Mathematisches Institut
Georg-August-Universität Göttingen
Prof. Dr. Ralf Meyer and Boris Bilich.

Bicategories in noncommutative geometry
Summer term 2022

Exercise sheet 5.

Name

Exercise	1	2	3	4	Σ
Points					

Deadline: **Wednesday, 25.05.2022, 16:00**.

Please use this page as a cover sheet and enter your name in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1 (Intertwiners as 2-arrows)**.** Let $f_1, f_2 \colon S \rightrightarrows R$ be ring homomorphisms. Find a bijection between bimodule homomorphisms $R_{f_1} \to R_{f_2}$ and elements $r \in R$ that satisfy $r \cdot f_1(s) = f_2(s) \cdot r$ for all $s \in S$ ("intertwiners"). The vertical product gives the multiplication in R for these intertwiners. Describe the horizontal product of intertwiners.

Exercise 2 (Uniters for the unit arrow)**.** Let $\mathcal{C}$ be a bicategory and $x, y \in \mathcal{C}^0$ be objects.

(i) Let $f \colon x \to y$ be a 1-arrow. Use the naturality of uniters to show that $l_{1_y \circ f} = 1_y \bullet l_f$ and $r_{f \circ 1_x} = r_f \bullet 1_x$.

(ii) Apply the triangle axiom twice to show that $1_x \bullet (l_{1_x} \bullet 1_x) = 1_x \bullet (1_x \bullet l_{1_x}) \circ 1_x \bullet \mathrm{ass}$.

(iii) Combine (i) and (ii) to prove that $l_{1_x} = r_{1_x}$.

Exercise 3 (Endomorphisms of the unit)**.** (i) Let $c_1, c_2 \colon 1_x \Rightarrow 1_x$ be 2-arrows. Use the naturality of uniters and the previous exercise to prove that $c_1 \circ c_2 = l_{1_x} \circ c_1 \bullet c_2 \circ l_{1_x}^{-1}$.

(ii) Show that $c_1 \circ c_2 = c_2 \circ c_1$. Hence, the endomorphism monoid of the unit object is commutative.

(iii) Apply this to the fundamental bigroupoid of a topological space X to prove that the second homotopy group $\pi_2(X)$ is abelian.

Exercise 4 (Homotopy as a transformation)**.** Let X, Y be topological spaces, let $f, g \colon X \rightrightarrows Y$ be continuous maps and let $\rho \colon X \times [0,1] \to Y$ be a homotopy from f to g. Consider the homomorphisms $\Pi_2(f), \Pi_2(g) \colon \Pi_2(X) \to \Pi_2(Y)$ defined in the previous sheet. For any object $x \in \Pi_2(X)^0 = X$, let $\Pi_2(\rho)_x \colon f(x) \to g(x)$ be the path $t \mapsto \rho(x,t)$. Show that these arrows are part of a transformation $\Pi_2(\rho) \colon \Pi_2(f) \Rightarrow \Pi_2(g)$, that is, define the natural 2-arrows in the definition of a transformation that are still missing and check the axioms.