Exercise sheet 3.

Exercise	1	2	3	Σ
Points				

Name

Deadline: **Wednesday, 12.11.2022, 16:00**.

Please use this page as a cover sheet and enter your name in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1 (Fundamental bigroupoid)**.** Let X be a topological space. We define the *fundamental bigroupoid* $\Pi_2(X)$ as follows. Objects of $\Pi_2(X)$ are points of X and 1-arrows between $x, y \in X$ are continuous paths $f\colon [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. Composition of arrows is given by concatenation: if $f\colon x \to y$ and $g\colon y \to z$, then $g \circ f$ is given by

$$g \circ f(t) = \begin{cases} f(2t), & \text{if } t \leq \frac{1}{2} \\ g(2t-1), & \text{if } t > \frac{1}{2}. \end{cases}$$

The 2-arrows are homotopy classes of basepoint-preserving homotopies. That is, for $f, g\colon x \to y$ the 2-arrow $\gamma\colon f \to g$ is an equivalence class of continuous map $\gamma\colon [0,1] \times [0,1] \to X$ such that $\gamma(t,0) = f(t), \gamma(t,1) = g(t), \gamma(0,s) = x$, and $\gamma(1,s) = y$ for all $t, s \in [0,1]$. Two maps γ, γ' are equivalent if and only if there is a basepoint preserving homotopy between them (i.e the map $[0,1]^3 \to X$ with analogous properties).

For an object x, the unit 1_x is just a trivial path $f(t) = x$. For 1-arrow γ, the unit 1_γ is a trivial homotopy.

Construct uniters and associators and verify the axioms of a bicategory. You may assume that the data given above satisifes the first 4 axioms. **This exercsise counts double.**

Exercise 2 (Decategorification)**.** Let $\mathcal{C}$ be a bicategory. Let $\mathcal{C}'$ be the set of isomorphism classes of 1-arrows. Show that there is a category with object set $\mathcal{C}^0$ and set of arrows $\mathcal{C}'$, with the product defined by $[f] \circ [g] := [f \circ g]$ for composable arrows f, g in $\mathcal{C}$. Show that an arrow f in $\mathcal{C}$ is an equivalence if and only if its image in $\mathcal{C}'$ is invertible.

Exercise 3 (Adjoint equivalence)**.** Let $\mathcal{C}$ be a bicategory. Let $\alpha\colon x \to y$ be an equivalence in $\mathcal{C}$. Choose $\beta\colon y \to x$ with $\beta \circ \alpha \cong 1_x$ and $\alpha \circ \beta \cong 1_y$. Then the invertible 2-arrows $1_x \Rightarrow \beta \circ \alpha$ and $\alpha \circ \beta \Rightarrow 1_y$ may be chosen so that the resulting composite 2-arrows

$$\alpha \cong \alpha \circ 1_x \Rightarrow \alpha \circ (\beta \circ \alpha) \cong (\alpha \circ \beta) \circ \alpha \Rightarrow 1_y \circ \alpha \cong \alpha,$$
$$\beta \cong 1_x \circ \beta \Rightarrow (\beta \circ \alpha) \circ \beta \cong \beta \circ (\alpha \circ \beta) \Rightarrow \beta \circ 1_y \cong \beta$$

are both unit 2-arrows. The arrows α and β together with 2-arrows with these properties are called an *adjoint equivalence*.