Exercise sheet 9.

Exercise	1	2	3	4	Σ
Points					

Name

Deadline: **Wednesday, 22.6.2022, 16:00**.

Please use this page as a cover sheet and enter your name in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Consider the ring $R = \mathbb{Z}[\sqrt{-5}]$ and the ideal $I = (2, 1 + \sqrt{-5})$ from Exercise 1, Sheet 7. Prove that the Cohn localisation of R at the inclusion $\iota\colon I \hookrightarrow R$ is isomorphic to the localisation of R by $2 \in R$.

Exercise 2. Let R be a commutative ring and let $s \in \mathbb{M}_{n\times n}(R)$. Show that the Cohn localisation of R at the matrix s is isomorphic to the localisation of R at $\det(s) \in R$.

Exercise 3. Let R be an algebra over a field of characteristic $\neq 2$ and let M be an R-bimodule. A linear map $\partial\colon R \to M$ is called a *derivation* if for any $r_1, r_2 \in R$ we have $\partial(r_1 r_2) = r_1\partial(r_2) + \partial(r_1)r_2$. Derivations from R to M form an abelian group which will be denoted by $\mathrm{Der}(R, M)$.

(i) Let $s \in R$ and let M be an $R[s^{-1}]$-module. Prove that any derivation $\partial \in \mathrm{Der}(R, M)$ extends uniquely to a derivation $\partial_s \in \mathrm{Der}(R[s^{-1}], M)$ by the rule $\partial_s(s^{-1}) = -s^{-1}\partial(s)s^{-1}$.

(ii) Let $\partial \in \mathrm{Der}(R, M)$ be a derivation. Consider the matrix ring $\mathbb{M}_n(R)$ and turn M into an $\mathbb{M}_n(R)$-bimodule $\mathbb{M}_n(M)$. Define a derivation $\mathbb{M}_n(\partial) \in \mathrm{Der}(\mathbb{M}_n(R), \mathbb{M}_n(M))$ which acts by application of ∂ to matrix entries: $(r_{i,j})_{1\le i,j\le n} \mapsto (\partial(r_{i,j}))_{1\le i,j\le n}$. Check that $\mathbb{M}_n(\partial)$ is indeed a derivation and that the assignment $\partial \mapsto \mathbb{M}_n(\partial)$ defines a bijection from $\mathrm{Der}(R, M)$ to $\mathrm{Der}(\mathbb{M}_n(R), \mathbb{M}_n(M))$.

(iii) Solve (i) with a matrix $s \in \mathbb{M}_n(R)$ instead of $s \in R$.

Exercise 4. Let $s \in R$ be a regular element, satisfying the right Ore condition (see Exercise 2, Sheet 8).

(i) Let M be a right R-module. Prove that any element of $M \otimes_R R[s^{-1}]$ has the form $m \otimes s^{-n}$ for $m \in M$ and $n \in \mathbb{N}$.

(ii) Right multiplication induces a right R-module structure on $M \otimes_R R[s^{-1}]$. Show that M is isomorphic to $M \otimes_R R[s^{-1}]$ if and only if s acts by an invertible transformation on M.

(iii) Prove that $R[s^{-1}]$ is a flat left R-module, that is, the functor $-\otimes_R R[s^{-1}]$ is exact.