{aligned}\lambda_{p,q}(U_1') &= (\lambda_{p,q} \circ \alpha^{t_1}_{p_1,p_2})(U_1) = \lambda_{p_1,p_2}(U_1)\\ &= \lambda_{q_1,q_2}(U_2) = \left(\lambda_{p,q} \circ \alpha^{t_2}_{p_1,p_2}\right)(U_2)\\ &= \lambda_{p,q}(U_2')\end{aligned}$$

for $(p, q) := (p_1 t_1, p_2 t_1) = (q_1 t_2, q_2 t_2)$. Furthermore, since the maps $\alpha^t_{p_1,p_2}$ are injective on U_1 and the maps $\alpha^t_{q_1,q_2}$ are injective on U_2 for all $t \in P$ (by Lemma 10.12), the maps $\alpha^t_{p,q}$ are also injective on U_1', U_2' for all $t \in P$.

Thus, we can, without loss of generality, assume that $U_1, U_2 \subset \mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^{\ast}_{p_2}$ are compact Hausdorff open subsets such that $\lambda_{p_1,p_2}(U_1) = \lambda_{p_1,p_2}(U_2)$ and the $\alpha^t_{p_1,p_2}$ are injective on U_1, U_2 for all $t \in P$.

Consider $u_1 \in U_1$, then there is $u_2 \in U_2$ such that $\lambda_{p_1,p_2}(u_1) = \lambda_{p_1,p_2}(u_2)$. Thus, there are $q, q' \in P$ such that $(p_1q, p_2q) = (p_1q', p_2q')$ and $\alpha^q_{p_1,p_2}(u_1) = \alpha^{q'}_{p_1,p_2}(u_2)$. By (O2) (in Definition 9.14), we find $z \in P$ such that $qz = q'z$. Hence, Lemma 10.12 implies

$$\alpha^t_{p_1,p_2}(u_1) = (\alpha^z_{p_1q,p_2q} \circ \alpha^q_{p_1,p_2})(u_1) = (\alpha^z_{p_1q',p_2q'} \circ \alpha^{q'}_{p_1,p_2})(u_2) = \alpha^t_{p_1,p_2}(u_2)$$

for $t := qz = q'z$. Now take $B_1, B_2 \in \mathcal{B}_{\mathsf{X}_{p_1} \circ_{\mathsf{G}} \mathsf{X}^{\ast}_{p_2}}$ so that $u_i \in B_i \subset U_i$ for $i = 1, 2$. Then,

$$\alpha^t_{p_1,p_2}(u_1) \in \alpha^t_{p_1,p_2}(B_1) \cap \alpha^t_{p_1,p_2}(B_2) \subset \mathcal{X}_{p_1t} \circ_{\mathsf{G}} \mathcal{X}^{\ast}_{p_2t}$$

is an open neighborhood and as $\alpha^t_{p_1,p_2}$ is continuous, we find $V \in \mathcal{B}_{\mathsf{X}_{p_1} \circ_{\mathsf{G}} \mathsf{X}^{\ast}_{p_2}}$ with $u_1 \in V \subset U_1$ such that $\alpha^t_{p_1,p_2}(V) \subset \alpha^t_{p_1,p_2}(B_1) \cap \alpha^t_{p_1,p_2}(B_2)$. Now, $\alpha^t_{p_1,p_2}$ is injective on B_i for $i = 1, 2$. Next, we define $W := (\alpha^t_{p_1,p_2}|_{B_2})^{-1}\big(\alpha^t_{p_1,p_2}(V)\big)$. That is, a compact open subset of $B_2 \in \mathcal{B}_{\mathsf{X}_{p_1} \circ_{\mathsf{G}} \mathsf{X}^{\ast}_{p_2}}$ and hence $W \in \mathcal{B}_{\mathsf{X}_{p_1} \circ_{\mathsf{G}} \mathsf{X}^{\ast}_{p_2}}$. Furthermore, we have $u_2 \in W$. So, we have found two neighborhoods $V, W \in \mathcal{B}_{\mathsf{X}_{p_1} \circ_{\mathsf{G}} \mathsf{X}^{\ast}_{p_2}}$ of u_1 and u_2, respectively, such that $\alpha^t_{p_1,p_2}(V) = \alpha^t_{p_1,p_2}(W)$.

Now, since we find such a V for every $u_1 \in U_1$, we cover U_1 with these V and use that U_1 is compact to find finitely many compact Hausdorff open subsets $V_1, \dots, V_n$ such that $U_1 = \bigcup_{i=1}^n V_i$. If we take the corresponding compact Hausdorff open subsets $W_1, \dots, W_n$ such that $\alpha^{t_i}_{p_1,p_2}(V_i) = \alpha^{t_i}_{p_1,p_2}(W_i)$ for $i = 1, \dots, n$ and fitting $t_i \in P$, we get $U_2 = \bigcup_{i=1}^n W_i$. After using (O1) (in Definition 9.14) $n - 1$ times on the $t_1, \dots, t_n$ (and Lemma 10.12) we can, without loss of generality, assume that $t := t_1 = \dots = t_n$. So we found $t \in P$ such that $\alpha^t_{p_1,p_2}(V_i) = \alpha^t_{p_1,p_2}(W_i)$ and hence we get

$$\begin{aligned}
\alpha^t_{p_1,p_2}(U_1) &= \alpha^t_{p_1,p_2}\Big(\bigcup_{i=1}^n V_i\Big) = \bigcup_{i=1}^n \alpha^t_{p_1,p_2}(V_i) \\
&= \bigcup_{i=1}^n \alpha^t_{p_1,p_2}(W_i) = \alpha^t_{p_1,p_2}\Big(\bigcup_{i=1}^n W_i\Big) \\
&= \alpha^t_{p_1,p_2}(U_2),
\end{aligned}$$

which is the desired result.

Finally, we define an ample groupoid that will turn out to be the groupoid model for the tight Ore diagram $\mathsf{X} = (P, \mathcal{G}, \mathcal{X}_p, \mu_{p,q})$ in $\mathsf{Gr_a}$.

Definition 10.18. Define the topological groupoid

$$\mathcal{H} := \bigsqcup_{g \in G} \mathcal{H}_g = \bigsqcup_{g \in G} \varinjlim_{(p_1,p_2) \in R_g} \mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^{\ast}_{p_2}$$

with

- object set $\mathcal{H}^0 := \mathcal{G}^0$;
- range map and source maps $r\big([x_1, x_2]\big) := r(x_1)$ and $s\big([x_1, x_2]\big) := r(x_2)$;
- composition $[x_1, x_2] \cdot [x_2, x_3] := [x_1, x_3]$;
- inversion $[x_1, x_2]^{-1} := [x_2, x_1]$; and
- units $[x, x]$.

Theorem 10.19. *The above-defined data $\mathcal{H}$ is indeed an ample topological groupoid and a groupoid model for X.*

Proof. By [Alb15, Proposition 3.10] the above-defined data $\mathcal{H}$ is indeed a locally compact, étale, topological groupoid. Since $\mathcal{G}$ is ample, the object set $\mathcal{H}^0 = \mathcal{G}^0$ of $\mathcal{H}$ is totally disconnected, and hence $\mathcal{H}$ is an ample groupoid. By [Mey22b, Theorem

8.18] $\mathcal{H}$ is a groupoid model of X in the bicategory GR and hence as $\mathcal{H} \in \mathsf{Gr}_{\mathsf{a}}^{0}$ it is a groupoid model of X in Gr_{a} as well.

Remark 10.20 (compare [Alb15, Remark 3.14]). For the neutral element $e \in G$, the topological space $\mathcal{H}_e$ is a clopen subgroupoid of $\mathcal{H}$ (and hence ample as well).

Finally, we want to find an ample base of compact slices on $\mathcal{H}$ to be able to compute the Steinberg algebra of this space in Section 11.

Proposition 10.21. *The set*

$$\mathcal{B}_{\mathsf{H}} := \bigcup_{g \in \mathsf{G}} \mathcal{B}_{\mathsf{H}_g}$$

is an ample base of compact slices on $\mathcal{H}$.

Proof. By the topology of the disjoint union and Proposition 10.16 the set indeed is an ample base for the topology on $\mathcal{H}$. So we only need to check that its elements are indeed slices, that is, that the range and source maps are injective on them. Consider $UV \in \mathcal{B}_{\mathsf{X}_{p_1} \circ_{\mathsf{G}} \mathsf{X}^{*}_{p_2}}$ and $[x, y], [x', y'] \in \lambda_{p_1,p_2}(UV)$ with $x, x' \in U$, $y, y' \in V$. First, assume that $s([x, y]) = s([x', y'])$, that is, $r(y) = r(y')$. As $\mathcal{X}_{p_2}$ is a tight correspondence there is a $g \in \mathcal{G}$ with $r(g) = s(y')$ such that $y = y'g$ and hence $p(y) = p(y'g) = p(y')$. Since V is a slice and $y, y' \in V$, it follows that $y = y'$. Furthermore, we get $s(x) = s(y) = s(y') = s(x')$ and as x, x' are in the same slice U, we get $x = x'$ as well. Hence, $[x, y] = [x', y']$ and thus s is injective on $\lambda_{p_1,p_2}(UV)$. The proof that r is injective on $\lambda_{p_1,p_2}(UV)$ is analogous.

11. Steinberg algebras of groupoid models

In this section, we combine all the results from earlier sections into our main result. As before, we fix a (commutative, unital) ring R with the discrete topology. We also fix a tight Ore diagram $\mathsf{X} = (P, \mathcal{G}, \mathcal{X}_p, \mu_{p,q})$ in $\mathsf{Gr_a}$ such that $\mathcal{G}$ is cocompact, that is, a strictly unital homomorphism in $\mathsf{Gr_{co,tight}} \subset \mathsf{Gr_{co,proper}}$ over an Ore monoid P. In Section 10, we constructed a groupoid model $\mathcal{H}$ of this diagram. We now want to show that the Steinberg algebra of the groupoid model $\mathcal{H}$ is the covariance ring of an induced Ore diagram in $\mathsf{Rings_{fgp}}$. In Section 7, we constructed a strictly unital homomorphism $A\colon \mathsf{Gr_{co,proper}} \to \mathsf{Rings_{fgp}}$ (by Remark 7.7) that we can compose this diagram with, to get an Ore diagram

$$\mathcal{F} := A * \mathsf{X} = \left(P, F_1 := A_R(\mathcal{G}), F_p := A_R(\mathcal{X}_p), \mu^F_{p,q} := A(\mu^{\mathsf{X}}_{p,q}) \circ \mu^A_{p,q}\right).$$

in $\mathsf{Rings_{fgp}}$ (using the usual composition of morphisms between bicategories as described in [Mey22a, Proposition 4.7.10]). Thus, we will apply our results from Section 9 to get that the covariance ring of $\mathcal{F}$ is given by $\mathcal{O}_\mathsf{F}$ (by Corollary 9.49). Note, that we have an additional R-module structure on all the F_p that behaves well with the relevant maps $\mu^F_{p,q}$ and thus also with $\phi_{p_1,p_2,q}$. Hence, we get a canonical R-module structure on the covariance ring $\mathcal{O}_\mathsf{F}$ (see Corollary 8.9) that turns it into an R-algebra.

Now, we want to prove that the Steinberg algebra of the groupoid model $A_R(\mathcal{H})$ is isomorphic to the covariance ring $\mathcal{O}_\mathsf{F}$ of the diagram $\mathcal{F} = A * \mathsf{X}$ as a unital ring and R-module. In other words, we prove that $A\colon \mathsf{Gr_a} \to \mathsf{Rings}$ preserves these particular bicategorical limits.

We start by taking a closer look at the groupoid model $\mathcal{H}$ of X. It is given by

$$\mathcal{H} = \bigsqcup_{g \in G} \mathcal{H}_g = \bigsqcup_{g \in G} \varinjlim_{(p_1,p_2) \in R_g} \mathcal{X}_{p_1} \circ_\mathsf{G} \mathcal{X}^*_{p_2}.$$

We might note a similarity with the covariance ring of $\mathcal{F}$ given by

$$\mathcal{O}_\mathsf{F} = \bigoplus_{g \in G} \mathcal{O}_g = \bigoplus_{g \in G} \varinjlim_{(p_1,p_2) \in R_g} \operatorname{Hom}_{-,A_R(\mathsf{G})}\left(A_R(\mathcal{X}_{p_2}), A_R(\mathcal{X}_{p_1})\right).$$

By Lemma 6.2, the Steinberg module of the disjoint union is given by the direct sum of the Steinberg modules, that is, we get an isomorphism

$$A_R(\mathcal{H}) = A_R\left(\bigsqcup_{g \in G} \mathcal{H}_g\right) \cong \bigoplus_{g \in G} A_R(\mathcal{H}_g)$$

of R-modules. So, if we first prove that

$$A_R\left(\mathcal{X}_{p_1} \circ_\mathsf{G} \mathcal{X}^*_{p_2}\right) \cong \operatorname{Hom}_{-,A_R(\mathsf{G})}\left(A_R(\mathcal{X}_{p_2}), A_R(\mathcal{X}_{p_1})\right)$$

are isomorphic as R-modules (see Proposition 11.1), and then that

$$A_R(\mathcal{H}_g) = A_R\left(\varinjlim \mathcal{X}_{p_1} \circ_\mathsf{G} \mathcal{X}^*_{p_2}\right) \cong \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_\mathsf{G} \mathcal{X}^*_{p_2}\right)$$

are isomorphic as R-modules, that is, that taking the Steinberg module commutes with filtered colimits (see Proposition 11.5), we get that the objects are isomorphic as R-modules. We also need to carefully handle the unital ring structure on both objects and prove that the isomorphism preserves it (see Proposition 11.3) to finally get our main result, namely, that

$$A_R(\mathcal{H}) \cong \mathcal{O}_\mathsf{F}$$

are isomorphic as R-modules and unital rings (by Theorem 11.6).

We start by rewriting the R-modules $\operatorname{Hom}_{-,A_R(\mathsf{G})}\left(A_R(\mathcal{X}_{p_2}), A_R(\mathcal{X}_{p_1})\right)$ for $g \in G$ and $(p_1, p_2) \in R_g$.

Proposition 11.1. *Consider two proper ample correspondences $\mathcal{X}\colon\mathcal{G}\leftarrow\mathcal{G}$, $\mathcal{Y}\colon\mathcal{G}\leftarrow\mathcal{G}$ for a cocompact groupoid $\mathcal{G}$. Then there are compact open subsets $K_1,\dots,K_n\subset\mathcal{G}^0$ such that*

$$\mathcal{Y}\cong\bigsqcup_{i=1}^{n} r_{\mathcal{G}}^{-1}(K_i)$$

as a right $\mathcal{G}$-space and the maps

$$\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}^{*}\to\bigsqcup_{i=1}^{n} s_{\mathcal{X}}^{-1}(K_i),\quad [x,y]\mapsto xy^{-1},\quad \big[x,s(x)\big]\mapsfrom x,$$

define a homeomorphism. Furthermore, the map

$$\mathcal{I}_{\mathcal{X},\mathcal{Y}}\colon A_R\left(\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}^{*}\right)\to\mathrm{Hom}_{-,A_R(\mathcal{G})}\big(A_R(\mathcal{Y}),A_R(\mathcal{X})\big),$$
$$f\mapsto\left[\beta\mapsto\Big[x\mapsto\sum_{\substack{y\in\mathcal{Y}\\ s(y)=s(x)}} f\big([x,y]\big)\beta(y)\Big]\right],$$

is an isomorphism of R-modules.

Proof. By Theorem 3.29 we find compact open subsets $K_1,\dots,K_n\subset\mathcal{G}^0$ such that the correspondence $\mathcal{Y}$ is given by

$$\mathcal{Y}\cong\bigsqcup_{i=1}^{n} r_{\mathcal{G}}^{-1}(K_i)$$

as a right $\mathcal{G}$-space. Hence, we get a chain of homeomorphisms

$$\begin{aligned}\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}^{*}&\cong\mathcal{X}\circ_{\mathcal{G}}\Big(\bigsqcup_{i=1}^{n} r_{\mathcal{G}}^{-1}(K_i)\Big)^{*}\cong\bigsqcup_{i=1}^{n}\mathcal{X}\circ_{\mathcal{G}}\big(r_{\mathcal{G}}^{-1}(K_i)\big)^{*}\\ &\cong\bigsqcup_{i=1}^{n}\mathcal{X}\circ_{\mathcal{G}} s_{\mathcal{G}}^{-1}(K_i)\cong\bigsqcup_{i=1}^{n} s_{\mathcal{X}}^{-1}(K_i),\end{aligned}$$

where the second equality is given by the canonical isomorphism, the third equality can be seen using Remark 10.9 and the fourth is given by the homeomorphism

$$\mathcal{X}\circ_{\mathcal{G}} s_{\mathcal{G}}^{-1}(K_i)\to s_{\mathcal{X}}^{-1}(K_i),\quad [x,g]\mapsto xg,\quad \big[x,s(x)\big]\mapsfrom x.$$

One can check that if we chase through the homeomorphisms, we get the defined maps. Next this gives a chain of isomorphisms of abelian groups

$$\begin{aligned}A_R\left(\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}^{*}\right)&\cong\bigoplus_{i=1}^{n}A_R\big(s_{\mathcal{X}}^{-1}(K_i)\big)\cong\bigoplus_{i=1}^{n}A_R(\mathcal{X})*1_{K_i}\\ &\cong\bigoplus_{i=1}^{n}A_R(\mathcal{X})\otimes_{A_R(\mathcal{G})}\big(A_R(\mathcal{G})*1_{K_i}\big)\\ &\cong\bigoplus_{i=1}^{n}A_R(\mathcal{X})\otimes_{A_R(\mathcal{G})}\mathrm{Hom}_{-,A_R(\mathcal{G})}\big(1_{K_i}*A_R(\mathcal{G}),A_R(\mathcal{G})\big)\\ &\cong A_R(\mathcal{X})\otimes_{A_R(\mathcal{G})}\mathrm{Hom}_{-,A_R(\mathcal{G})}\Big(\bigoplus_{i=1}^{n}1_{K_i}*A_R(\mathcal{G}),A_R(\mathcal{G})\Big)\\ &\cong A_R(\mathcal{X})\otimes_{A_R(\mathcal{G})}\mathrm{Hom}_{-,A_R(\mathcal{G})}\big(A_R(\mathcal{Y}),A_R(\mathcal{G})\big)\\ &\cong A_R(\mathcal{X})\otimes_{A_R(\mathcal{G})}A_R(\mathcal{Y})^{*}\\ &\cong\mathrm{Hom}_{-,A_R(\mathcal{G})}\big(A_R(\mathcal{Y}),A_R(\mathcal{X})\big),\end{aligned}$$

where the first equality is given by Lemma 6.2 and the homeomorphism above, the second works analogously to Lemma 6.23, the third is given by Lemma 5.14, the fourth is given by Lemma 5.15, the fifth is given by Lemma 5.16 together with Lemma 5.17, the sixth is Theorem 6.24, the seventh is by the definition of the dual module and the eighth is given by Theorem 5.10. Now, if we prove that this chain of isomorphisms is actually given by the above-defined R-module homomorphism, we are done. We start with $f \in A_R(\mathcal{X} \circ_{\mathsf{G}} \mathcal{Y})^{\square}$ such that we can find some $\alpha \in A_R(\mathcal{X})$ and $\xi_i \in A_R\left(s_{\mathsf{G}}^{-1}(K_i)\right)$ for $i = 1, \ldots, n$ such that

$$f\big([x, y]\big) = (\alpha * \xi_i)(xy^{-1})$$

for all $x \in \mathcal{X}, y \in s_{\mathsf{G}}^{-1}(K_i)$ such that $s(x) = s(y)$. Now, if we map f through the chain of isomorphisms and evaluate it at $\beta \in A_R\left(r_{\mathsf{G}}^{-1}(K_i)\right) \subset A_R(\mathcal{Y})$ and $x \in \mathcal{X}$, we get $\left(\alpha * (\mathrm{mult}_{\xi_i})_{i=1}^n(\beta)\right)(x)$. Now, we also have

$$\begin{aligned}
\left(\alpha * (\mathrm{mult}_{\xi_i})_{i=1}^n(\beta)\right)(x) &= \big(\alpha * (\xi_i * \beta)\big)(x) \\
&= \sum_{g \in \mathsf{G}_{s(x)}} \alpha(xg^{-1})(\xi_i * \beta)(g) \\
&= \sum_{g \in \mathsf{G}_{s(x)}} \alpha(xg^{-1}) \sum_{y \in \mathsf{G}_{s(g)}} \xi_i(gy^{-1})\beta(y) \\
&= \sum_{y \in \mathsf{G}_{s(x)}} \sum_{g \in \mathsf{G}_{s(x)}} \alpha(xg^{-1})\xi_i(gy^{-1})\beta(y) \\
&= \sum_{y \in \mathsf{G}_{s(x)}} \sum_{g \in \mathsf{G}_{r(y)}} \alpha(xy^{-1}g^{-1})\xi_i(g)\beta(y) \\
&= \sum_{y \in \mathsf{G}_{s(x)}} (\alpha * \xi_i)(xy^{-1})\beta(y) \\
&= \sum_{\substack{y \in \mathsf{Y} \\ s(y) = s(x)}} f\big([x, y]\big)\beta(y) \\
&= \mathcal{I}_{\mathsf{X},\mathsf{Y}}(f)(\beta)(x)
\end{aligned}$$

and hence the chain of isomorphisms is indeed given by the above-defined map, as an arbitrary map $f \in A_R(\mathcal{X} \circ_{\mathsf{G}} \mathcal{Y})^{\square}$ is given by a finite sum of these. Finally, it is easy to check that the above-defined map is an R-module homomorphism.

Thus, we have $\mathrm{Hom}_{-,A_R(\mathsf{G})}\left(A_R(\mathcal{X}_{p_2}), A_R(\mathcal{X}_{p_1})\right) \cong A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}_{p_2}^{\square}\right)$. To properly extend this isomorphism to $\mathcal{O}_g = \varinjlim \mathrm{Hom}_{-,A_R(\mathsf{G})}\left(A_R(\mathcal{X}_{p_2}), A_R(\mathcal{X}_{p_1})\right)$, we need to understand how it behaves with the maps

$$\phi_{p_1,p_2,q} \colon \mathrm{Hom}_{-,A_R(\mathsf{G})}\left(A_R(\mathcal{X}_{p_2}), A_R(\mathcal{X}_{p_1})\right) \to \mathrm{Hom}_{-,A_R(\mathsf{G})}\left(A_R(\mathcal{X}_{p_2 q}), A_R(\mathcal{X}_{p_1 q})\right)$$

that define this filtered colimit in Definition 9.26.

Proposition 11.2. *The local homeomorphisms* $\alpha^q_{p_1,p_2} \colon \mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}_{p_2}^{\square} \to \mathcal{X}_{p_1 q} \circ_{\mathsf{G}} \mathcal{X}_{p_2 q}^{\square}$ *from Definition 10.11 induce R-module homomorphisms (via Definition 6.5 and Proposition 6.6)*

$$\begin{aligned}
(\alpha^q_{p_1,p_2})_{\square} \colon A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}_{p_2}^{\square}\right) &\to A_R\left(\mathcal{X}_{p_1 q} \circ_{\mathsf{G}} \mathcal{X}_{p_2 q}^{\square}\right), \\
f &\mapsto \left[y \mapsto \sum_{x \in (\alpha^q_{p_1,p_2})^{-1}(y)} f(x) \right],
\end{aligned}$$

so that the diagram

$$\begin{array}{ccc}
A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}_{p_2}^{\square}\right) & \xrightarrow{(\alpha^q_{p_1,p_2})_{\square}} & A_R\left(\mathcal{X}_{p_1 q} \circ_{\mathsf{G}} \mathcal{X}_{p_2 q}^{\square}\right) \\
{\scriptstyle \mathrm{I}_{\mathsf{X}_{p_1},\mathsf{X}_{p_2}}} \downarrow {\scriptstyle \square} & & {\scriptstyle \mathrm{I}_{\mathsf{X}_{p_1 q},\mathsf{X}_{p_2 q}}} \downarrow {\scriptstyle \square} \\
\mathrm{Hom}_{-,A_R(\mathsf{G})}\left(A_R(\mathcal{X}_{p_2}), A_R(\mathcal{X}_{p_1})\right) & \xrightarrow{\phi_{p_1,p_2,q}} & \mathrm{Hom}_{-,A_R(\mathsf{G})}\left(A_R(\mathcal{X}_{p_2 q}), A_R(\mathcal{X}_{p_1 q})\right)
\end{array}$$

commutes. Furthermore, the $\mathcal{I}_{\mathsf{X}_{p_1},\mathsf{X}_{p_2}}$ *descend to R-module isomorphisms*

$$\mathcal{I}_g \colon \mathcal{O}_g \to \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}_{p_2}^{\square}\right)$$

for all $g \in G$ that assemble into an isomorphism

$$\mathcal{I}\colon \mathcal{O}_{\mathsf{F}} \to \bigoplus_{g \in G} \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^{\square}_{p_2}\right)$$

of R-modules.

Proof. The induced map is given by Definition 6.5 since $\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^{\square}_{p_2}$ has an ample base $\mathcal{B}_{\mathsf{X}_{p_1} \circ_{\mathsf{G}} \mathsf{X}^{\square}_{p_2}}$ such that $\alpha^q_{p_1,p_2}$ is injective on it (by Lemma 10.12) and hence it is indeed a well-defined R-module homomorphism (by Proposition 6.6). Consider $f \in A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^{\square}_{p_2}\right)$, $\varphi \in A_R\left(\mathcal{X}_{p_2}\right)$, $\psi \in A_R\left(\mathcal{X}_q\right)$ and $[x_{p_1}, x_q] \in \mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^{\square}_q$. We denote $\varphi\psi := \mu^{\mathsf{F}}_{p_2,q}(\varphi \otimes \psi)$ and $x_{p_1 q} := \mu^{\mathsf{X}}_{p_1,q}\big([x_{p_1}, x_q]\big)$. Then we have

$$\begin{aligned}
&\big(\mathcal{I}_{\mathsf{X}_{p_1 q}, \mathsf{X}_{p_2 q}} \circ (\alpha^q_{p_1,p_2})_{\square}\big)(f)(\varphi\psi)(x_{p_1 q}) \\
&= \sum_{\substack{x_{p_2 q} \in \mathsf{X}_{p_2 q} \\ s(x_{p_2 q}) = s(x_{p_1 q})}} (\alpha^q_{p_1,p_2})_{\square}(f)\big([x_{p_1 q}, x_{p_2 q}]\big) \cdot \varphi\psi(x_{p_2 q}) \\
&= \sum_{\substack{x_{p_2 q} \in \mathsf{X}_{p_2 q} \\ s(x_{p_2 q}) = s(x_{p_1 q})}} \sum_{\substack{x \in \mathsf{X}_{p_1} \circ_{\mathsf{G}} \mathsf{X}^{\square}_{p_2} \\ \alpha^q_{p_1,p_2}(x) = [x_{p_1 q}, x_{p_2 q}]}} f(x) \cdot \varphi\psi(x_{p_2 q}) \\
&= \sum_{\substack{x_{p_2} \in \mathsf{X}_{p_2} \\ s(x_{p_2}) = s(x_{p_1})}} f\big([x_{p_1}, x_{p_2}]\big) \cdot \varphi\psi\Big(\mu^{\mathsf{X}}_{p_2,q}\big([x_{p_2}, x_q]\big)\Big) \\
&= \sum_{\substack{x_{p_2} \in \mathsf{X}_{p_2} \\ s(x_{p_2}) = s(x_{p_1})}} f\big([x_{p_1}, x_{p_2}]\big) \sum_{g \in \mathsf{G}_{s(x_{p_2})}} \varphi(x_{p_2} g^{-1}) \cdot \psi(g x_q) \\
&= \sum_{g \in \mathsf{G}_{s(x_{p_1})}} \sum_{\substack{x_{p_2} \in \mathsf{X}_{p_2} \\ s(x_{p_2}) = s(g)}} f\big([x_{p_1} g^{-1}, x_{p_2} g^{-1}]\big) \cdot \varphi(x_{p_2} g^{-1}) \cdot \psi(g x_q) \\
&= \sum_{g \in \mathsf{G}_{s(x_{p_1})}} \sum_{\substack{x_{p_2} \in \mathsf{X}_{p_2} \\ s(x_{p_2}) = r(g)}} f\big([x_{p_1} g^{-1}, x_{p_2}]\big) \cdot \varphi(x_{p_2}) \cdot \psi(g x_q) \\
&= \sum_{g \in \mathsf{G}_{s(x_{p_1})}} \mathcal{I}_{\mathsf{X}_{p_1}, \mathsf{X}_{p_2}}(f)(\varphi)(x_{p_1} g^{-1}) \cdot \psi(g x_q) \\
&= \mu^A_{p_1,q}\left(\mathcal{I}_{\mathsf{X}_{p_1}, \mathsf{X}_{p_2}}(f)(\varphi) \otimes \psi\right)\left([x_{p_1}, x_q]\right) \\
&= \mu^A_{p_1,q}\Big(\big(\mathcal{I}_{\mathsf{X}_{p_1}, \mathsf{X}_{p_2}}(f) \otimes \mathrm{id}\big)(\varphi \otimes \psi)\Big)\big([x_{p_1}, x_q]\big) \\
&= \Big(\mu^{\mathsf{F}}_{p_1,q} \circ \big(\mathcal{I}_{\mathsf{X}_{p_1}, \mathsf{X}_{p_2}}(f) \otimes \mathrm{id}\big) \circ (\mu^{\mathsf{F}}_{p_2,q})^{-1}(\varphi\psi)\Big)(x_{p_1 q}) \\
&= \Big(\phi_{p_1,p_2,q} \circ \mathcal{I}_{\mathsf{X}_{p_1}, \mathsf{X}_{p_2}}\Big)(f)(\varphi\psi)(x_{p_1 q})
\end{aligned}$$

where the third equality is true because the sets

$$\left\{x_{p_2} \in \mathcal{X}_{p_2} \mid s(x_{p_2}) = s(x_{p_1})\right\}$$

$$\left\{(x_{p_2 q}, x) \in \mathcal{X}_{p_2 q} \times (\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^{\square}_{p_2}) \mid s(x_{p_2 q}) = s(x_{p_1 q}), \alpha^q_{p_1,p_2}(x) = [x_{p_1 q}, x_{p_2 q}]\right\}$$

are bijective via the map

$$x_{p_2} \mapsto \Big(\mu^{\mathsf{X}}_{p_2,q}\big([x_{p_2}, x_q]\big), [x_{p_2}, x_{p_1}]\Big)$$

and hence the diagram commutes. Thus, the R-module isomorphisms $\mathcal{I}_{\mathsf{X}_{p_1}, \mathsf{X}_{p_2}}$ descend to R-module isomorphisms

$$\mathcal{I}_g\colon \mathcal{O}_g = \varinjlim \operatorname{Hom}_{-,A_R(\mathsf{G})}\left(A_R\left(\mathcal{X}_{p_2}\right), A_R\left(\mathcal{X}_{p_1}\right)\right) \cong \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^{\square}_{p_2}\right)$$

of the colimits of the respective diagrams, and we can assemble them into an R-module isomorphism $\mathcal{I}\colon \mathcal{O}_{\mathsf{F}} \to \bigoplus_{g \in G} \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^{\square}_{p_2}\right)$.

Now, the multiplication on $\mathcal{O}_{\mathsf{F}}$ is defined via $\omega_{g,h}$, which is induced by the concatenation of maps given by the map

$$\begin{aligned}\mathrm{Hom}_{-,A_R(\mathsf{G})}\left(A_R(\mathcal{X}_{p_2}),A_R(\mathcal{X}_{p_1})\right)\times\mathrm{Hom}_{-,A_R(\mathsf{G})}\left(A_R(\mathcal{X}_{p_3}),A_R(\mathcal{X}_{p_2})\right)&\longrightarrow\mathrm{Hom}_{-,A_R(\mathsf{G})}\left(A_R(\mathcal{X}_{p_3}),A_R(\mathcal{X}_{p_1})\right),\\(f_1,f_2)&\longmapsto f_1\circ f_2.\end{aligned}$$

After applying the fitting isomorphisms $\mathcal{I}_{\mathsf{X}_p,\mathsf{X}_q}$, this induces a well-defined map

$$\begin{aligned}\star\colon A_R\left(\mathcal{X}_{p_1}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_2}\right)\times A_R\left(\mathcal{X}_{p_2}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_3}\right)\to A_R\left(\mathcal{X}_{p_1}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_3}\right),&\\(f_1,f_2)\mapsto f_1\star f_2:=\mathcal{I}^{-1}_{p_1,p_3}\big(\mathcal{I}_{p_1,p_2}(f_1)\circ\mathcal{I}_{p_2,p_3}(f_2)\big),&\end{aligned}$$

that has the same properties as $\omega_{g,h}$ and hence induces a unital ring structure on $\bigoplus_{g\in G}\varinjlim A_R\left(\mathcal{X}_{p_1}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_2}\right)$.

Proposition 11.3. *The above-defined map $\star$ induces a multiplication on the R-module $\bigoplus_{g\in G}\varinjlim A_R\left(\mathcal{X}_{p_1}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_2}\right)$ that turns it into a unital ring such that the R-module isomorphism $\mathcal{I}\colon\mathcal{O}_{\mathsf{F}}\to\bigoplus_{g\in G}\varinjlim A_R\left(\mathcal{X}_{p_1}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_2}\right)$ is also an isomorphism of unital rings. Furthermore, it is explicitly given by*

$$(f_1\star f_2)\big([x_1,x_3]\big):=\sum_{\substack{x_2\in\mathsf{X}_{p_2}\\ s(x_2)=s(x_1)}}f_1\big([x_1,x_2]\big)f_2\big([x_2,x_3]\big)$$

for $f_1\in A_R\left(\mathcal{X}_{p_1}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_2}\right)$, $f_2\in A_R\left(\mathcal{X}_{p_2}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_3}\right)$ and $[x_1,x_3]\in\mathcal{X}_{p_1}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_3}$.

Proof. Using the isomorphism of abelian groups $\mathcal{I}\colon\mathcal{O}_{\mathsf{F}}\to\bigoplus_{g\in G}\varinjlim A_R\left(\mathcal{X}_{p_1}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_2}\right)$ the ring structure on $\mathcal{O}_{\mathsf{F}}$ induces a ring structure on $\bigoplus_{g\in G}\varinjlim A_R\left(\mathcal{X}_{p_1}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_2}\right)$ such that $\mathcal{I}$ is by definition a unital ring homomorphism. Since the isomorphism $\mathcal{I}$ is given by the isomorphisms $\mathcal{I}_{\mathsf{X}_p,\mathsf{X}_q}$ and the multiplication on the unital ring $\mathcal{O}_{\mathsf{F}}$ is defined via the map $\omega_{g,h}\colon\mathcal{O}_g\otimes_{\mathcal{O}_e}\mathcal{O}_h\to\mathcal{O}_{gh}$, which is induced by concatenation, this definition breaks down exactly to the definition of $\star$ above.

The explicit formula for $\star$ follows immediately from

$$\begin{aligned}\mathcal{I}_{1,3}(f_1\star f_2)(\beta)(x_1)&=\sum_{\substack{x_3\in\mathsf{X}_{p_3}\\ s(x_3)=s(x_1)}}(f_1\star f_2)\big([x_1,x_3]\big)\beta(x_3)\\&=\sum_{\substack{x_3\in\mathsf{X}_{p_3}\\ s(x_3)=s(x_1)}}\sum_{\substack{x_2\in\mathsf{X}_{p_2}\\ s(x_2)=s(x_1)}}f_1\big([x_1,x_2]\big)f_2\big([x_2,x_3]\big)\beta(x_3)\\&=\sum_{\substack{x_2\in\mathsf{X}_{p_2}\\ s(x_2)=s(x_1)}}f_1\big([x_1,x_2]\big)\sum_{\substack{x_3\in\mathsf{X}_{p_3}\\ s(x_3)=s(x_2)}}f_2\big([x_2,x_3]\big)\beta(x_3)\\&=\sum_{\substack{x_2\in\mathsf{X}_{p_2}\\ s(x_2)=s(x_1)}}f_1\big([x_1,x_2]\big)\mathcal{I}_{2,3}(\beta)(x_2)\\&=(\mathcal{I}_{1,2}\circ\mathcal{I}_{2,3})(\beta)(x_2)\end{aligned}$$

for $f_1\in A_R\left(\mathcal{X}_{p_1}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_2}\right)$, $f_2\in A_R\left(\mathcal{X}_{p_2}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_3}\right)$, $\beta\in A_R\left(\mathcal{X}_{p_3}\right)$ and $x_1\in\mathcal{X}_{p_1}$.

Remark 11.4. Note that since $\mathcal{X}_1=\mathcal{G}$ we get $\mathcal{X}_1\circ_{\mathsf{G}}\mathcal{X}^{*}_1\cong\mathcal{G}$ and, it is easy to check that the above-defined multiplicative structure $\star$ on $A_R\left(\mathcal{X}_1\circ_{\mathsf{G}}\mathcal{X}^{*}_1\right)$ is exactly the known ring structure on $A_R(\mathcal{G})$.

Next, we want to relate the R-module $\varinjlim A_R\left(\mathcal{X}_{p_1}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_2}\right)$ to the Steinberg module of $\mathcal{H}_g$. Note that since $\mathcal{H}_g=\varinjlim\mathcal{X}_{p_1}\circ_{\mathsf{G}}\mathcal{X}^{*}_{p_2}$ we prove that taking the Steinberg module commutes with sufficiently well-behaved filtered colimits of topological spaces.

Proposition 11.5. *The Steinberg module of $\mathcal{H}_g$ is isomorphic to the direct limit*

$$A_R\left(\mathcal{H}_g\right) \cong \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_{\mathrm{G}} \mathcal{X}_{p_2}^{\square}\right)$$

as an R-module.

Proof. We start with the diagram in R-$\mathtt{Mod}$, given by the R-module homomorphisms

$$(\alpha_{p_1,p_2}^{q})_{\square}: A_R\left(\mathcal{X}_{p_1} \circ_{\mathrm{G}} \mathcal{X}_{p_2}^{\square}\right) \to A_R\left(\mathcal{X}_{p_1 q} \circ_{\mathrm{G}} \mathcal{X}_{p_2 q}^{\square}\right)$$

for all $(p_1, p_2) \in R_g$ and $q \in P$. The cone maps $\lambda_{p_1,p_2}: \mathcal{X}_{p_1} \circ_{\mathrm{G}} \mathcal{X}_{p_2}^{\square} \to \mathcal{H}_g$ are local homeomorphisms that are injective on $\mathcal{B}_{\mathrm{X}_{p_1} \circ_{\mathrm{G}} \mathrm{X}_{p_2}^{\square}}$ (by Lemma 10.15) and hence by Proposition 6.6 they induce R-module homomorphisms

$$(\lambda_{p_1,p_2})_{\square}: A_R\left(\mathcal{X}_{p_1} \circ_{\mathrm{G}} \mathcal{X}_{p_2}^{\square}\right) \to A_R\left(\mathcal{H}_g\right).$$

Now, since $\lambda_{p_1,p_2} = \lambda_{p_1 q, p_2 q} \circ \alpha_{p_1,p_2}^{q}$ we get $(\lambda_{p_1,p_2})_{\square} = (\lambda_{p_1 q,p_2 q})_{\square} \circ (\alpha_{p_1,p_2}^{q})_{\square}$ (by Proposition 6.6) and hence the $(\lambda_{p_1,p_2})_{\square}$ indeed define a cone under the diagram with nadir $A_R\left(\mathcal{H}_g\right)$. Thus, we get a unique R-module homomorphism

$$\mathcal{J}_g: \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_{\mathrm{G}} \mathcal{X}_{p_2}^{\square}\right) \to A_R\left(\mathcal{H}_g\right), \qquad \left[f, (p_1, p_2)\right] \mapsto (\lambda_{p_1,p_2})_{\square}(f),$$

that sends

$$\left[1_U, (p_1, p_2)\right] \mapsto (\lambda_{p_1,p_2})_{\square}(1_U) = 1_{\lambda_{p_1,p_2}(U)}$$

for all $U \in \mathcal{B}_{\mathrm{X}_{p_1} \circ_{\mathrm{G}} \mathrm{X}_{p_2}^{\square}}$.

Next, we want to define an inverse to this map. By Proposition 10.16 we find an ample base $\mathcal{B}_{\mathrm{H}_g}$ on $\mathcal{H}_g$ and thus by Proposition 6.4 the Steinberg module $A_R\left(\mathcal{H}_g\right)$ is given by the quotient of the direct sum

$$\bigoplus_{V \square \mathrm{B}_{\mathrm{H}_g}} R \cdot 1_V$$

by

$$\langle 1_{U \square V} - 1_U - 1_U \mid U, V, U \sqcup V \in \mathcal{B}_{\mathrm{H}_g} \rangle$$

as an R-module. Now, we can define the R-module homomorphism

$$\bigoplus_{V \square \mathrm{B}_{\mathrm{H}_g}} R \cdot 1_V \to \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_{\mathrm{G}} \mathcal{X}_{p_2}^{\square}\right), \qquad 1_V \mapsto \left[1_U, (p_1, p_2)\right],$$

where $V \in \mathcal{B}_{\mathrm{H}_g}$ is given by $V := \lambda_{p_1,p_2}(U)$ for $U \in \mathcal{B}_{\mathrm{X}_{p_1} \circ_{\mathrm{G}} \mathrm{X}_{p_2}^{\square}}$. This is well-defined, since for $(p_1, p_2), (q_1, q_2) \in R_g$ and $U_1 \in \mathcal{B}_{\mathrm{X}_{p_1} \circ_{\mathrm{G}} \mathrm{X}_{p_2}^{\square}}, U_2 \in \mathcal{B}_{\mathrm{X}_{q_1} \circ_{\mathrm{G}} \mathrm{X}_{q_2}^{\square}}$ such that $\lambda_{p_1,p_2}(U_1) = \lambda_{q_1,q_2}(U_2)$ we find $t_1, t_2 \in P$ such that $(p_1 t_1, p_2 t_1) = (q_1 t_2, q_2 t_2)$ and $\alpha_{p_1,p_2}^{t_1}(U_1) = \alpha_{p_1,p_2}^{t_2}(U_2)$ (by Lemma 10.17). Hence, we get

$$\begin{aligned}
\left[1_{U_1}, (p_1, p_2)\right] &= \left[(\alpha_{p_1,p_2}^{t_1})_{\square}(1_{U_1}), (p_1 t_1, p_2 t_2)\right] = \left[1_{\alpha_{p_1,p_2}^{t_1}(U_1)}, (p_1 t_1, p_2 t_1)\right] \\
&= \left[1_{\alpha_{q_1,q_2}^{t_2}(U_2)}, (q_1 t_2, q_2 t_2)\right] = \left[(\alpha_{q_1,q_2}^{t_2})_{\square}(1_{U_2}), (q_1 t_2, q_2 t_2)\right] \\
&= \left[1_{U_2}, (q_1, q_2)\right],
\end{aligned}$$

that is, the map is well-defined.

Next, we want to show that this R-module homomorphism descends to the quotient $A_R\left(\mathcal{H}_g\right)$. Consider $V = V_1 \sqcup V_2, V_1, V_2 \in \mathcal{B}_{\mathrm{H}_g}$. Thus, $V = \lambda_{p_1,p_2}(U)$ for some $U \in \mathcal{B}_{\mathrm{X}_{p_1} \circ_{\mathrm{G}} \mathrm{X}_{p_2}^{\square}}$. As $\lambda_{p_1,p_2}|_U$ is a homeomorphism onto its image we can define the compact open subsets $U_i := \lambda_{p_1,p_2}|_U^{-1}(V_i) \subset U$ and hence we have $U_i \in \mathcal{B}_{\mathrm{X}_{p_1} \circ_{\mathrm{G}} \mathrm{X}_{p_2}^{\square}}$ (as $\mathcal{B}_{\mathrm{X}_{p_1} \circ_{\mathrm{G}} \mathrm{X}_{p_2}^{\square}}$ is stable under taking compact open subsets) and $V_i = \lambda_{p_1,p_2}(U_i)$ by definition for $i = 1, 2$. Hence, our R-module homomorphism maps 1_V to

$$\left[1_U, (p_1, p_2)\right] = \left[1_{U_1 \square U_2}, (p_1, p_2)\right] = \left[1_{U_1}, (p_1, p_2)\right] + \left[1_{U_2}, (p_1, p_2)\right],$$

which is exactly the image of $1_{V_1} + 1_{V_2}$. Thus, the R-module homomorphism indeed descends to the quotient and gives us a well-defined R-module homomorphism

$$A_R(\mathcal{H}_g) \to \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^*_{p_2}\right), \qquad 1_{\lambda_{p_1,p_2}(U)} \mapsto \left[1_U, (p_1, p_2)\right]$$

for all $U \in \mathcal{B}_{\mathsf{X}_{p_1} \circ_{\mathsf{G}} \mathsf{X}^*_{p_2}}$. Now, this homomorphism is inverse to $\mathcal{J}_g$ and hence $\mathcal{J}_g$ is indeed an R-module isomorphism.

Finally, we get our main result that the covariance ring of a diagram of bimodules $\mathcal{F}$ obtained from an Ore diagram of correspondences X in $\mathsf{Gr}_{\mathsf{co,tight}}$ is given by the Steinberg algebra of the groupoid model $\mathcal{H}$ of X.

Theorem 11.6. *Consider an Ore diagram* $\mathsf{X} = (P, \mathcal{G}, \mathcal{X}_p, \mu_{p,q})$ *in* $\mathsf{Gr}_{\mathsf{co,tight}}$ *with a groupoid model* $\mathcal{H}$*. Then the Steinberg algebra of the groupoid model* $\mathcal{H}$ *gives the covariance ring of the diagram* $\mathcal{F} := A * \mathsf{X}$ *in* $\mathsf{Rings}_{\mathsf{fgp}}$*, that is, we have an isomorphism*

$$A_R(\mathcal{H}) \cong \mathcal{O}_{\mathsf{F}}$$

of R*-modules and unital rings.*

Proof. Using Lemma 6.2 and Proposition 11.5 we get an isomorphism

$$A_R(\mathcal{H}) \cong \bigoplus_{g\in G} A_R(\mathcal{H}_g) \cong \bigoplus_{g\in G} \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^*_{p_2}\right)$$

of R-modules given by

$$\mathcal{J} \colon \bigoplus_{g\in G} \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^*_{p_2}\right) \to A_R(\mathcal{H}), \qquad \left[f, (p_1, p_2)\right] \mapsto (\lambda_{p_1,p_2})_*(f)$$

for $\left[f, (p_1, p_2)\right] \in \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^*_{p_2}\right)$ with $g = p_1 p_2^{-1}$, $f \in A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^*_{p_2}\right)$ and $(\lambda_{p_1,p_2})_*(f) \in A_R(\mathcal{H}_g) \subset A_R(\mathcal{H})$.

Now, we want to show that this is also an isomorphism of unital rings. We take $\varphi, \psi \in \bigoplus_{g\in G} \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^*_{p_2}\right)$. Since $\mathcal{J}$ is additive and independent of the representative, it is sufficient to consider $\varphi = \left[f_1, (p_1, p_2)\right]$, $\psi = \left[f_2, (q_1, q_2)\right]$ for $f_1 \in A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^*_{p_2}\right)$, $f_2 \in A_R\left(\mathcal{X}_{p_2} \circ_{\mathsf{G}} \mathcal{X}^*_{p_3}\right)$ and $p_1, p_2, p_3 \in P$. Now, for all $z \in \mathcal{H}$ we get

$$\begin{aligned}
\mathcal{J}(\varphi \star \psi)(z) &= \mathcal{J}\Big(\left[f_1 \star f_2, (p_1, p_3)\right]\Big)(z) \\
&= (\lambda_{p_1,p_3})_*(f_1 \star f_2)(z) \\
&= \sum_{\substack{[x_1,x_3]\in \mathsf{X}_{p_1}\circ_{\mathsf{G}} \mathsf{X}^*_{p_3} \\ \lambda_{p_1,p_3}([x_1,x_3])=z}} (f_1 \star f_2)\big([x_1, x_3]\big) \\
&= \sum_{\substack{[x_1,x_3]\in \mathsf{X}_{p_1}\circ_{\mathsf{G}} \mathsf{X}^*_{p_3} \\ \lambda_{p_1,p_3}([x_1,x_3])=z}} \; \sum_{\substack{x_2\in \mathsf{X}_{p_2} \\ s(x_2)=s(x_1)}} f_1\big([x_1, x_2]\big) f_2\big([x_2, x_3]\big) \\
&= \sum_{\substack{x\in \mathsf{X}_{p_1}\circ_{\mathsf{G}} \mathsf{X}^*_{p_2},\, y\in \mathsf{X}_{p_2}\circ_{\mathsf{G}} \mathsf{X}^*_{p_3} \\ \lambda_{p_1,p_2}(x)\lambda_{p_2,p_3}(y)=z}} f_1(x) \cdot f_2(y) \\
&= \sum_{xy=z} (\lambda_{p_1,p_2})_*(f_1)(x) \cdot (\lambda_{p_2,p_3})_*(f_2)(y) \\
&= \big((\lambda_{p_1,p_2})_*(f_1) * (\lambda_{p_2,p_3})_*(f_2)\big)(z) \\
&= \big(\mathcal{J}(\varphi) * \mathcal{J}(\psi)\big)(z),
\end{aligned}$$

and hence $\mathcal{J}(\varphi \star \psi) = \mathcal{J}(\varphi) * \mathcal{J}(\psi)$. Now, the multiplicative unit of

$$\bigoplus_{g\in G} \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_{\mathsf{G}} \mathcal{X}^*_{p_2}\right)$$

is $[1_{G^0}, (1,1)]$ with $1_{G^0} \in A_R(\mathcal{G}) \cong A_R(\mathcal{X}_1 \circ_G \mathcal{X}_1^{\square})$ (using $\mathcal{G} \cong \mathcal{X}_1 \circ_G \mathcal{X}_1^{\square}$ and Remark 11.4). It is mapped under $\mathcal{J}$ to

$$\mathcal{J}\Big(\big[1_{G^0}, (1,1)\big]\Big) = \lambda_{1,1}(1_{G^0}) = 1_{\lambda_{1,1}(G^0)} = 1_{H^0},$$

where the last equality follows from the definition of $\mathcal{H}^0$ as it is defined in [Alb15]. Hence, the R-module isomorphism $\mathcal{J}$ is also a unital ring isomorphism giving

$$\bigoplus_{g \in G} \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_G \mathcal{X}_{p_2}^{\square}\right) \cong A_R(\mathcal{H}).$$

Additionally, by Proposition 11.2 and Proposition 11.3 we have an isomorphism

$$\mathcal{O}_F \cong \bigoplus_{g \in G} \varinjlim A_R\left(\mathcal{X}_{p_1} \circ_G \mathcal{X}_{p_2}^{\square}\right)$$

of R-modules and unital rings. So finally, we can compose these two isomorphisms of R-modules and unital rings to get the desired isomorphism.

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