Exercise sheet 1.

Exercise	1	2	3	4	Σ
Points					

Name

Deadline: **Wednesday, 27.4.2022, 16:00**.

Please use this page as a cover sheet and enter your name in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Let **Ab** be the category of abelian groups and group homomorphisms. Let A and B be abelian groups. The *tensor product* $A \otimes B$ is an Abelian group defined follows. It is the quotient of the free abelian group on the set of symbols $\{a \otimes b \mid a \in A,\ b \in B\}$ modulo the relations

$$(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b \text{ for all } a_1, a_2 \in A,\ b \in B,$$
$$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2 \text{ for all } a \in A,\ b_1, b_2 \in B.$$

(i) Let $f\colon A \to A'$ and $g\colon B \to B'$ be two homomorphisms of abelian groups. Show that there is a unique homomorphism $f \otimes g\colon A \otimes B \to A' \otimes B'$ that satisfies

$$f \otimes g(a \otimes b) = f(a) \otimes f(b) \qquad \text{for all } a \in A,\ b \in B. \tag{1}$$

Prove that this makes the construction of $A \otimes B$ to a bifunctor $\mathbf{Ab} \times \mathbf{Ab} \to \mathbf{Ab}$.

(ii) Show that for any abelian group A, the tensor product $A \otimes \mathbb{Z}^n$ is isomophic to A^n.

(iii) Calculate $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$ for natural numbers $n, m \in \mathbb{N}$.

Exercise 2. Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be an exact sequence of abelian groups. This means that f is injective and that g descends to an isomorphism $B/f(A) \cong C$. Suppose that D is another abelian group.

(i) Prove that the sequence

$$A \otimes D \xrightarrow{f \otimes \mathrm{id}} B \otimes D \xrightarrow{g \otimes \mathrm{id}} C \to 0$$

is exact. That is, $g \otimes \mathrm{id}$ induces an isomorphism from $B \otimes D/(f \otimes \mathrm{id})(A \otimes D)$ onto $C \otimes D$.

(ii) Give an example where $f \otimes \mathrm{id}$ is not injective (**Hint:** you may take $D = \mathbb{Z}/n\mathbb{Z}$.)

Exercise 3. Let R be an associative ring. We denote by R-**mod** (resp. **mod**-R) the category of left and right R-modules, respectively. For $M \in$ **mod**-R and $N \in R$-**mod** define the abelian group $M \otimes_R N$ as a quotient of $M \otimes N$ by the subgroup generated by the elements of the form $(m \cdot r) \otimes n - m \otimes (r \cdot n)$ for $r \in R$, $m \in M$, $n \in N$.

(i) For R-module maps $f\colon M \to M'$, $g\colon N \to N'$ check that the map $f \otimes g$ induces a map $M \otimes_R N \to M' \otimes_R N'$.

(ii) Let $Z(R)$ be the center of R. Show that the formula $z \cdot (m \otimes n) = (m \cdot z) \otimes n$ for $z \in Z(R)$, $m \in M$, $n \in N$ defines a $Z(R)$-module structure on $M \otimes_R N$. In particular, if R is a k-algebra for some field k, then $M \otimes_R N$ is naturally a k-vector space.

(iii) Let $R = \mathbb{C}[x]$ and $M = \mathbb{C}[x]/(x^n)$, $N = \mathbb{C}[x]/(x^m)$ for $n, m \geq 1$. Describe $M \otimes_R N$ as an R-module (since R is commutative, we have $R = Z(R)$).

Exercise 4. Prove that the multiplication map $r \otimes m \mapsto r \cdot m$ defines an isomorphism $R \otimes_R M \cong M$ for any left R-module M. Similarly, $n \otimes r \mapsto n \cdot r$ defines an isomorphism $N \otimes_R R \cong N$ for any right R-module N.