Prof. Dr. Ralf Meyer and Boris Bilich.

Exercise sheet 4.

Name ______________________

Exercise	1	2	3	4	Σ
Points					

Deadline: **Wednesday, 18.05.2022, 16:00**.

Please use this page as a cover sheet and enter your name in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a morphism. Show that F maps invertible 2-arrows to invertible 2-arrows. Thus F preserves the isomorphism relation on arrows.

Exercise 2. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a homomorphism. Show that F descends to a functor between the 1-categories associated to $\mathcal{C}$ and $\mathcal{D}$ (see Exercise 2, Sheet 3). Deduce that F maps an equivalence $f\colon x \to y$ in $\mathcal{C}$ to an equivalence in $\mathcal{D}$. Thus $F^0(x)$ and $F^0(y)$ are equivalent in $\mathcal{D}$ if x and y are equivalent in $\mathcal{C}$.

Exercise 3. Let X and Y be topological spaces. Consider the fundamental bigroupoids $\Pi_2(X)$ and $\Pi_2(Y)$. We are going to define a homomorphism $\Pi_2(f)\colon \Pi_2(X) \to \Pi_2(Y)$ for a continuous map $f\colon X \to Y$. On objects, it acts by $\Pi_2(f)(x) := f(x) \in \Pi_2(Y)^0$ for any $x \in \Pi_2(X)^0 = X$. For an arrow $\varphi\colon [0,1] \to X$ and a 2-arrow $\gamma\colon [0,1]^2 \to X$ we define $\Pi_2(f)(\varphi) = f \circ \varphi$ and $\Pi_2(f)(\gamma) = f \circ \gamma$. Extend this data to a morphism of bicategories and check the axioms. Show that it is even a homomorphism.

Exercise 4. Let R be a ring. We will construct a graded ring from the action of a discrete group on R.

Let $f\colon R \to Q$ be a ring homomorphism from R to some other ring Q. Define a Q,R-bimodule Q_f such that it is the same as Q as an abelian group, the action of Q is given by the left multiplication, and the action of $r \in R$ is given by the right multiplication by $f(r)$.

(i) Suppose that $g\colon Q \to S$ is another ring homomorphism. Prove that the map $\mu_{g,f}\colon S_g \otimes_Q Q_f \to S_{g\circ f}$ defined by $\mu_{g,f}(s \otimes q) = s \cdot g(q)$ for $s \in S_g$, $q \in Q_f$ is an isomorphism of S,R-bimodules.

(ii) Let G be a discrete group acting by automorphisms on R. That is, we have a map $g \mapsto \gamma_g \in \mathrm{Aut}(R)$ such that $\gamma_g \circ \gamma_h = \gamma_{gh}$. Let $e \in G$ be the unit element. Define $G \ltimes R = \bigoplus_{g\in G} R_{\gamma_g}$ as a graded R-bimodule with multiplication of homogeneous elements given by

$$r \cdot s = \mu_{\gamma_g,\gamma_h}(r \otimes s) \in R_{\gamma_{gh}} \subset G \ltimes R$$

for $r \in R_g$ and $s \in R_h$. Check that this defines a G-graded ring with $(G \ltimes R)_e$ isomorphic to R.