Mathematisches Institut
Georg-August-Universität Göttingen
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Exercise sheet 7.

Exercise	1	2	3	4	Σ
Points					

Name

Deadline: **Wednesday, 8.6.2022, 16:00**.

Please use this page as a cover sheet and enter your name in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1 (Example from number theory)**.** Consider the ring $R = \mathbb{Z}[\sqrt{-5}] = \mathbb{Z}[x]/(x^2+5)$ and the ideal $I = (2, 1+\sqrt{-5}) \subset R$. One can show that $I \oplus I = R \oplus R$, so I is a projective generator. Give I the symmetric R-bimodule structure where $x \cdot y = y \cdot x$ for all $x \in I \subseteq R$, $y \in R$.

(i) Prove that I is not principal, that is, it is not generated by a single element. In particular, I is not isomorphic to R as an R-module.

(ii) Show that the map $\mu_I \colon I \otimes_R I \to R$, $a \otimes b \mapsto \frac{ab}{2}$ for $a, b \in I$ is a well defined bimodule isomorphism.

(iii) We can think about I as an arrow $R \to R$ in the bicategory $\mathfrak{Rings}$. By (ii), the composition of I with itself is isomorphic to R. Show that this gives a homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathfrak{Rings}$.

(iv) The covariance ring $R[I]$ of this diagram is $R \oplus I$ as an R-bimodule and the multiplication is defined as $(a, b) \cdot (c, d) = (ac + \mu_I(b, d), ad + bc)$. Find an isomorphism between $R[I]$ and $R[s, t]/(s^2 - 2, t^2 - \sqrt{-5} - 2, st - 1 - \sqrt{-5})$.

Exercise 2 (Diagrams over $\mathbb{N}^2$)**.** We already know that a homomorphism $\mathbb{N}^2 \to \mathfrak{Rings}$ is "equivalent" to a strongly $\mathbb{N}^2$-graded ring. Show that such homomorphisms are "equivalent" to the following data:

- a ring R;
- R, R-bimodules X and Y for the two generators of $\mathbb{N}^2$;
- an R, R-bimodule isomorphism $X \otimes_R Y \cong Y \otimes_R X$ that allows to define a bimodule for $(1, 1, 0)$ and $(1, 0, 1)$.

Exercise 3 (Invertible arrow diagram)**.** Consider the category $\mathcal{C}$ with two objects x, y and the only nontrivial arrows $f \colon x \to y$ and $g \colon y \to x$ such that $fg = 1_y$, $gf = 1_x$. Let $\mathcal{D}$ be a bicategory and let $F \colon \mathcal{C} \to \mathcal{D}$ be a homomorphism. Let $D \in \mathcal{D}^0$. Prove that the categories $\mathrm{Cone}(D, F)$ and $\mathcal{D}(D, F^0(x))$ are equivalent.

Exercise 4 (Twisted action)**.** Let G be a group viewed as a category with one object $G^0 = \{*\}$. Let R be a commutative ring. We are going to classify strictly unital homomorphisms $F \colon G \to \mathfrak{Rings}$ with $F^0(*) = R$ and $F(g) = 1_R$ for any $g \in G$. To completely define a strictly unital homomorphism, we also need invertible 2-arrows $\mu_{f,g} \colon F(f) \circ F(g) = R \otimes_R R \Rightarrow F(f \circ g) = R$ for $f, g \in G$. Then $\mu_{f,g} \circ l_R^{-1}$ is an invertible 2-arrow from R to R. As such, it is equal to multiplication by a unique invertible element $u(f, g) \in R^\times$. Since R is commutative, any element is allowed.

(i) Show that the arrows $\{\mu_{f,g}\}$ define a strictly unital homomorphism if and only if the following holds:
   - $u(f, g) \cdot u(fg, h) = u(g, h) \cdot u(f, gh)$;
   - $u(1, g) = u(g, 1) = 1$.

   Check that u is a normalised 2-cocycle $G \times G \to R^\times$ (nothing to write here). If you do not know group cohomology, look up its definition.

(ii) Extend the above results to the situation when $F(g) = R$ as a right module, with left action given by $a \cdot b := \alpha_g(a)b$ for some automorphisms $\alpha_g \colon R \to R$ for $g \in G$.