Mathematisches Institut
Georg-August-Universität Göttingen
Prof. Dr. Ralf Meyer and Boris Bilich.

Exercise sheet 8.

Name

Exercise	1	2	3	4	Σ
Points					

Deadline: **Wednesday, 15.6.2022, 16:00**.

Please use this page as a cover sheet and enter your name in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Let K be a field. Consider the following graphs.

$$R_1 = \bullet^{v} \circlearrowleft e,$$

$$A_n = \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n},$$

$$E_T = e \circlearrowright \bullet^{u} \xrightarrow{f} \bullet^{v}.$$

Show that the Leavitt path algebras of those graphs are:

(i) $L_K(R_1) \cong K[x, x^{-1}]$, the algebra of Laurent polynomials;

(ii) $L_K(A_n) \cong \mathbb{M}_n(K)$;

(iii) $L_K(E_T) \cong K\langle x, y\rangle/(xy = 1)$, the algebraic Toeplitz algebra (**Hint:** take $y = s_e + s_f$).

Exercise 2. Let R be a ring and $s \in R$ be a regular element, i.e., a non-zero divisor. We say that s satisfies the *right Ore condition* if for every element $r \in R$ there are $n \in \mathbb{N}$ and $r' \in R$ such that $sr' = rs^n$. Consider the set Rs^{-1} of right fractions rs^{-n} for $r \in R$ modulo the relation $r_1s^{-n} = r_2s^{-m}$ if and only if $r_1s^m = r_2s^n$ in R. Define the product and sum on Rs^{-1} by the formula

$$r_1s^{-n} + r_2s^{-m} = (r_1s^m + r_2s^n)s^{-n-m}, \; (r_1s^{-n})(r_2s^{-m}) = r_1r_2's^{-n'-m}$$

with $r_2' \in R$ and $n' \in \mathbb{N}$ such that $s^nr_2' = r_2s^{n'}$ (we can find such r_2' and n' by iterating the Ore condition).

(i) Show that the formula above defines a ring structure on Rs^{-1} and R is a subring in Rs^{-1}. This ring is called the *Ore localization*.

(ii) By the universal property, there is a unique ring homomorphism $R_s \to Rs^{-1}$ from the Cohn localization by s to the Ore localization. Prove that this is an isomorphism.

(iii) Let $\mathbb{C}\langle x, y\rangle/(xy - yx - 1)$ be the Weyl algebra. Show that x satisfies the right Ore condition.

Exercise 3. Consider the *coequalizer category* $\mathcal{C}$:

$$1_x \circlearrowright x \overset{f}{\underset{g}{\rightrightarrows}} y \circlearrowleft 1_y .$$

Let $m, n \in \mathbb{N}$ be natural numbers. Define a diagram $L_{m,n}\colon \mathcal{C} \to \mathfrak{Rings}$ by $L^0_{m,n}(x) = L^0_{m,n}(y) = \mathbb{C}$, $L_{m,n}(f) = \mathbb{C}^m$, and $L_{m,n}(g) = \mathbb{C}^n$.

(i) Prove that the lax covariance ring of $L_{m,n}$ is finite-dimensional and find its dimension.

(ii) Show that the strong covariance ring of $L_{m,n}$ is infinite-dimensional unless $m = n = 1$.

Exercise 4. Let M be a finitely generated projective bimodule over the ring R. Consider the diagram $F\colon (\mathbb{N}, +) \to \mathfrak{Rings}$ generated by M. Prove that there is a unique $\mathbb{Z}$-grading on the strong covariance ring of F such that the elements of R have degree 0 and elements of M have degree 1. Describe this grading in case of Leavitt path algebras (i) and (iii) from Exercise 1.