Exercise sheet 10.

Name

Exercise	1	2	3	4	Σ
Points					

Deadline: **Wednesday, 29.6.2022, 16:00**.

Please use this page as a cover sheet and enter your name in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Let G and H be discrete groups. A *group correspondence* $X\colon H \leftarrow G$ is a set X together with commuting actions of H on the left and G on the right. The composite of two correspondences $Y\colon K \leftarrow H$ and $X\colon H \leftarrow G$ is the correspondence $Y \circ X$ defined as the quotient $(Y \times X)/H$ with respect to the H-action given by $h \cdot (y, x) = (y \cdot h^{-1}, h \cdot x)$ for $h \in H, y \in Y, x \in X$. The actions of K and G are then given by $k \cdot [y, x] = [k \cdot y, x]$ and $[y, x] \cdot g = [y, x \cdot g]$ for $k \in K, g \in G$, $(x, y) \in X \times Y$. Here $[y, x]$ denotes the class of (y, x) in the quotient.

A morphism between two correspondences $X, Y\colon H \leftarrow G$ is an H-G-equivariant mapping $f\colon X \to Y$.

(i) Define a bicategory $\mathfrak{Grp}$ with discrete groups as objects, correspondences as arrows and morphisms between correspondences as 2-arrows.

(ii) A group correspondence $X\colon H \leftarrow G$ is called a *covering permutational bimodule* if the action of G is free and the orbit space $G\backslash X$ is finite. Prove that covering permutational bimodules define a subbicategory $\mathfrak{Grp}_c \subset \mathfrak{Grp}$.

Exercise 2. Let R be a commutative ring. We are going to define a homomorphism $F_R\colon \mathfrak{Grp} \to \mathfrak{Rings}$ as follows.

- For $G \in \mathfrak{Grp}^0$ we set $F_R^0(G) = R[G]$.
- For a correspondence $X\colon H \leftarrow G$ we let $F_R(X) = R[X]$ be the free R-module on the set X with the obvious $R[H]$-$R[G]$-bimodule structure.

Prove that this data can be extended to a homomorphism. Show that for a permutational bimodule $X\colon G \leftarrow G$, the $\mathbb{Z}[G]$-bimodule $F_{\mathbb{Z}}(X)$ is the one which was discussed in the lectures.

Exercise 3. Let $A \times G\colon G \leftarrow G$ be a permutational bimodule which comes from a finite left G-set A and a cocycle $G \times A \to G$, $(g, a) \mapsto g|_a$. Consider the bimodule $\mathbb{Z}[A \times G]$ and the corresponding $(\mathbb{N}, +)$-shaped diagram in $\mathfrak{Rings}$.

Show that the strong covariance ring U of this diagram is generated by elements δ_g for $g \in G$ and S_a and S_a^* for $a \in A$ subject to the following relations:

- $\delta_g \delta_h = \delta_{gh}$ for $g, h \in G$;
- $S_a^* S_a = 1$ for $a \in A$, $S_a^* S_b = 0$ for $a, b \in A$ with $a \neq b$, and $\sum_{a \in A} S_a S_a^* = 1$ (the Leavitt relations);
- $\delta_g S_a = S_{g(a)} \delta_{g|_a}$ for $g \in G$ and $a \in A$.

Prove that the equations $S_a^* \delta_g = \delta_{g|_{g^{-1}a}} S_{g^{-1}(a)}^*$ for $a \in A$, $g \in G$ follow from the relations above.

Exercise 4. Let $X\colon G \leftarrow G$ be a group correspondence. A (G, X)-action is a G-set Y (that is, a group correspondence $Y\colon G \leftarrow \{e\}$ from the trivial group to G) and an isomorphism $\tau_X\colon X \circ Y \to Y$ of correspondences.

Suppose now that $X = A \times G$ is a permutational bimodule. Prove that a (G, X)-action on Y is equivalent to the following data:

- a partition $Y = \bigsqcup_{a \in A} Y_a$ into subsets;
- group actions of G on Y_a for all $a \in A$;
- bijections $L_a\colon Y \xrightarrow{\sim} Y_a$ for $a \in A$;

such that for any $y \in Y$ we have $g \cdot L_a(y) = L_{g(a)}(g|_a \cdot y)$.