Exercise sheet 13.

Exercise	1	2	3	4	Σ
Points					

Name

Deadline: **Wednesday, 11.2.2022, 16:00**.

Please use this page as a cover sheet and enter your name in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories. An *adjunction* between them is a pair of homomorphisms $L\colon \mathcal{D} \to \mathcal{C}$ and $R\colon \mathcal{C} \to \mathcal{D}$ together with an equivalence of categories $\varphi_{d,c}\colon \mathcal{C}(L(d), c) \to \mathcal{D}(d, R(c))$, which is a part of a transformation between the homomorphisms $(d, c) \mapsto \mathcal{C}(L(d), c), \mathcal{D}(d, R(c))$ on the product $\mathcal{D}^{op} \times \mathcal{C} \to \mathfrak{Cat}$ bicategory.

Let $d \in \mathcal{D}^0$. A pair (c, g) with $c \in \mathcal{C}^0$ and $g\colon d \to R(c)$ is called a *universal arrow* from d to R if, for every $x \in \mathcal{C}^0$, the following functor is an equivalence of categories:

$$g^*\colon \mathcal{C}(c, x) \to \mathcal{D}(d, R(x)),\ f \mapsto R(f) \cdot g, \qquad w \mapsto R(w) \bullet 1_g.$$

Show that for any $d \in \mathcal{D}^0$, the pair $(L(d), \varphi_{d,L(d)}(1_{L(d)}))$ is a universal arrow from d to R.

Exercise 2. Let $\mathcal{R}_c(2) \subset \mathcal{R}(2)$ be the subbicategory of commutative rings and denote the inclusion homomorphism by R. Find a homomorphism $L\colon \mathcal{R}(2) \to \mathcal{R}_c(2)$ and a collection of equivalences of categories $\varphi_{d,c}$ as in the previous exercise such that the data (L, R, φ) defines an adjunction. Is there an analogous adjunction for the inclusion $\mathfrak{Rings}_c \to \mathfrak{Rings}$?

Exercise 3. A skeletal bicategory is a bicategory where equivalent objects are equal and isomorphic 1-arrows are equal. Let $\mathcal{A}$ be a bicategory with only one object (a monoidal category). Show that it is equivalent to a skeletal bicategory (**Hint:** every category is equivalent to a skeletal category. Use the existence of an adjoint equivalence, which we have established earlier.)

Exercise 4. Let $G\colon \mathcal{C} \to \mathcal{D}$ be a homomorphism. Construct a dual morphism $G^*\colon \operatorname{Hom}(\mathcal{D}^{op}, \mathfrak{Cat}) \to \operatorname{Hom}(\mathcal{C}^{op}, \mathfrak{Cat})$ that acts on objects by $(G^*)^0(F) = F \circ G$ for every homomorphism $F\colon \mathcal{D}^{op} \to \mathfrak{Cat}$.

Suppose that G is an equivalence. Show that in this case for any $c \in \mathcal{C}^0$, the two objects $\mathbb{Y}_{\mathcal{C}}(c)$ and $G^*(\mathbb{Y}_{\mathcal{D}}(G(c)))$ of $\operatorname{Hom}(\mathcal{C}^{op}, \mathfrak{Cat})$ are equivalent.