Exercise sheet 6.

Exercise	1	2	3	4	Σ
Points					

Name

Deadline: **Thursday, 2.6.2022, 16:00**.

Please use this page as a cover sheet and enter your name in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Describe two morphisms $\mathcal{C} \rightrightarrows \mathfrak{Rings}$ through $S = \bigoplus_{\gamma\in\mathcal{C}} S_\gamma$ with $R \to S|_{\mathcal{C}^0}$ and $T = \bigoplus_{\gamma\in\mathcal{C}} T_\gamma$ with $U \to T|_{\mathcal{C}^0}$. Describe two transformations between these morphisms through $\mathcal{C}^0$-graded R,U-bimodules M and N with suitable $\mathcal{C}$-graded S,T-bimodule structures on $M\otimes_U T$ and $N\otimes_U T$. Show that a modification between these transformations is equivalent to an R,U-bimodule map $\varphi\colon M \to N$ such that the induced grading-preserving right T-module map $\varphi\otimes_U \mathrm{id}_T\colon M\otimes_U T \to N\otimes_U T$ is also a left S-module map. If $U \to T|_{\mathcal{C}^0}$ is an isomorphism, then show that modifications are equivalent to grading-preserving S,T-bimodule maps $M\otimes_U T \to N\otimes_U T$.

Exercise 2. Let $E = (E^0, E^1, s, r)$ be a directed graph. Here, E^0 is a finite set of vertices, E^1 is a set of edges, and $s, r\colon E^1 \rightrightarrows E^0$ are the source and range maps on edges. Consider the ring $R_0 = \oplus_{v\in E^0}\mathbb{C}$ and denote by $p_v \in R_0$ the idempotent corresponding to p_v. Consider the vector space R_1 with basis $\{s_e\colon e\in E^1\}$. Define a R_0-bimodule structure on R_1 as

$$s_e\cdot p_v = \delta_{s(e)v} s_e,\ p_v\cdot s_e = \delta_{r(e)v} s_e.$$

This data defines a homomorphism $F\colon (\mathbb{N},+) \to \mathfrak{Rings}$. Denote the corresponding graded ring by R. Show that the n-th graded component of R has a natural basis indexed by paths of length n in E such that the product is given by the concatenation of paths.

Exercise 3. We use the notation of the previous exercise. Let B be some other $\mathbb{C}$-algebra. Consider the category $\mathcal{E}(B)$ whose objects are pairs $(\{M_v\}_{v\in E^0}, \{\varphi_e\}_{e\in E^1})$, where M_v are B-modules and $\varphi_e\colon M_{s(e)} \to M_{r(e)}$ are module homomorphisms. A morphisms f between $(\{M_v\}_{v\in E^0}, \{\varphi_e\}_{e\in E^1})$ and $(\{N_v\}_{v\in E^0}, \{\psi_e\}_{e\in E^1})$ in $\mathcal{E}(B)$ is a collection of module homomorphism $f_v\colon M_v \to N_v$ for $v\in E^0$ such that $f_{r(e)}\circ\varphi_e = \psi_e\circ f_{s(e)}$ for all $e\in E^1$. These morphisms are composed simply by composing the homomorphisms in the collection. Show that $\mathcal{E}$ is equivalent to the category $\mathfrak{Rings}(R, B)$.

Exercise 4. A *crossed module* consists of two groups G and H and group homomorphisms $\partial\colon H\to G$ and $c\colon G \to \mathrm{Aut}(H)$ such that $\partial(c_g(h)) = g\partial(h)g^{-1}$ and $c_{\partial(h)}(k) = hkh^{-1}$ for all $g\in G$, $h,k\in H$.

Let (G, H, ∂, c) be a crossed module. We are going to build a 2-group $\mathcal{C}$ (that is, a strict bicategory with one object and all 1- and 2-arrows being invertible). The arrow set of $\mathcal{C}$ is G with the given group law. The set of 2-arrows is $H\times G$, and the horizontal product is the product in the semidirect product group $H\rtimes_c G$. A pair (h,g) is a 2-arrow $h\colon g\Rightarrow \partial(h)g$, that is, $s(h,g) = g$ and $r(h,g) = \partial(h)g$. The unit 2-arrow on $g\in G$ is $(1,g)$. The horizontal product is

$$(h_1, g_1)\bullet(h_2, g_2) := (h_1\cdot c_{g_1}(h_2), g_1g_2),$$

and the vertical product is

$$(h_1, \partial(h_2)g)\circ(h_2, g) := (h_1\cdot h_2, g).$$

Prove that this data satisfies the axioms of a 2-category.