Exercise sheet 11.

Name

Exercise	1	2	3	Σ
Points				

Deadline: **Wednesday, 6.7.2022, 16:00**.

Please use this page as a cover sheet and enter your name in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. In this exercise, we use the notation of Exercise 3 on Sheet 10 and assume that $|A| > 1$. Let G_A be the multiplicative subsemigroup of the strong covariance ring U generated by the elements δ_g for $g \in G$ and S_a, S_a^* for $a \in A$.

(i) Let $I = (I_1 I_2 \dots I_k) \in A^k$ be a word in the alphabet A. We write $S_I := S_{I_1} S_{I_2} \cdots S_{I_k}$ and $S_I^* := S_{I_k} \cdots S_{I_1}$. Show that any nonzero element of G_A has the form $S_I \delta_g S_J^*$ for some words I, J and some $g \in G$.

(ii) Show that for any element $s \in G_A$ there is a unique $s^* \in G_A$ with $ss^*s = s$ and $s^*ss^* = s^*$. (A semigroup with this extra property is called an *inverse semigroup*.)

(iii) Let Y be a set. A partial bijection on Y is a bijection between subsets of Y. Partial bijections form an inverse semigroup $\mathcal{I}(Y)$ under composition. Show that a (G, A)-action on Y (see Exercise 4, Sheet 10) is equivalent to a homomorphism $G_A \to \mathcal{I}(Y)$.

Exercise 2. A morphism $F : \mathcal{C} \to \mathcal{D}$ between two bicategories is invertible in the category of bicategories and morphisms defined in the lecture if and only if it is an isomorphism in the naive sense, that is, F is a homomorphism, and it is bijective on objects, arrows and 2-arrows.

Exercise 3 (Counts double). Let $\mathcal{R}(2)$ be the 2-category with unital rings as objects, ring homomorphisms as 1-arrows and invertible intertwiners as 2-arrows. Here, an intertwiner between two homomorphisms $f, g : R \rightrightarrows Q$ is an invertible element $q \in Q^\times$ with $qf(r) = g(r)q$ for all $r \in R$. The vertical product $q_1 \circ q_2$ of intertwiners is just the product $q_1 q_2$. For $f_1, f_2 : R \rightrightarrows Q$, $g_1, g_2 : Q \rightrightarrows S$ and intertwiners $q : f_1 \to f_2$ and $s : g_1 \to g_2$, the horizontal product is given by $s \bullet q = s \cdot g_1(q) = g_2(q) \cdot s$.

(i) Let G be a group. Show that a strictly unital homomorphism $G \to \mathcal{R}(2)$ is equivalent to a unital map $\alpha : G \to \operatorname{Aut}(R)$ and a collection of intertwiners $\mu_{g,h} : \alpha_g \alpha_h \to \alpha_{gh}$ such that $\mu_{kg,h} \mu_{k,g} = \mu_{k,gh} \alpha_k(\mu_{g,h})$ and $\mu_{1,g} = \mu_{g,1} = 1$ (this is actually a simplified version of Exercise 4, Sheet 7 but with other conventions on the direction of arrows).

(ii) Let $F_1, F_2 : G \rightrightarrows \mathcal{R}$ be two strictly unital homomorphisms with $F_1^0(*) = R$ and $F_2^0(*) = Q$. These are described as above by collections $(\alpha_g^1, \mu_{g,h}^1)$ and $(\alpha_g^2, \mu_{g,h}^2)$, respectively. A transformation $F_1 \Rightarrow F_2$ is described by a ring homomorphism $\sigma : R \to Q$ and a collection of intertwiners σ_g between $\alpha_g^2 \circ \sigma$ and $\sigma \circ \alpha_g^1$ for any $g \in G$. Show that this data satisfies the axioms of a transformation if and only if the equality

$$\sigma(\mu_{g,h}^1) \sigma_g \alpha_g^2(\sigma_h) = \sigma_{gh} \mu_{g,h}^2$$

holds for all $g, h \in G$.

(iii) For two such transformations $\sigma : F_1 \Rightarrow F_2$ and $\sigma' : F_2 \Rightarrow F_3$, describe the intertwiners for the composition $\sigma' \circ \sigma$.

(iv) Describe a modification between two transformations $\sigma^1 \Rrightarrow \sigma^2$ analogously.

(v) Find necessary and sufficient condition for the transformation $\sigma : F_1 \to F_2$ to be an equivalence. When is it even an isomorphism?