Mathematisches Institut
Georg-August-Universität Göttingen
Prof. Dr. Ralf Meyer and Boris Bilich.

Exercise sheet 2.

Name

Exercise	1	2	3	4	Σ
Points					

Deadline: **Wednesday, 4.5.2022, 16:00**.

Please use this page as a cover sheet and enter your name in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1 (Leavitt algebras)**.** We say that a ring R has the *Invariant Basis Number (IBN)* property if $R^m \cong R^{m'}$ for $m, m' \in \mathbb{N}$ implies $m = m'$. The Leavitt algebra $L_n = L_n(\mathbb{Z})$ is an example of a ring without the IBN property.

(i) Prove that commutative rings have the IBN property.

(ii) Conclude that the ideal generated by elements of the form $xy - yx$ in L_n is the whole L_n. In particular, there are no homomorphisms from L_n to a commutative ring.

(iii) Let $\mathbb{K}$ be a field and let V be an infinite-dimensional $\mathbb{K}$-vector space. For $R = \operatorname{End}_{\mathbb{K}}(V)$ show that $R^n \cong R^m$ for any $m, n \in \mathbb{N}$ (**Hint:** it is enough to consider $m = 1$, $n = 2$. For this case, represent the potential isomorphism as a column vector as in the first lecture).

Exercise 2 (The centre is Morita invariant)**.**

(i) Let R be a ring and let E be an R-module. Denote by $Z(R)$ the center of R. For $c \in Z(R)$ define a map $m_c\colon E \to E$ as $m_c(e) = c \cdot e$ for all $e \in E$. Show that m_c is an R-module homomorphism and $c \mapsto m_c$ is a ring homorphism $Z(R) \to \operatorname{End}_R(E)$.

(ii) Suppose that rings R and Q are Morita equivalent. Show that $Z(R) \cong Z(Q)$. Conclude that commutative rings are Morita equivalent if and only if they are isomorphic.

Exercise 3 (Simple examples of Morita equivalences)**.**

(i) Let $R = \mathbb{Z}$ or a field $\mathbb{K}$. Prove that if a ring Q is Morita equivalent to R then $Q \cong \mathrm{M}_n(R)$ for some $n \in \mathbb{N}_{\geq 1}$.

(ii) Let G and H be finite groups. Show that $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are Morita equivalent if and only if G and H have the same number of conjugacy classes. You may use that every representation of a finite group over $\mathbb{C}$ is a sum of irreducible finite-dimensional representations and the number of isomorphism classes of irreducible representations is the same as the number of conjugacy classes.

Exercise 4 (A nontrivial Morita equivalence)**.** Let $U \subset \mathrm{M}_2(\mathbb{C})$ be the subalgebra of upper-triangular matrices. Denote by $e_1 = (1,0)^T$ and $e_2 = (0,1)^T$ the canonical basis of $\mathbb{C}^2$. Then $P_1 = \langle e_1 \rangle \subseteq \mathbb{C}^2$ and $P_2 = \langle e_1, e_2 \rangle = \mathbb{C}^2$ are left U-modules with respect to the matrix multiplication.

(i) Show that $P_1 \oplus P_2$ is the free U-module of rank 1. Deduce that P_1 and P_2 are projective and finitely generated.

(ii) Is P_1 or P_2 a projective generator?

(iii) Prove that $P = P_1^2 \oplus P_2$ is a projective generator and explicitly describe $U' = \operatorname{End}_U(P)$ as a subalgebra of $\mathrm{M}_4(\mathbb{C})$.

(iv) Show that the algebras U and U' are Morita equivalent but U' is not isomorphic to $\mathrm{M}_n(U)$ for any $n \in \mathbb{N}$.